Lecture 3: Quantum Field Theory

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Central Extensions of Groups

Let G be a group. By an extension of G by another group A we mean a short exact sequence:

$$1 \longrightarrow A \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \longrightarrow 1$$

There is an obvious equivalence relation: two central extensions $\widetilde{G}_1$ and $\widetilde{G}_2$ are equivalent if there is a extensions are equivalent if there is a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & \widetilde{G}_1 \arrow[r, "p"] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & A \arrow[r, "i"] & \widetilde{G}_2 \arrow[r, "p"] & G \arrow[r] & 1 \end{tikzcd}$$

Classification of central extensions

If A is abelian, then extensions

$$1 \longrightarrow A \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \longrightarrow 1$$

can be classified by group cohomology. The simplest case is when A is abelian, and the homomorphism i is required to embed A in the center of $\widetilde{G}$. Then we say $\widetilde{G}$ is a central extension. With A is a trivial G-module the equivalence classes of central extensions are in bijection with $H^2(G, A)$.

We won't need this but we'll prove an analog for Lie algebras. See Lang's Algebra, Exercise 5 in Chapter 20.

Projective Representations of Groups

A projective representation of a group G is a homomorphism $\pi : G \to \mathbb{P}(V)$ where V is a complex vector space and $\mathbb{P}(V)$ is the corresponding projective space. Let $\gamma : V - 0 \to \mathbb{P}(V)$ be the canonical projection map. Given a projective representation, there exists a central extension

$$1 \longrightarrow \mathbb{C}^{\times} \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \longrightarrow 1$$

and a complex representation $\widetilde{p}$ of $\widetilde{G}$ that lifts π in the sense that if $g \in \widetilde{G}$ then $\pi(p(g)) = \gamma(\widetilde{p}(g))$.

The construction requires producing a cohomology class in $H^2(G, \mathbb{C}^{\times})$ from the projective representation.

Spin

For connected semisimple Lie groups, there is another way to classify central extensions. Indeed, a connected extension is a topological cover, and these are in bijection with subgroups of the fundamental group. They are always central.

For $SO(p,q)^\circ$ the fundamental group agrees with the fundamental group of the maximal compact subgroup $SO(p) \times SO(q)$, and can be determined by the following table.

p	$\pi_1 SO(p)$
1	1
2	$\mathbb{Z}$
> 2	$\mathbb{Z}/2\mathbb{Z}$

If $p > 2$, the double cover of $SO(p)$ is called the spin group.

The Poincaré group

It is commonly believed that we live in a 4-dimensional world with 3 spacelike dimensions and one timelike. Thus the connected isotropy subgroup of a point is $SO(3,1)^{\circ}$, the Lorentz group. Elements have a physical interpretation as Lorentz transformations.

The semidirect product of this with translations is the Poincaré group $\mathbb{R}^{(3,1)} \rtimes SO(3,1)^{\circ}$. In order to accomodate fermions it is necessary to replace $SO(3,1)^{\circ}$ by its double cover $\mathrm{spin}(3,1) \cong SL(2,\mathbb{C})$. Thus for Streater and Wightman the Poincaré group is $\mathcal{P} = R^{(3,1)} \rtimes \mathrm{SL}(2,\mathbb{C})$.

In their book they gave axioms for quantum field theories with this symmetry group.

Other dimensions

The Wightman axioms may be transported without much change to other dimensions, except that the exploitation of the isomorphism $\mathrm{spin}(3,1) \cong SL(2,\mathbb{C})$, which is only a matter of convenience, is special to $d = 4$.

However the case $d = 2$ is special for two reasons.

- Since $\mathrm{spin}(2) \cong \mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$, the division of particles into bosons and fermions is not clear; there can be particles whose interchange introduces a phase change that is not ± 1, called anyons and indeed these have been observed in the fractional quantum Hall effect.
- It is possible for a theory to have infinitesimal conformal symmetries, leading to a role for the Virasoro algebra.

Lie algebra H^2

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an abelian Lie algebra, meaning $[X, Y] = 0$ for all $X, Y \in \mathfrak{a}$. We wish to classify central extensions of $\mathfrak{g}$ by $\mathfrak{a}$:

$$1 \longrightarrow \mathfrak{a} \xrightarrow{i} \widetilde{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \longrightarrow 1$$

We require that $i(\mathfrak{a})$ lies in the center of $\widetilde{\mathfrak{g}}$ To this end we define a group $H^2(\mathfrak{g}, \mathfrak{a})$ that classifies such extensions.

A bilinear map $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ is called a 2-cocycle if it is skew-symmetric and satisfies

$$\sigma([X, Y], Z) + \sigma([X, Y], Z) + \sigma([X, Y], Z) = 0, \qquad X, Y, Z \in \mathfrak{a}.$$

Let $Z^2(\mathfrak{g}, \mathfrak{a}$ be the group of 2-cocycles. Let $f : \mathfrak{g} \to \mathfrak{a}$ be any linear map. Then $\sigma(X, Y) = f([X, Y])$ defines a 2-cocycle. Cocycles of this type are called coboundaries, forming a group $B^2(\mathfrak{g}, \mathfrak{a})$. Let $H^2(\mathfrak{g}, \mathfrak{a}) = Z^2(\mathfrak{g}, \mathfrak{a})/B^2(\mathfrak{g}, \mathfrak{a})$.

Central extensions of Lie algebras

Given a 2-cocyle $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ define a bracket operation on $\mathfrak{g} \times \mathfrak{a}$ by

$$[(X, A), (Y, B)] = \big([X, Y], \sigma(X, Y)\big).$$

Since the bracket on $\mathfrak{a}$ and σ are skew-symmetric, this bracket is skew-symmetric and the cocycle condition

$$\sigma([X, Y], Z) + \sigma([X, Y], Z) + \sigma([X, Y], Z) = 0$$

implies the Jacobi identity on this Lie algebra to be denoted $\widetilde{\mathfrak{g}}$. If two cocycles differ by a coboundary they produce the same extension, leading to the classification of central extensions by $H^2(\mathfrak{g}, \mathfrak{a})$.

Projective representations of Lie algebras

Recall that a representation of a Lie algebra $\mathfrak{g}$ is a linear map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ such that

$$\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x).$$

We weaken this to obtain the notion of a projective representation; then we assume that

$$\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x) + \alpha(x, y)I_V$$

where $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a bilinear map. Let us check that α is a 2-cocycle. It is easy to see that α is skew-symmetric. Now

$$\pi([[x, y], z]) = \pi([x, y])\pi(z) - \pi(z)\pi([x, y]) + \alpha([x, y], z)I$$

$$= \pi(x)\pi(y)\pi(z) - \pi(y)\pi(x)\pi(z) - \pi(z)\pi(x)\pi(y) + \pi(z)\pi(y)\pi(x) + \alpha([x, y], z)I$$

Summing this over cyclic permutations of x, y, z and using the Jacobi relation gives the cocycle condition.

The Virasoro cocycle

Recall that the Witt Lie algebra $\mathfrak{d}_F$ (where $F = \mathbb{R}$ or $\mathbb{C}$) has a basis $\{d_n\}$ with $n \in \mathbb{Z}$, and bracket

$$[d_n, d_m] = (n-m)d_{n+m}.$$

For definiteness we will denote $\mathfrak{d} = \mathfrak{d}_{\mathbb{C}}$. Regard $\mathbb{C}$ as an abelian one-dimensional Lie algebra. The cohomology group $H^2(\mathfrak{d}, \mathbb{C})$ was computed by Feigin and Fuchs and it is one-dimensional. This means that $\mathfrak{d}$ has a unique nontrivial central extension by $\mathbb{C}$, called the Virasoro algebra.

The cocycle may be defined by

$$\sigma(d_n, d_m) = \delta_{n,-m} \cdot \frac{1}{12}(n^3 - n).$$

The cocycle relation may be checked with a little work.

The Virasoro Lie algebra

Hence we come to the (complex) Virasoro Lie algebra **Vir** spanned by d_k ($k \in \mathbb{Z}$) and a central element C subject to

$$[d_n, d_m] = (n-m)d_{n+m} + \delta_{n,-m} \cdot \frac{1}{12}(n^3 - n).$$

We are interested in the irreducible representations of **Vir**.

Quantum Mechanics

Formal axioms of QFT were figured out in the 1950s by Wightman and Gårding, and were described in the book PCT, Spin and Statistics and All That by Streater and Wightman, which aims to give a rigorous mathematical treatment of certain topics.

We have seen that a quantum mechanical system involves a Hilbert space $\mathcal{H}$. States are represented by rays in $\mathbb{P}^1(\mathcal{H})$. Observables are Hermitian operators. Inner products between vectors (amplitudes) have a probabilistic interpretation: the inner product $|\langle\psi_1, \psi_2\rangle|$ is the probability that the system prepared in state ψ_1 will transition to ψ_2 when a measurement is conducted.

Bosons and Fermions

In $d > 2$ dimensions, there are two kinds of fields or particles, called bosons and fermions. If we interchange two identical bosons, the state of the system will be unchanged. However if we interchange two identical fermions, the state changes from Ψ to $-\Psi$.

It is useful to generalize the notion of a Lie algebra to a Lie superalgebra. This is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The superalgebra paradigm is that in any formula involving x and y, in any place of any formula where they appear in reversed order, we introduce a sign $(-1)^{\deg(x)\deg(y)}$. There is a Lie bracket $[\ ,\ ]$ and it is to satisfy

$$[y, x] = -(-1)^{\deg(x)\deg(y)}[x, y].$$

There is a similar modification to the Jacobi identity.

Bosons and Fermions, continued

In a quantum field theory there are Hermitian operators that are associated with particles or fields. These are given a superalgebra grading, with the bosonic fields of degree 0, and fermionic fields of degree 1.

There are field theories (supersymmetric) in which bosons and fermions can interact in a nontrivial way. This is called supersymmetry. Supersymmetry has not been observed in nature, so the use of the superalgebra grading is a more superficial use.

Probabilistic interpretation

Suppose that A is a classical observable, such as angular momentum in a particular direction. As in Lecture 1 this corresponds to a Hermitian operator $\hat{A}$ on $\mathcal{H}$. Suppose that the spectrum of $\hat{A}$ is discrete, so that $\mathcal{H}$ has an orthonormal basis Φ_i of eigenvectors of $\hat{A}$, with eigenvalues λ_i. Assume that the system is in a state Ψ that is possibly in a superposition:

$$\Psi = \sum_i a_i \Phi_i, \qquad \sum |a_i|^2 = 1.$$

Then $a_i = (\Psi, \Phi_i)$.

Assume for simplicity that $\lambda = \lambda_i$ is an eigenvalue with multiplicity one. Then if the observable A is measured when the system is in state Ψ, then $|a_i|^2$ is the probability that the measurement will give the value λ_i.

Wigner on Symmetries

A theorem of Wigner asserts that a symmetry that respects such probabilities is represented by a unitary or antiunitary transformation of $\mathcal{H}$. (Weinberg, Quantum Theory of Fields, Vol.1, Appendix A to Chapter 2.)

Now suppose we have a one-parameter subgroup of transformations of the ambient space. By Wigner's theorem, we have a corresponding one-parameter family of unitary transformations of $\mathcal{H}$. Such a one-parameter family is of the form $t \mapsto e^{itP}$, where P is a Hermitian operator.

Symmetry by translation

We consider a Lorentzian system on $\mathbb{R}^{(d-1,1)}$. Let $x_1, \cdots, x_{d-1}, x_d$ be the coordinate functions. Thus x_d is the unique timelike coordinate. The positive cone consists of

$$x_1^2 + \cdots + x_{d-1}^2 < x_d^2.$$

Let $T_i(a)$ be the transformation $x_i \to x_i + a$. Assuming Poincaré invariance of the system, this gives a one parameter family of symmetries so by Wigner's theorem the symmetries comprise a one parameter family of unitary operators e^{iaP_i}, where P_i is a Hermitian operator. The operator $H = P_d$ is the nonrelativistic Hamiltonian.

Momentum and energy

The operators P_i commute, so they have a joint spectrum. Let us imagine that the spectrum is discrete, so there is a state with eigenvalues $(p_1, \cdots, p_{d-1}, e)$. This is a state with definite momentum and energy, and $\mathbf{p} = (p_1, \cdots, p_{d-1}, e)$ are the components of the relativistic momentum-energy vector.

For physical reasons, we expect $\mathbf{p}$ to lie in the positive cone.

If the spectrum is not discrete, there is still a joint spectrum of the operators P_i (since they commute). If $\mathbf{p} = (p_1, \cdots, p_{d-1}, e)$ is in the joint spectrum, then we still expect $\mathbf{p}$ to lie in the positive cone.

Wightman's Axiom 0

We will state the Wightman axioms for $d = 4$, following Streater and Wightman. See also Folland, Quantum Field Theory, Section 5.5.

The first Wightman axiom is that we a unitary representation U of the Poincaré group $\mathcal{P} = R^{(3,1)} \rtimes \mathrm{SL}(2, \mathbb{C})$ on a Hilbert space $\mathcal{H}$, subject to the following positivity assumption. By Wigner's theorem $U(T_i(a)) = e^{iaP_i}$ where $T_i(a)$ is the translation $x_i \to x_i + a$. Here P_i are commuting Hermitian operators. It is assumed that if (p_1, p_2, p_3, e) are in the joint spectrum of $P_1, P_2, P_3, P_4 = H$ then $e^2 - p_1^2 - p_2^2 - p_3^2 \geqslant 0$.

It is also assumed that there is a vacuum vector $|0\rangle$ that is invariant under all the operators $U(\gamma)$ ($\gamma \in \mathcal{P}$).

Distributions as generalized functions

It will be convenient to maintain the notational fiction that a distribution is a function. Thus a tempered distribution α is a functional $f \to (f, \alpha)$ on the Schwartz space $\mathcal{S}(\mathbb{R})$ that is continuous in the Fréchet topology. According to the notational fiction we write

$$(f, \alpha) = \int_{\mathbb{R}} f(x)\alpha(x)\,dx.$$

An example is the Dirac distribution δ which maps a Schwartz function f to $f(0)$. According to the notational fiction we write

$$\int_{\mathbb{R}} f(x)\delta(x)\,dx = f(0).$$

The Schwartz space and distribution theory extend to $\mathbb{R}^{(d-1,1)}$.

Fields

In classical physics a field such as the electromagnetic field is a section of a vector bundle. So in quantum field theory, a field ϕ should attach an observable at every point $x \in \mathbb{R}^{(d-1,1)}$, that is, a Hermitian operator $\phi(x)$.

This point of view is not quite adequate. In practice it is necessary for ϕ is an operator valued distribution. If $f \in \mathcal{S}(\mathbb{R}^{(d-1,1)})$ then there is an operator $\phi(f)$. Let us maintain the notational fiction that ϕ is a function on $\mathbb{R}^{(3,1)}$ and not just a distribution and write

$$\phi(f) = \int_{\mathbb{R}^{(3,1)}} f(x)\phi(x)\,dx.$$

Wightman Axiom 1

It is assumed that there are given fields $\phi_1, \cdots, \phi_n$ that assign to a Schwartz function f operators $\phi_i(f)$ on $\mathcal{H}$. It is assumed that there exists a dense linear subspace D of $\mathcal{H}$ that contains the vacuum $|0\rangle$, and that is closed under the $\phi_i(f)$ and their adjoints.

Moreover it is assumed that if $\Phi, \Psi \in D$ then the functional on $f \in \mathcal{S}(\mathbb{R}^{(3,1)})$ given by $\langle\Phi|\phi_i(f)|\Psi\rangle$ is a tempered distribution.

Representation of $\operatorname{spin}(d-1,1)$

Now we can consider how the Poincaré group $\mathcal{P} = \mathbb{R}^{(3,1)} \rtimes SL(2,\mathbb{C})$ (or more generally $\mathbb{R}^{(d-1,1)} \rtimes \operatorname{spin}(d-1,1)^\circ$) acts on fields. In order to accomodate fermions we use $\operatorname{spin}(d-1,1)$ instead of $SO(d-1,1)^\circ$.

We choose a representation $\pi : \mathrm{SL}(2,\mathbb{C}) \to GL(n)$, where n is the number of fields. Since $\mathcal{P}$ is a semidirect product of $\mathrm{SL}(2,\mathbb{C})$ by $\mathbb{R}^{(3,1)}$ the group of translations is normal, and there is a homomorphism $\mathcal{P} \to \mathrm{SL}(2,\mathbb{C})$ with kernel $\mathbb{R}^{(3,1)}$. Thus we can extend π to $\mathcal{P}$.

Wightman Axiom 2

Now we assemble the fields into a vector $\mathbf{phi} = (\phi_1, \cdots, \phi_n)$. We let $\in \mathcal{P}$ act on the fields by the representation π.

Then Wightman Axiom 2 states that if $\gamma \in \mathcal{P}$ and $f \in \mathcal{S}(\mathbb{R}^{(3,1)})$

$$U(\gamma)\Phi_f U(\gamma)^{-1} = \pi(\gamma)\Phi_f.$$

That is, these two operators have the same effect on any vector in D.

Each field is assigned a degree 0 or 1 so that $-I \in \mathrm{SL}(2,\mathbb{C})$ acts by $(-1)^{\deg(\phi_i)}$. The even fields (degree 0) are called bosons and the odd fields (degree 1) are called fermions.

Wightman Axiom 3: locality

Let f and g be test functions whose supports are spacelike separated. This means that if $x = (x_1, x_2, x_3, t)$ and $x' = (x_1', x_2', x_3', t')$ are such that $f(x) \neq 0$ and $g(y) \neq 0$ then

$$(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 - (t - t')^2 > 0.$$

Then the Hermitian operators associated with these fields commute in the superalgebra sense:

$$\phi_i(f)\phi_j(g) = (-1)^{\deg(\phi_1)\deg(\phi_2)}\phi_j(g)\phi_i(f).$$

Or in the fiction that distributions are functions,

$$\phi_i(x)\phi_j(x') = (-1)^{\deg(\phi_1)\deg(\phi_2)}\phi_j(x')\phi_i(x)$$

if the points x and x' are spacelike separated.

The meaning of locality

Locality is the manifestation of the idea that information cannot travel faster than light. This means that amplitudes corresponding to processes that have a spacelike separation should be independent. The corresponding Hermitian operators correspond to observables that can both be simultaneously measured and they must therefore commute.

The fact that fermionic fields must anticommute is proved in Streater and Wightman, who assume a slightly weaker axiom

$$\phi_i(x)\phi_j(x') = \pm\phi_j(x')\phi_i(x).$$

One of their theorems is that the sign must be $(-1)^{\deg(\phi_i)\deg(\phi_j)}$.

Conformal Field Theories

Recall axiom 2:

$$U(\gamma)\Phi_f U(\gamma)^{-1} = \pi(\gamma)\Phi_f.$$

This relates to an action of the Poincaré group on the fields $\Phi = (\phi_1, \cdots, \phi_n)$, coming from a representation of the Lorentz group.

The the Poincaré group is contained in the larger conformal group $SO(4,2)^\circ$ or more generally $SO(d,2)^\circ$. Potentially we may extend the action of $\mathcal{P}$ on fields and extend this axiom to include conformal transformations.