Lecture 6: Affine Lie Algebras

Daniel Bump

January 1, 2020

References

- Kac, Infinite-Dimensional Lie Algebras
- DMS, Chapter 14

Sage has considerable algorithms for computing with affine Lie algebras, including methods computing with highest weight integrable representations, which are the most important class. Below are links to the Thematic Tutorial, Lie Methods and Related Combinatorics in Sage by Bump, Salisbury and Schilling. Note that this includes a section on affine root systems that is of interest independent of the Sage material.

- Affine Root System Basics (Web link to Sage documentation)
- Integrable Highest Weight Representations of Affine Lie algebras

Kac-Moody Lie Algebras

Kac-Moody Lie algebras are generalizations of finite-dimensional simple Lie algebras. They include finite-dimensional simple Lie algebras as special cases but are usually infinite-dimensional. Many concepts and results from the representation theory of finite-dimensional Lie groups and Lie algebras extend to Kac-Moody Lie algebras. This includes the root system, Weyl group, weight lattice, the parametrization of important representations (the integrable highest weight ones) by dominant weights and the Weyl character formula for these representations.

Among Kac-Moody Lie algebras, affine Lie algebras are an important infinite-dimensional class. Each affine Lie algebra $\mathfrak{g}$ is related to a finite-dimensional Lie algebra $\mathfrak{g}_0$. We will only consider untwisted affine Lie algebras.

Generalized Cartan Matrix

The basic data defining a Kac-Moody Lie algebra is a (generalized) Cartan matrix. This is a square matrix $A = (a_{ij})$ with diagonal entries equal to 2 and nonpositive off diagonal entries such that $a_{ij} = 0$ if and only if $a_{ji} = 0$. It is useful to assume that it is indecomposable and symmetrizable. Indecomposable means that it cannot be arranged into two diagonal blocks by permuting the rows and columns; and symmetrizable means that DA is symmetric for some invertible diagonal matrix D.

Simple Roots and Coroots

Given a generalized Cartan matrix there is a vector space $\mathfrak{h}$ containing vectors $\alpha_i^\vee$ (called simple coroots) and vectors $\alpha_i \in \mathfrak{h}^*$ (called simple roots) such that $\langle \alpha_i^\vee, \alpha_j \rangle = \alpha_i^\vee(\alpha_j) = a_{ij}$. Moreover there exists a Kac-Moody Lie algebra $\mathfrak{g}$ containing $\mathfrak{h}$ as an abelian subalgebra that is generated by $\mathfrak{h}$ and elements e_i and f_i such that

$$[e_i, f_i] = \delta_{ij}\alpha_i^\vee, \qquad [h, e_i] = \alpha_i(h) e_i, \qquad [h, f_i] = -\alpha_i(h) f_i.$$

(These conditions do not quite characterize $\mathfrak{g}$, but they do if supplemented by the Serre relations, which we will not need or state.)

Symmetrizable Cartan Types and their dual types

The significance of the symmetrizability assumption is that $\mathfrak{g}$ admits an invariant symmetric bilinear form, and hence has a Casimir operator and a good representation theory.

The transpose of A is also a symmetrizable indecomposable generalized Cartan matrix, so there is a dual Cartan type in which the roots and coroots are interchanged.

If A is the Cartan type of a finite semisimple Lie group, then the dual Cartan type is the type of the Langland dual group.

The Cartan Matrix in Sage

In Sage, we may recover the Cartan matrix as follows:

```
sage: RootSystem(['B',2]).cartan_matrix()
[ 2 -1]
[-2  2]
sage: RootSystem(['B',2,1]).cartan_matrix()
[ 2  0 -1]
[ 0  2 -1]
[-2 -2  2]
```

The first example is the finite Cartan type B_2, which is the Cartan type of $\mathfrak{so}(5)$.

If $\det(A) = 0$ and its nullspace is one-dimensional, then $\mathfrak{g}$ is an affine Lie algebra as in the second example (Type $B_2^{(1)}$).

A note on notation

Untwisted affine Lie algebras are associated with finite-dimensional semisimple Lie algebras. We will follow the notation of Kac, Infinite-dimensional Lie algebras, and denote the finite-dimensional Lie algebra as $\mathfrak{g}^\circ$ and the associated affine Lie algebra as $\mathfrak{g}$.

But in future lectures we will probably follow the convention of denoting the finite-dimensional Lie algebra as $\mathfrak{g}$ and its affinization as $\widehat{\mathfrak{g}}$.

Review: central extensions

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an abelian Lie algebra. A bilinear map $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ is called a 2-cocycle if it is skew-symmetric and satisfies

$$\sigma([X,Y],Z) + \sigma([X,Y],Z) + \sigma([X,Y],Z) = 0, \qquad X,Y,Z \in \mathfrak{a}.$$

In this case we may define a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{a}$ by

$$[(X,a),(Y,b)] = ([X,Y], \sigma(X,Y)).$$

Denoting this Lie algebra $\mathfrak{g}'$ we have a central extension

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Adding a derivation

Suppose we have a derivation d of a Lie algebra $\mathfrak{g}$. This means

$$d([x,y]) = [dx,y] + [x,dy].$$

We may then construct a Lie algebra $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}d$ in which $[d,x] = d(x)$ for $x \in \mathfrak{g}$.

This is a special case of a more general construction, the semidirect product.

Affine Lie algebras as Central Extensions

Although the affine Lie algebra $\mathfrak{g}$ may be constructed from its Cartan matrix, another construction described in Chapter 7 of Kac begins with the finite-dimensional simple Lie algebra $\mathfrak{g}^\circ$ of rank ℓ. Tensoring with the Laurent polynomial ring gives the loop Lie algebra $\mathfrak{g}^\circ \otimes \mathbb{C}[t, t^{-1}]$. This is the Lie algebra of vector fields in $\mathfrak{g}^\circ$ on the circle. Then one may make a central extension:

$$0 \to \mathbb{C} \cdot K \to \mathfrak{g}' \to \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}^\circ \to 0.$$

After that, one usually adjoins another basis element, which acts on $\mathfrak{g}'$ as a derivation d. This gives the full affine Lie algebra $\mathfrak{g}$.

The cocycle

To describe the central extension we need a 2-cocycle on $\mathfrak{g}^\circ \otimes \mathbb{C}[t, t^{-1}]$. (See Lecture 3.) We will denote by $(\ |\ )$ the unique (up to scalar) ad-invariant bilinear form on $\mathfrak{g}^\circ$. We extend it to a bilinear form $(\ |\ )_t$ on $\mathfrak{g}^\circ \otimes \mathbb{C}[t, t^{-1}]$ taking values in $\mathbb{C}[t, t^{-1}]$ by

$$(t^n \otimes x | t^m \otimes y)_t = t^{n+m}(x|y).$$

If $a = \sum t^n \otimes a_n$ (a finite sum) let

$$\frac{da}{dt} = \sum n t^{n-1} \otimes a_n,$$

and define the cocycle

$$\psi(a, b) = \mathrm{Res}\left(\frac{da}{dt} | b\right)_t ,$$

where the residue is the coefficient of t^{-1}.

The central extension

Now that we have the cocycle we can define the central extension $\mathfrak{g}' = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C} \cdot K$ where K is to be a central element and

$$[a + \lambda K, b + \mu K] = [a,b]_0 + \psi(a,b)K.$$

Here we are denoting $[a,b]_0$ the Lie bracket on $\mathbb{C} \otimes [t,t^{-1}] \otimes \mathfrak{g}$, to distinguish it from the new bracket that is being defined.

To $\mathfrak{g}'$ we may adjoin a derivation $d = t\, d/dt$ to obtain the full affine Lie algebra $\mathfrak{g}$.

Enlargement by Vir

Instead of enlarging $\mathfrak{g}'$ by the derivation d we may form the semidirect product of $\mathfrak{g}'$ by the entire Virasoro algebra

$$[d_i, d_j] = (i-j)d_{i+j} + \frac{1}{12}(i^3 - i)\delta_{i,-j} \cdot C$$

where d_i acts as the derivation $-t^{i+1}d/dt$. This contains the affine Lie algebra $\mathfrak{g}$ with $d = -d_0$.

The affine root system

The roots may be defined to be the nonzero weights in $\mathfrak{h}^*$ in the adjoint representation of $\mathfrak{g}$.

Let $\mathfrak{h}^\circ$ be the Cartan subalgebra of the finite-dimensional Lie algebra $\mathfrak{h}^\circ$. We enlarge it by adding K and $\mathfrak{d}$. Thus the simple roots $\alpha_1, \cdots, \alpha_\ell$ become roots of $\mathfrak{g}$.

There is an “imaginary root” δ which is defined to be zero on $\mathfrak{h}$ and K but $\delta(d) = 1$. Now we may describe all the roots. There are two kinds.

The affine root system (continued)

Real Roots: These have the form $\alpha + n\delta$ where α is in the root system Δ° of $\mathfrak{g}^\circ$ and $n \in \mathbb{Z}$. They have multiplicity one in that the α-eigenspace of $\mathfrak{h}$ in $\mathfrak{g}$ is one-dimensional.

Imaginary Roots: These have the form $n\delta$ where $n \in \mathbb{Z}$ and $n \neq 0$. They have multiplicity ℓ.

Here is the $\widehat{\mathfrak{sl}}(2)$ affine root system. Positive roots: $\bullet$ Negative roots: $\circ$

$$
\begin{array}{ccccccc}
\cdots & \circ & \overset{-\alpha_0}{\circ} & \overset{\alpha_1}{\bullet} & \overset{\alpha_1+\delta}{\bullet} & \bullet & \cdots \\
\\
\cdots & \circ & \circ^{-\delta} & \cdot & \bullet\ \delta & \bullet 2\delta & \cdots \\
\\
\cdots & \circ & \circ & \underset{-\alpha_1}{\circ} & \underset{\alpha_0}{\bullet} & \bullet & \cdots
\end{array}
$$

The root α_0 and the Coxeter numbers

Let θ be the highest root in the finite root system $\mathfrak{g}^\circ$. Then $\delta - \theta$ is a root, the affine root α_0. The simple roots are

$$\{\alpha_0, \alpha_1, \cdots, \alpha_\ell\}.$$

We write

$$\delta = \sum_{i=0}^{\ell} a_i \alpha_i,$$

where the labels or marks a_i may be found in tables in Kac's book or many other places. For $\widehat{\mathfrak{sl}}(n)$ they are all equal to 1. There are also dual marks $a_i^\vee$. The numbers

$$h = \sum a_i, \qquad h^\vee = \sum a_i^\vee$$

are called the Coxeter number and dual Coxeter number, respectively.

The Weyl vector ρ

In the representation theory of a finite-dimensional semisimple Lie group, the vector ρ can be defined as half the sum of the positive roots. It appears everywhere, for example in the Weyl character formula:

$$\chi_\lambda(\mathbf{z}) = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\lambda+\rho)}.$$

Here λ is a dominant weight and the Weyl denominator

$$\Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\rho)} = \prod_{\alpha \in \Delta^+} \mathbf{z}^{\rho} (1 - \mathbf{z}^{-\alpha}).$$

For affine Weyl groups, or more generally infinite-dimensional Kac-Moody groups, ρ still exists but cannot be defined as half the sum of the positive roots. It can be characterized by $(\rho|\alpha_i^\vee) = 1$.

Triangular decomposition

The root system Δ may be partitioned into positive roots and negative roots. Let Δ^+ and Δ^- be the positive and negative roots.

If α is a root, let $\mathfrak{g}_\alpha$ be the α eigenspace of $\mathfrak{h}$ on $\mathfrak{g}$. Thus

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Moreover we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

So there is a Category $\mathcal{O}$, Verma modules, etc. (Kac Chapter 9.)

The affine Weyl group

There is an affine Weyl group, an infinite Coxeter group that can optionally be enlarged to an "extended" group that is not a Coxeter group but sometimes important to work with.

We recall that the Weyl group W° of $\mathfrak{g}^\circ$ acts on $(\mathfrak{h}^\circ)^*$ as follows. There are generators $s_1, \cdots, s_\ell$ and s_i is the reflection

$$x \to x - \langle x|\alpha_i^\vee\rangle\alpha_i,$$

where $(\ ,\ )$ is an invariant inner product on $(\mathfrak{h}^\circ)^*$. The inner product $(\ |\ )$ is positive definite.

The inner product may be enlarged to an inner product on $\mathfrak{h}^*$. It is no longer definite since $(\delta|\delta) = 0$. The affine Weyl group adds a single reflection s_0.

Overview

In conformal field theory, we encounter theories whose fields are modular forms. Such a theory depends on two things: a finite-dimensional Lie algebra $\mathfrak{g}^\circ$ with corresponding affine Lie algebra $\mathfrak{g}$, and a level k. The primary fields in such a theory are in bijection with certain representations of $\mathfrak{g}$ – those of level k – and the characters of these representations are modular forms. This was proved by Kac and Peterson (1984) and is the topic of Chapter 14 of FMS. Chapters 15-17 of FMS discuss the role of affine Lie algebras in the WZW conformal field theories.

Modular forms

The group $SL(2,\mathbb{R})$ acts on the upper half plane by linear fractional transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}.$$

The group $\Gamma_0(N)$ is the subgroup where $c \equiv 0 \bmod N$.

A function f on the upper half plane is called a (weakly) modular form of weight k and level N if it is holomorphic and satisfies

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (cz+d)^k f(z), \qquad \gamma \in \Gamma_0(N).$$

This implies $f(z+1) = f(z)$ so f has a Fourier expansion

$$f(z) = \sum_n a_n q^n, \qquad q = e^{2\pi i z}.$$

Dominant integral weights

A integral weight λ is an element of $\mathfrak{h}^*$ such that $(\alpha_i^\vee|\lambda) \in \mathbb{Z}$ for all simple coroots. We will denote by P the weight lattice of integral weights. If $(\alpha_i|\lambda) \geqslant 0$ we call the weight dominant. The dominant integral weights form a cone P^+.

As in Lecture 4, if $\lambda \in \mathfrak{h}^*$ then the Verma module $M(\lambda)$ has a unique irreducible quotient $V = L(\lambda)$. If $\lambda \in P^+$, then V is called integral. These are the most important irreducible representations, infinite-dimensional but analogous to the finite-dimensional irreducible representations of a Lie group. Their characters are given by Kac's generalization of the Weyl character formula.

Integrable representations

The term integral means that these representations "integrate" to representations of the loop group. The weight multiplicities are invariant under the affine Weyl group.

Let $\Lambda \in P^+$ and let $V = L(\Lambda)$ be the integrable representation with highest weight Λ. If μ is another weight, let $\mathrm{mult}(\mu)$ denote the multiplicity of the weight μ in $L(\Lambda)$. Define the *support* of the representation $\mathrm{supp}(V)$ to be the set of μ such that $\mathrm{mult}(\mu) > 0$.

If $\mathrm{mult}(\mu) > 0$ then $\lambda - \mu$ is a linear combination of the simple roots with nonnegative integer coefficients. Moreover $\mathrm{supp}(V)$ is contained in the paraboloid

$$(\Lambda + \rho|\Lambda + \rho) - (\mu + \rho|\mu + \rho) \geqslant 0$$

We organize the weight multiplicities into sequences called string functions or strings as follows. By Kac Proposition 11.3 or Corollary 11.9, for fixed μ the function $\mathrm{mult}(\mu - k\delta)$ of k is an increasing sequence. We adjust μ by a multiple of δ to the beginning of the positive part of the sequence. Thus we define μ to be maximal if $\mathrm{mult}(\mu) \neq 0$ but $\mathrm{mult}(\mu + \delta) = 0$.

Since δ is fixed under the action of the affine Weyl group, and since the weight multiplicities are Weyl group invariant, the function $k \mapsto \mathrm{mult}(\mu - k\delta)$ is unchanged if μ is replaced by $w(\mu)$ for some Weyl group element w. Now every Weyl orbit contains a dominant weight. Therefore in enumerating the string we may assume that the weight μ is dominant. There are only a finite number of dominant maximal weights. Thus there are only a finite number of such strings to be computed.

Modular characteristics

In 1984, Kac and Peterson showed that each string is the set of Fourier coefficients of a weakly holomorphic modular form; see also Kac Chapters 12 and 13. Here weakly holomorphic modular means that the form is allowed to have poles at cusps.

To this end we define the modular characteristic, which originally appeared in string theory (I think):

$$m_\Lambda = \frac{|\Lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2h^\vee}.$$

Here $k = (\Lambda|\delta)$ is the level of the representation and $h^\vee$ is the dual Coxeter number, defined above. If μ is a weight, define

$$m_{\Lambda,\mu} = m_\Lambda - \frac{|\mu|^2}{2k}.$$

String Functions as modular forms

Let Λ be a dominant integral weight, and let μ be maximal weight, defined above. Then Kac and Peterson defined the string function

$$c_\mu^\Lambda = q^{m_{\Lambda,\mu}} \sum_{n \in \mathbb{Z}} \mathrm{mult}(\mu - n\delta) q^n.$$

Although these do arise as partition functions in string theory, the term "string" here does not refer to physical strings.

The string function c_μ^Λ is a weakly holomorphic modular form, possibly of half-integral weight. See Kac, Corollary 13.10, or Kac and Peterson (1984). It can have poles at infinity, but multiplying c_μ^Λ by $\eta(\tau)^{\dim \mathfrak{g}^\circ}$ gives a holomorphic modular form (for some weight and level). Here $\eta = q^{1/24} \prod_{k=1}^\infty (1 - q^k)$ is the Dedekind eta function.

Advertisement: Sage methods

Sage has methods for working with integrable representations of affine Lie algebras.

Integrable Highest Weight Representations of Affine Lie algebras

In the following example, we work with the integrable representation with highest weight $2\Lambda_0$ for $\widehat{\mathfrak{sl}}_2$, that is, $A_1^{(1)}$. First we create a dominant weight in the extended weight lattice, then create the `IntegrableRepresentation` class.

We compute the string functions. There are two, since there are two dominant maximal weights. One of them is the highest weight $2\Lambda_0$, and the other is $2\Lambda_1 - \delta$.

Example

```
sage: L = RootSystem("A1~").weight_lattice(extended=True)
sage: Lambda = L.fundamental_weights()
sage: delta = L.null_root()
sage: W = L.weyl_group(prefix="s")
sage: s0, s1 = W.simple_reflections()
sage: V = IntegrableRepresentation(2*Lambda[0])
sage: V.strings()
{2*Lambda[0]: [1, 1, 3, 5, 10, 16, 28, 43, 70, 105, 161, 236],
 2*Lambda[1] - delta: [1, 2, 4, 7, 13, 21, 35, 55, 86, 130, 196, 287]}
sage: mw1, mw2 = V.dominant_maximal_weights(); mw1, mw2
(2*Lambda[0], 2*Lambda[1] - delta)
```

Further examples may be found in the tutorial.