Lecture 4: Representation Theory

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January 1, 2020

References

Much of the material in this section is standard. It applies not only to the Virasoro algebra but to any Lie algebra with a triangular decomposition. This includes all finite-dimensional semisimple (or reductive) Lie algebras, indeed all Kac-Moody Lie algebras including the affine ones that will concern us, and the Virasoro algebra.

A good reference is Kac, Infinite-Dimensional Lie Algebras, Chapter 9.

The universal enveloping algebra

If A is an associative algebra, it obtains a Lie algebra structure by defining $[a, b] = ab - ba$. Let $\mathrm{Lie}(A)$ be A with this Lie algebra structure.

If $\mathfrak{g}$ is a Lie algebra, there is a universal associative algebra $U(A)$ with a homomorphism $i : \mathfrak{g} \to \mathrm{Lie}(A)$. It can be defined by quotienting the tensor algebra $T(A)$ by the ideal generated by expressions $a \otimes b - b \otimes a - [a, b]$.

The Poincaré-Birkhoff-Witt theorem asserts that if $X_1, \cdots, X_d$ is a basis of $\mathfrak{g}$ then $U(A)$ has a basis consisting of monomials $X_1^{k_1} \cdots X_d^{k_d}$ for $(k_1, \cdots, k_d) \in \mathbb{N}^d$.

Weight decompositions

Let $\mathfrak{h}$ be an abelian Lie algebra, and V a module. If $\lambda \in \mathfrak{h}^*$ let

$$V(\lambda) = \{v \in V \mid H \cdot v = \lambda(H)v, H \in \mathfrak{h}\}.$$

We say that V has a weight space decomposition if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V(\lambda).$$

If $V(\lambda) \neq 0$ we say λ is a weight of V.

Proposition

If V has a weight space decomposition then so does any submodule.

For a proof, see Kac, Infinite-Dimensional Lie Algebras Proposition 1.5.

Lie algebras with triangular decomposition

Let $\mathfrak{g}$ be a complex Lie algebra that can be written as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, where $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$ are Lie subalgebras with $\mathfrak{h}$ abelian, such that

$$[\mathfrak{h}, \mathfrak{n}_+] \subseteq \mathfrak{n}_+, \qquad [\mathfrak{h}, \mathfrak{n}_-] \subseteq \mathfrak{n}_-.$$

We require that

$$[\mathfrak{h}, \mathfrak{n}_+] \subset \mathfrak{n}_+, \qquad [\mathfrak{h}, \mathfrak{n}_0] \subset \mathfrak{n}_-.$$

This implies that $\mathfrak{n}_\mu \oplus \mathfrak{h}$ are Lie algebras, denoted $\mathfrak{b}$ and $\mathfrak{b}_-$.

We assume that $\mathfrak{n}_\pm$ have weight space decompositions with respect to the adjoint representation under $\mathfrak{h}$ and that 0 is not a weight. Moreover we assume there is a closed convex cone $D \subset \mathfrak{h}^*$ such that D (resp. $-D$) contains the weights of $\mathfrak{n}_+$ ($\mathfrak{n}_-$) and that $D \cap (-D) = \{0\}$.

Examples. Enveloping Algebra

Lie algebras having triangular decomposition include all Kac-Moody Lie algebras. That includes finite semisimple Lie algebras, and affine Lie algebras. It also includes the Virasoro algebra.

Using the PBW theorem it is easy to see that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$$

implies

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \otimes U(\mathfrak{n}_-).$$

Relation with Bruhat decomposition

If $G = \mathrm{GL}(n)$ then $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are the Lie algebras of the diagonal torus T, and the Lie subgroups N and N_- of upper and lower triangular matrices respectively. We may also consider the Borel subgroups $B = TN$ and $B_- = TN_-$. One version of the Bruhat decomposition states that

$$G = \bigcup_{w \in W} BwB_-,$$

where $W \cong S_n$ is the Weyl group, and representatives w are chosen. This is a cell decomposition with one dense open cell BB_- corresponding to $w = 1$. Writting $BB_- = NTN_-$ the multiplication map $N \times T \times N_- \to G$ is a homeomorphism onto its image, which is dense in G. The triangular decomposition as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$a is a reflection of this fact.

BGG Category $\mathcal{O}$

Let Φ_- be the set of weights in $\mathfrak{n}_-$ which is an $\mathfrak{h}$-module under the adjoint representation. Let Q_- be the set of finite sums of elements of Φ_- (with repetitions allowed). This is a discrete subset of $-D$.

The Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ of modules can be defined for any Lie algebra with triangular decomposition. A module V in this category is assumed to have a weight space decomposition with finite-dimensional weight spaces. Furthermore, it is assumed that there is a finite set of weights $\lambda_1, \cdots, \lambda_N$ such that the weights of V lie in the set

$$\bigcup_i (\lambda_i + Q_-).$$

Highest weight modules

A module V is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there is a vector $v \in V(\lambda)$ such that $X \cdot v = 0$ for $X \in \mathfrak{n}_+$, and such that $V = U(\mathfrak{g}) \cdot v$. Since

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \otimes U(\mathfrak{n}_-)$$

this is equivalent to $V = U(\mathfrak{n}_n) \cdot v$.

Any highest weight module is in Category $\mathcal{O}$.

Verma modules

There exists a unique highest weight module M_λ such that if V is a highest weight module with highest weight λ then V is isomorphic to a quotient of M_λ. To construct M_λ, note that λ extends to a character of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ by letting $\mathfrak{n}_+$ act by zero. Let $\mathbb{C}_\lambda$ be $\mathbb{C}$ with this $\mathfrak{b}$-module structure, with generator 1_λ. Define

$$M_\lambda = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda.$$

In view of

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \otimes U(\mathfrak{n}_-),$$

the map $\xi \to \xi \otimes 1_\lambda$ is a vector space isomorphism $U(\mathfrak{n}_- \to M_\lambda$. Let $v_\lambda = 1 \otimes 1_\lambda$ be the highest weight element of M_λ, unique up to scalar.

Irreducibles

The module $M = M_\lambda$ may or may not be irreducible. A vector $x \in M_\lambda(\mu)$ is called primitive if there exists a submodule U of V such that $x \notin U$ but $n_+ \cdot x \subset U$. Thus if $n_+ \cdot x = 0$, then x is primitive. If M_λ has a primitive vector x whose weight $\mu \neq \lambda$ then $U(\mathfrak{g}) \cdot x$ is a proper submodule, so V is not irreducible if such primitive vectors may be found.

Theorem

If $\lambda \in \mathfrak{h}^*$ then there is a unique irreducible highest weight representation with highest weight λ. It is a quotient of M_λ.

Proof

By a universal property of M_λ, any highest weight module is a quotient, so we need to show M_λ has a unique irreducible quotient.

Lemma

Let V be a highest weight module, with highest weight λ and highest weight vector u_λ. If U is a submodule, then U is proper if and only if $u_\lambda \notin U$.

If $u_\lambda \notin U$ then obviously U is proper. If $u_\lambda \in U$ then $U = V$ since u_λ generates V.

From this, it is clear that the sum of the proper submodules of V is a proper submodule. Thus M_λ has a unique largest proper submodule Q, and M_λ / Q is irreducible, and it is the only quotient that is irreducible.

References

References for this section are:

- Kac and Raina, Chapters 1-10
- Di Francesco, Mathieu and Senechal, Conformal Field Theory, Chapter 7
- Schottenloher, A mathematical introduction to Conformal Field Theory, Chapter 6.

Triangular decomposition

Recall that **Vir** is spanned by a central element C and d_n ($n \in \mathbb{Z}$) such that

$$[d_n, d_m] = (n-m)d_{n+m} + \delta_{n,-m}\frac{1}{12}(n^3 - n)\,C.$$

We take $\mathfrak{h}$ to be the span of C and d_0. This is an abelian subalgebra. We take $\mathfrak{n}_+$ to be the span of d_n with $n > 0$ and $\mathfrak{n}_-$ to be the span of d_n with $n < 0$. This gives us a triangular decomposition.

The important representations of **Vir** are in Category , and the irreducibles are highest weight representations. If V is a highest weight representation with highest weight λ, the weight $\lambda \in \mathfrak{h}^*$ is determined by its eigenvalues on C and d_0. These are denoted c and h.

Real forms of Lie algebras

If $\mathfrak{g}$ is a complex Lie algebra, by a real form of $\mathfrak{g}$ we mean a real subalgebra $\mathfrak{g}_0$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_\circ$. For example, let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. This is the (complex) n^2-dimensional Lie algebra of $n \times n$ complex matrices. We will describe two distinct real forms.

The first is the real Lie algebra $\mathfrak{gl}(, n\mathbb{R})$ of $n \times n$ real matrices. It is the Lie algebra of the noncompact Lie group $GL(n, \mathbb{R})$.

Another real form is the Lie algebra $\mathfrak{u}(n)$ of $n \times n$ complex skew-Hermitian matrices. It is not a complex vector space, but it is a real Lie algebra and a real form of $\mathfrak{gl}(n, \mathbb{C})$.

Real forms of the Witt Lie algebra

One real form of $\mathfrak{d}_{\mathbb{C}}$ is $\mathfrak{d}_{\mathbb{R}}$, the real span of the d_n and C.

There is another real form of $\mathfrak{d}_{\mathbb{C}}$, namely $\mathfrak{w}$, the Lie algebra of real vector fields on the circle. Recall

$$d_n = -z^{n+1}\frac{d}{dz} = ie^{in\theta}\frac{d}{d\theta}, \qquad z = e^{i\theta}.$$

So

$$\cos(n\theta) = \frac{1}{2i}(d_n + d_{-n}), \qquad \sin(n\theta) = -\frac{1}{2}(d_n - d_{-n}), \qquad \frac{d}{d\theta} = -id_0.$$

Note that $\mathfrak{w}$ contains a copy of $\mathfrak{su}(2)$ where $\mathfrak{d}_{\mathbb{R}}$ contains a copy of $\mathfrak{sl}(2,\mathbb{R})$, both real forms of $\mathfrak{sl}(2,\mathbb{C})$.

Unitary representations

If G is a Lie group, and $\pi : G \to \mathrm{GL}(V)$ is a unitary representation on a Hilbert space V, then the corresponding representation π of the Lie algebra $\mathfrak{g}$ is skew-Hermitian with respect to the inner product.

$$\langle \pi(X) \cdot u, v \rangle = -\langle u, \pi(X) \cdot v \rangle.$$

Thus $\pi(X) = -\pi(X)^*$ where $\pi(X)^*$ is the adjoint.

For the Virasoro algebra, we define the representation π to be unitary if $\pi(d_n)^* = -\pi(d_n)$ and $\pi(C)^* = \pi(C)$.

In an irreducible representation, C has constant eigenvalue c because it is central. The generator d_0 is not central and has different values on different weight spaces. If there is a highest weight space $V(\lambda)$ we denote the d_0 eigenvalue on it h.

Highest weight modules for Vir

Suppose that V is an irreducible representation that is a highest weight representation with highest weight vector $v_{c,h}$. Then $C\, v_{c,h} = cv_{c,h}$ and $d_0\, v_{c,h} = h\, v_{c,h}$. The eigenvalues c and h of C and d_0 determine a weight $\lambda_{c,h}$ in $\mathfrak{h}^*$ since $\mathfrak{h}$ is the span of C and d_0.

From the general theory, there is a unique Verma module (universal highest weight module) with given weight λ, and that it has a unique irreducible quotient. We will denote these $M(c,h)$ and $L(c,h)$. They are the same unless $M(c,h)$ has primitive vectors besides $v_{c,h}$.

Basis of $M(c,h)$

We have noted for general Lie algebras with triangular decomposition that the map $U(\mathfrak{n}_-) \to M(c,h)$ applying an element of the enveloping algebra to $v_{c,h}$ is a vector space isomorphism. By the PBW theorem, it follows that a basis of $M(c,h)$ consists of the vectors

$$d_{-i_t} \cdots d_{-i_1} v_{c,h}, \qquad 0 < i_1 \leqslant i_2 \leqslant \cdots \leqslant i_t.$$

Kac found a method based on the Kac determinant for finding primitive vectors.

Summary of results

A key problem is to classify the irreducible unitary representations of the Virasoro algebra, since these control the possible CFT. We will look at this topic in future lectures. Right now, here are the facts, proved by Kac, Friedan-Qiu-Shenkar and Goddard-Kent-Olive.

A main tool is the Kac determinant, which detects primitive vectors. The main issue is to determine the pairs (c, h) such that $L(c, h)$ is unitary. The principle issue is to determine what happens when $0 \leqslant c < 1$, where there are discrete values of (c, h) that give unitary $L(c, h)$.

Summary of results: $c \geqslant 1$

In order for $L(c,h)$ to be unitary, we must have $c \geqslant 0$ and $h \geqslant 0$. If $c = h = 0$, then $L(c,h)$ is the trivial representation.

If $c > 1$ and $h > 0$ then $M(c,h) = L(c,h)$ and this representation is unitary.

If $c = 1$ and $h \geqslant 0$ then $L(c,h)$ is unitary, and $M(c,h) = L(c,h)$ unless $h = m^2/4$ ($m \in \mathbb{Z}$).

Summary of results: $0 \leqslant c < 1$

In the region $0 \leqslant c < 1$, Friedan-Qiu-Shenkar proved by careful analysis of the Kac determinant that $L(c, h)$ can be unitary only if there exist integers $1 \leqslant p \leqslant q < m$ with

$$c = 1 - \frac{6}{m(m+1)}, \qquad h = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}.$$

The proof that these values actually do correspond to unitary $L(c, h)$ was proved by Goddard, Kent and Olive using affine Lie algebras as a tool.

The corresponding conformal field theories were constructed by Belavin, Polyakov and Zamalodchikov (BPZ) and are called minimal models.