Lecture 5: The Free Boson

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Review: Wightman Axioms

Axiom 0. We have a unitary representation U of the Poincaré group $\mathcal{P} = R^{(3,1)} \rtimes \mathrm{SL}(2,\mathbb{C})$ on a Hilbert space $\mathcal{H}$, subject to the following positivity assumption. By Wigner's theorem $U(T_i(a)) = e^{iaP_i}$ where $T_i(a)$ is the translation $x_i \to x_i + a$. Here P_i are commuting Hermitian operators. It is assumed that if (p_1, p_2, p_3, e) are in the joint spectrum of $P_1, P_2, P_3, P_4 = H$ then $e^2 - p_1^2 - p_2^2 - p_3^2 \geqslant 0$.

It is also assumed that there is a vacuum vector $|0\rangle$ that is invariant under all the operators $U(\gamma)$ ($\gamma \in \mathcal{P}$).

Wightman Axiom 1

Axiom 1. It is assumed that there are given fields $\phi_1, \cdots, \phi_n$ that assign to a Schwartz function f operators $\phi_i(f)$ on $\mathcal{H}$. It is assumed that there exists a dense linear subspace D of $\mathcal{H}$ that contains the vacuum $|0\rangle$, and that is closed under the $\phi_i(f)$ and their adjoints.

Moreover it is assumed that if $\Phi, \Psi \in D$ then the functional on $f \in \mathcal{S}(\mathbb{R}^{(3,1)})$ given by $\langle \Phi | \phi_i(f) | \Psi \rangle$ is a tempered distribution.

Wightman Axiom 2

We choose a representation $\pi : \mathrm{SL}(2,\mathbb{C}) \to GL(n)$, where n is the number of fields. Since $\mathcal{P}$ is a semidirect product of $\mathrm{SL}(2,\mathbb{C})$ by $\mathbb{R}^{(3,1)}$ the group of translations is normal, and there is a homomorphism $\mathcal{P} \to \mathrm{SL}(2,\mathbb{C})$ with kernel $\mathbb{R}^{(3,1)}$. Thus we can extend π to $\mathcal{P}$.

Axiom 2. If $\gamma \in \mathcal{P}$ and $f \in \mathcal{S}(\mathbb{R}^{(3,1)})$

$$U(\gamma)\Phi_f U(\gamma)^{-1} = \pi(\gamma)\Phi_f.$$

Remark: Potentially the Poincaré group $\mathcal{P}$ can be expanded to the conformal group, leading to conformal field theories. In the $d = 2$ case we can also include the action of the local conformal operators, that is the Witt algebra.

Wightman Axiom 3: locality

Let f and g be test functions whose supports are spacelike separated. This means that if $x = (x_1, x_2, x_3, t)$ and $x' = (x_1', x_2', x_3', t')$ are such that $f(x) \neq 0$ and $g(y) \neq 0$ then

$$(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 - (t - t')^2 > 0.$$

Then the Hermitian operators associated with these fields commute in the superalgebra sense:

$$\phi_i(f)\phi_j(g) = (-1)^{\deg(\phi_1)\deg(\phi_2)}\phi_j(g)\phi_i(f).$$

Or in the fiction that distributions are functions,

$$\phi_i(x)\phi_j(x') = (-1)^{\deg(\phi_1)\deg(\phi_2)}\phi_j(x')\phi_i(x)$$

if the points x and x' are spacelike separated.

References

We have stated the Wightman axioms of QFT and motivated them to some extent. However we will not given an example. The simplest example is the free boson. The Hilbert space that emerges is the Bosonic Fock space.

- Schottenloher, Sections 8.1 and 8.3.
- FMS, Sections 2.1 and 2.3
- Folland, QFT, A Tourist Guide for Mathematicians, Section 5.1

We will follow Schottenloher rather closely. But we begin with another approach.

The classical setup

We start with the Klein-Gordan equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + m^2\right) \phi(\mathbf{x}, t) = 0.$$

This describes a massive scalar field.

Note that the equation is Lorentz invariant. We will encounter the energy-momentum vector $\mathbf{p} = (p_0, p_1, p_2, p_3)$ where in a fixed inertial frame p_0 is energy and p_1, p_2, p_3 are the components of momentum. We will write $p = (p_1, p_2, p_3)$ to denote momentum per se. The Mass shell Γ_m consists of solutions to $|\mathbf{p}|^2 = m^2$, $p_0 > 0$.

One approach

One approach proceeds nonrelativistically but results in a Lorentz invariant operator. For the free boson of mass m

$$(\partial_t^2 - \nabla_{\mathbf{x}} + m^2)\phi(t,x) = 0$$

is to make a Fourier transform in the spacelike variables and write:

$$\phi(t,x) = \int_{\mathbb{R}^3} \psi_p(t) e^{ip\cdot x}\, dp.$$

Since $\nabla e^{ip\cdot x} = -|p|^2 e^{ip\cdot x}$ we must have

$$(\partial_t^2 + \omega_{\mathbf{p}}^2)\psi_p(t) = 0, \qquad \omega_p = \sqrt{m^2 + |p|^2}.$$

Enter the harmonic oscillators

To quantize this system, we note that

$$(\partial_t^2 + \omega_p^2)\psi_p(t) = 0$$

is the differential equation of the classical harmonic oscillator of frequency $\omega_p/2\pi$. Thus we have one (classical) harmonic oscillator for each momentum, so the quantum system will have one quantum harmonic oscillator for each momentum.

Particularly, this approach introduces creation and annihilation operators $a^\dagger(p)$, $a(p)$ for every p, creating or annihilating a particle of given momentum. To just state the answer

$$\phi(t,x) = \int_{\mathbb{R}^3} (a(p)e^{i(p\cdot x - \omega_p t)} + a^\dagger(p)e^{-i(p\cdot x - \omega_p t)})d\lambda_m(p)$$

Here $-p\cdot x + \omega_p t$ lies on the Mass shell $|p|^2 = m^2$ and λ_m is the $SO(1,3)^\circ$ invariant measure on Γ_m.

Another approach

We will show how the operator-valued distributions satisfying the Wightman axioms appear by a different method, following Schottenloher.

To reiterate, the free boson is the quantization of the classical Klein-Gordan equation

$$(\square + m^2)\phi = 0, \qquad \square = \frac{\partial^2}{\partial t^2} - \nabla, \qquad \nabla = \sum \frac{\partial^2}{\partial x_i^2}.$$

We are using coordinates $x_0, x_1, \cdots, x_{d-1}$ where $x_0 = t$ is time. The conjugate coordinates $p_0, \cdots, p_{d-1}$ have the interpretation as the energy-momentum vector.

Fourier transform

Consider the Fourier transform

$$\mathcal{F}\phi(\mathbf{p}) = \int_{\mathbb{R}^{1,d-1}} \phi(\mathbf{x}) e^{i\mathbf{p}\cdot\mathbf{x}}\, d\mathbf{x}$$

where $\mathbf{p}\cdot\mathbf{x} = p_0x_0 - p_1x_1 - \cdots p_{D-1}x_{D-1}$. Integrating by parts,

$$(\mathcal{F}\Box\phi)(\mathbf{p}) = -|\mathbf{p}|^2\mathcal{F}\phi(\mathbf{p})$$

so any solution to the Klein-Gordan equation has its Fourier transform contained in $\Gamma_m \cup (-\Gamma_m)$ where Γ_m is the Mass shell $\{|\mathbf{p}|^2 = m^2, p_0 > 0\}$.

Inverse Fourier Transform

Conversely we may obtain solutions to the Klein-Gordan equation by taking the inverse Fourier transform of a distribution whose support is in Γ_m. For example the Pauli-Jordan distribution

$$D_m(\mathbf{x}) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^{1,d-1}} \operatorname{sgn}(p_0)\delta(|\mathbf{p}|^2 - m^2)e^{-i\mathbf{p}\cdot\mathbf{x}}\, d\mathbf{p}.$$

Remembering that the Fourier transform interchanges convolution with pointwise multiplication, and function that is a convolution with D_m is a solution of the Klein-Gordan equation.

Vanishing of D_m

The function

$$D_m(\mathbf{x}) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^{1,d-1}} \operatorname{sgn}(p_0)\delta(|\mathbf{p}|^2 - m^2)e^{-i\mathbf{p}\cdot\mathbf{x}}\, d\mathbf{p}$$

vanishes if x is spacelike, i.e. $x = (x_0, x_1, x_2, x_3)$ and $x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0$. To see this, note that if $x_0 = 0$, then the transformation $(p_0, p_1, p_2, p_3) \to (-p_0, p_1, p_2, p_3)$ demonstrates cancellation. The general case follows since the expression is Lorentz invariant.

Fock space

Here is a description of the field theory satisfying the Wightman axioms. First let us describe the Hilbert space $\mathcal{H}$. Note that Γ_m is a homogeneous space for $SO(1,3)^\circ$ and so it has an invariant measure λ_m.

Let $\mathcal{H}_N$ be the space of functions on Γ_m^N that are symmetric under the permutations of the variables, and square integrable with respect to the inner product

$$\langle u, v\rangle = \frac{1}{N!}\int_{\Gamma_m^N} \overline{u(\xi_1,\cdots,\xi_N)}v(\xi_1,\cdots,\xi_N)\,d\lambda_m(\xi_1)\cdots d\lambda_m(\xi_N).$$

Now let

$$\mathcal{H} = \bigoplus_{N=0}^{\infty} H_N.$$

This is the bosonic Fock space.

The Wightman Free Boson QFT

The action of the Lorentz group $SO(1,d-1)^\circ$ on $\mathcal{H}$ is via its action on the cone Γ_m. There is exactly one field ϕ, which we recall is to be an operator-valued distribution on $\mathbb{R}^{1,d-1}$. The distribution ϕ will satisfy the Klein-Gordan equation in the sense that

$$\phi(\Box f + m^2 f) = 0, \qquad f \in \mathcal{S}(\mathbb{R}^{1,d-1}).$$

If $g = (g_0, g_1, g_2, \cdots) \in \mathcal{H}$ then we define

$$(\phi(f)g)_N(\xi_1, \cdots, \xi_N) = \int_{\Gamma_m} \hat{f}(\xi) g_{N+1}(\xi, \xi_1, \cdots, \xi_N) d\lambda_m(\xi)$$

$$+ \sum_{j=1}^{N} \hat{f}(-\xi_j) g_{N-1}(\xi_1, \cdots, \widehat{\xi_j}, \cdots, \xi_N)$$

where $\widehat{\xi_j}$ denotes an omitted variable in the last expression.

The Wightman Axioms

The Wightman axioms may be verified. The joint spectrum of the momentum operators is contained in Γ_m, which is contained in the positive cone.

To prove locality, one uses the identity

$$[\phi(f), \phi(g)] = -i \int_{\mathbb{R}^{(1,d-1} \times \mathbb{R}^{(1,d-1)}} f(x) D_m(x-y) g(y)\, dx\, dy,$$

which may be proved by a calculation, where (we recall) the Pauli-Jordan function

$$D_m(\mathbf{x}) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^{1,d-1}} \operatorname{sgn}(p_0) \delta(|\mathbf{p}|^2 - m^2) e^{-i\mathbf{p}\cdot\mathbf{x}}\, d\mathbf{p}.$$

Locality follows since $D_m(\mathbf{x}-\mathbf{y}) = 0$ for spacelike separation.