Lecture 2: Conformal mappings

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Quadratic forms

Over a field F of characteristic $\neq 2$, a quadratic space is a vector space V with a quadratic form $Q : V \to F$. Choosing a basis we may identify $V = F^d$ for some d and

$$Q(x) = \sum_{i,j} a_{i,j} x_i x_j = x^t \cdot A \cdot x$$

where $A = (a_{i,j})$ is a symmetric matrix.

If $F = \mathbb{R}$ a change of basis puts any quadratic form in form

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, \qquad p + q = d.$$

Then we say the signature of Q is (p, q). Example: if $d = 2$, the quadratic form $x_1 x_2$ is equivalent to $x_1^2 - x_2^2$ by the change of basis $(x_1, x_2) \to (x_1 + x_2, x_1 - x_2)$ so its signature is $(1, -1)$.

Orthognal maps

Let $d = p + q$. Let $V = \mathbb{R}^{(p,q)}$ be $\mathbb{R}^d$ with the inner product of signature p, q. Thus with $(\eta) = (\eta^{(p,q)})$ let

$$\langle x, y\rangle = \sum \eta_{i,j} x_i y_j, \qquad \eta_{i,j} = \begin{cases} 1 & \text{if } 1 \leqslant i = j \leqslant p, \\ -1 & \text{if } p+1 \leqslant i = j \leqslant p+q, \\ 0 & \text{if } i \neq j. \end{cases} .$$

An orthogonal transformation is one that preserves this inner product. A dilation is a map of the form $x \mapsto \lambda x$ where λ is a positive constant. An orthogonal similitude is an orthogonal transformation times a similitude. Thus it is a map $g : V \to V$ that satisfies

$$\langle gv, gw\rangle = \lambda\langle v, w\rangle.$$

Orthogonal groups

The group of transformations of $\mathbb{R}^{(p,q)}$ that preserve the quadratic form η is denoted $O(p,q)$. The special orthogonal group $SO(p,q)$ consists of elements of determinant 1. This group is not connected if $q > 0$: if $q > 0$ then the connected component $SO^{\circ}(p,q)$ has index 2.

The three groups $O(2,1)$, $O(3,1)$ and $O(2,2)$ will be especially important for us. We will show

$$SO(2,1)^{\circ} \cong SL(2,\mathbb{R})/\{\pm I\},$$

$$SO(3,1)^{\circ} \cong SL(2,\mathbb{C})/\{\pm I\},$$

$$SO(2,2)^{\circ} \cong (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\{\pm(I,I)\}.$$

$SO(2,1)$

The group $SL(2,\mathbb{R})$ acts on the 3-dimensional space $\mathrm{Mat}_2^\circ(\mathbb{R})$ of real matrices of trace zero by conjugation. An invariant quadratic form of signature (2,1) on $\mathrm{Mat}_2^\circ(\mathbb{R})$ is

$$-\det(X) = a^2 - bc, \qquad X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

so we obtain a homomorphism $SL(2,R) \to SO(2,1)$. The kernel is $\{\pm I\}$. Both groups have dimension 3 so the image is open. But this homomorphism is not surjective because

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}$$

cannot be obtained by conjugation by an element of determinant 1. The image $SO(2,1)^\circ$ of this homomorphism is connected since $SL(2,\mathbb{R})$ is connected. Since the homomorphism is not surjective, $SO(2,1)$ is not connected.

$SO(3,1)$ and $SO(2,2)$

The group $SL(2,\mathbb{C})$ acts on the 4-dimensional real vector space of 2×2 Hermitian matrices

$$H = \left\{ X \in \mathrm{Mat}_2(\mathbb{C}) \,|\, X = \overline{X}^t \right\}$$

by

$$g : X \to gX\overline{g}^t.$$

The quadratic form $-\det(X)$ has signature $(3,1)$. This gives an isomorphism $\mathrm{SL}(2,\mathbb{C})/\{\pm I\} \cong SO(3,1)^\circ$.

The group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ acts $\mathrm{Mat}_2(\mathbb{R})$ by

$$(g_1, g_2) : X \to g_1 X g_2^t.$$

(Alternatively, $g_1 X g_2^{-1}$.) The invariant determinant quadratic form is of signature (2,2) giving an isomorphism $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\{\pm(I,I)\} \cong SO(2,2)^\circ$.

The case $(p,q) = (2,0)$

Let us classify orthogonal similitudes when $(p,q) = (2,0)$. If $(p,q) = (2,0)$ we have the usual orthogonal group:

$$SO(2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}.$$

The group may be parametrized by the circle, taking

$$a = \cos(\theta), \qquad b = \sin(\theta), \qquad \theta \in \mathbb{R} \text{ mod } 2\pi.$$

The group of special orthogonal similitudes $GSO(2)$ removes the condition $a^2 + b^2 = 1$. (We have to assume $a^2 + b^2 \neq 0$ since it is the determinant.) For the orthogonal group $O(2)$ or the full group $GO(2)$ of orthogonal similitides we add another coset

$$\left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \right\}.$$

The case $(p, q) = (1, 1)$

The indefinite special orthogonal group $SO(1,1)$ consists of transformations

$$SO(1,1) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{R}, a^2 - b^2 = 1 \right\}.$$

These appear in the theory of special relativity as Lorentz transformations. Again, we obtain the group $GSO(1,1)$ of special orthogonal similitudes by removing the condition $a^2 - b^2 = 1$, and both $SO(1,1)$ and $GSO(1,1)$ are of index two in larger groups $O(1,1)$ and $GO(1,1)$, respectively.

The groups $O(2)$ and $O(1,1)$ are related by the fact that they have the same complexification: if we allow $a, b \in \mathbb{C}$ instead of $a, b \in \mathbb{R}$ we obtain isomorphic groups.

Conformal Mappings

Now let U be an open subset of $V = \mathbb{R}^{(p,q)}$ and $\phi : U \to V$ a smooth map. We may identify the tangent space TU_u with V ($u \in U$). The map ϕ is conformal if the differential $d\phi_u : V \to V$ is an orthogonal similitude for all $u \in U$.

Conformal maps are most interesting if $d = 2$ so we will only consider in detail the cases $(p, q) = (2, 0)$ and $(p, q) = (1, 1)$. The case $q = 1$ will be called Lorentzian and the case $q = 0$ will be called Euclidean.

- In the Euclidean case identify $\mathbb{R}^{(2,0)} = \mathbb{C}$. A map is conformal if and only if it is holomorphic or antiholomorphic with nonvanishing derivative.
- Lorentzian fields relate to Euclidean ones by a process called Wick rotation (continuation to imaginary time).

Cauchy-Riemann equations

Let us consider conformal maps when $(p, q) = (2, 0)$. In this case we introduce complex coordinates and identify $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$. We consider a smooth map $f : U \to \mathbb{C}$ where $(x, y) \in U$, and open set. Write $f(z) = u + iv$ where u, v are real. If f is orientation preserving, the condition that it be conformal (locally at z) is that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

We recognize these as the Cauchy-Riemann equations, and conclude that the condition for f to be conformal is that it is a holomorphic function, with nonvanishing derivative.

If f is orientation-reversing, the condition is that it is antiholomorphic.

The $(1,1)$ case

Let us consider conformal maps when $(p,q)=(1,1)$. In this "Lagrangian" case if $f(x,y)=(u,v)$ the condition that it be conformal (locally at z) is that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}.$$

These are reminiscent of the Cauchy-Riemann equations but also somewhat different.

They still carry a powerful amount of information, but the nature of that information (we will see) is slightly different.

A conventional fiction

It is common in complex analysis to pretend that z and $\bar{z}$ are independent variables. If $z = x + iy$ and $\bar{z} = x - iy$ then we define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z}\right).$$

Then for the purpose of “calculus,” e.g. the chain rule in multiple variables, z and $\bar{z}$ behave as independent variables, e.g. if w is a function of z and ϕ is a function of w then

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial \phi}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial z}, \qquad \frac{\partial \phi}{\partial \bar{z}} = \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial \bar{z}} + \frac{\partial \phi}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial w}.$$

Cauchy-Riemann equations revisited

Let $\phi = \phi(z)$ be a smooth function of the complex variable $z = x + iy$. We will write $\phi(z) = u + iv$, u, v real. The condition

$$\frac{\partial\phi}{\partial\bar{z}} = 0 \qquad (*)$$

then reads

$$\frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x} + i\frac{\partial(u+iv)}{\partial y}\right) = 0.$$

Separating the real and imaginary parts, this boils down to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the Cauchy-Riemann equations. Therefore (*) is equivalent to ϕ being holomorphic.

Global conformal maps

The idea of a global conformal map is that we embed $\mathbb{R}^{(p,q)}$ into a suitable completion or compactification, X, such that there is a sufficiently large collection of conformal automorphisms of X. In practice, X turns out to be a homogeneous space of $SO(p+1, q+1)$.

Let us see how this works if $(p, q) = (2, 0)$. In this case we embed $\mathbb{R}^2$ into the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$which is a homomogenous space for $SL(2, \mathbb{C})$ acting by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}.$$

Conformal completion

Reference: Schottenloher, A Mathematical Introduction to Conformal Field Theory.

If p, q are arbitrary, the conformal compactification $N^{p,q}$ is a compact space containing $\mathbb{R}^{(p,q)}$ as a dense subspace. If $p+q>2$ it has the property that every germ of a conformal map in $\mathbb{R}^{(p,q)}$ can be extended to an automorphism of $N^{(p,q)}$. This property fails if $p+q=2$, but the space $N^{(p,q)}$ can still be defined. for example if $(p,q)=(2,0)$ it is the Riemann sphere.

The conformal compactification goes back to Veblin (1935) and Dirac (1936) with the case $q=0$ earlier (Riemann, Klein).

Conformal completion (continued)

The space $N^{(p,q)}$ is the image of the isotropic cone in $\mathbb{R}^{(p+1,q+1)}$ under the natural map $\gamma : (\mathbb{R}^{(p+1,q+1)} - 0) \to \mathbb{P}(\mathbb{R}^{(p+1,q+1)})$, where $\mathbb{P}(\mathbb{R}^{(p+1,q+1)})$ is projective space. By Witt's Theorem (Lang's *Algebra*, Theorem 10.2 on page 591) the group $SO(p+1, q+1)$ acts transitively on the nonzero isotropic vectors in $\mathbb{R}^{(p+1,q+1)}$. Let $\iota : \mathbb{R}^{(p,q)} \to N^{(p,q)}$ be the map

$$\iota(x_1, \cdots, x_{p+1}) = \gamma\left(\frac{1-|x|}{2}, x_1, \cdots, x_{p+q}, \frac{1+|x|}{2}\right).$$

Identify $\mathbb{R}^{(p,q)}$ with its image, a dense subset of $N^{(p,q)}$.

Local conformal transformations

We may ask for vector fields X on $\mathbb{R}^{(p,q)}$ such that the geodesic flow tangent to X is through conformal maps. Thus by the theory of first order systems of differential equations (assuming a mild Lipschitz condition) there is a family of maps $\phi_t : \mathbb{R}^{(p,q)} \to \mathbb{R}^{(p,q)}$ defined for $t \in (-\varepsilon, \epsilon)$ for small ϵ such that ϕ_0 is the identity map, and the vector field tangent to the curves $t \to \phi_t(x)$ is X.

We ask that the family ϕ_t gives a family of conformal maps. We do not ask that they extend to all t or to all of $N^{(p,q)}$. Schottenloher calls these conformal killing fields, and others call them local conformal transformations. They are classified in the general theory. See Schottenloher Section 1.3 or Di Francesco, Mathieu and Senechal (DMS) Section 4.1. We will use shortcuts to handle the cases $(2, 0)$ and $(1, 1)$.

Conformal completion ($d > 2$)

Theorem

Assume $d = p + q > 2$. Let $U \subset R^{(p,q)}$ be open. Then any conformal map $U \to R^{(p,q)}$ extends uniquely to a conformal map $N^{(p,q)} \to N^{(p,q)}$.

Reference: Schottenloher, A Mathematical Introduction to Conformal Field Theory, Theorem 2.9.

This fails if $d = 2$. Nevertheless we can compute the Lie algebras of local conformal transformations in both cases $(2, 0)$ and $(1, 1)$. These will be be infinite-dimensional Lie algebras containing the finite-dimensional Lie algebras of the global conformal groups.

The Witt Lie algebra

The Witt Lie algebra $\mathfrak{d}_{\mathbb{R}}$ is the Lie algebra of polynomial vector fields on the circle. A vector field is then a function $f(\theta)d/d\theta$ where $f(\theta+2\pi)=f(\theta)$; polynomial means that f is a finite linear combination of terms $e^{ni\theta}$. If $z=e^{i\theta}$ then a basis for $\mathfrak{d}$ consists of

$$d_n = ie^{in\theta}\frac{d}{d\theta} = -z^{n+1}\frac{d}{dz}.$$

The Lie bracket is

$$[d_n, d_m] = (n-m)d_{n+m}.$$

So $\mathfrak{d}_{\mathbb{R}}$ is the real span of the d_n.

A finite-dimensional subalgebra of $\mathfrak{d}$

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $SL(2,\mathbb{R})$ consists of 2×2 real matrices of trace zero. It has a basis

$$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with relations

$$[H, X] = 2x, \qquad [H, Y] = -2Y, \qquad .$$

Thus the subalgebra $\mathfrak{d}_{\mathbb{R}}$ contains a copy of $\mathfrak{sl}_2(\mathbb{R})$ via

$$\begin{array}{ccc} H & \Leftrightarrow & -2d_0 \\ X & \Leftrightarrow & d_1 \\ Y & \Leftrightarrow & d_{-1} \end{array} \quad .$$

Local conformal maps: the $(2,0)$ case

We take $(p,q) = (2,0)$. Then the space $N^{(p,q)}$ can still be constructed: it is the Riemann sphere, and we have seen that the global conformal group is $SO(3,1)^\circ \cong SL(2,\mathbb{C})/\{\pm I\}$. The action on the Riemann sphere is by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}.$$

We also have local conformal maps, which are vector fields that exponentiate locally to conformal maps. Obviously these should be holomorphic, so they are complex linear combinations of

$$d_n = -z^{n+1}\frac{d}{dz}.$$

We see that the Lie algebra of local conformal transformations is the complexified Witt Lie algebra $\mathfrak{d}_{\mathbb{C}}$.

The finite-dimensional subalgebra

Within the Witt algebra $\mathfrak{d}_{\mathbb{C}}$ we have noted that there is a finite-dimensional subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. This is the Lie algebra of $SL(2, \mathbb{C})/\{\pm I\} = SO(3, 1)$, as expected since the global conformal transformations should give rise to local ones.

Local conformal maps: the $(1,1)$ case

Now we consider $\mathbb{R}^{(1,1)}$ with the Lorentzian metric. We have seen that the conformal maps send $(x,y) \to (u,v)$ subject to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}. \qquad (*)$$

We may produce solutions to these differential equations as follows. Let f and g be arbitrary smooth functions on $\mathbb{R}$ and define

$$u(x,y) = \frac{1}{2}(f(x+y)+g(x-y)), \qquad v(x,y) = \frac{1}{2}(f(x+y)-g(x-y)).$$

It is easy to check (*).

Local conformal maps: the $(1,1)$ case (continued)

We saw that

$$u(x,y) = \frac{1}{2}(f(x+y)+g(x-y)), \qquad v(x,y) = \frac{1}{2}(f(x+y)-g(x-y))$$

gives solutions to the conformal conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}. \qquad (*)$$

To see that every solution is of this form, note that (*) implies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2}.$$

Thus u satisfies the wave equation and may be expressed as the superposition of left and right moving waves:

$$u(x,y) = \frac{1}{2}(f(x+y)+g(x-y)).$$

The expression for v may be deduced.

Local conformal maps: the $(1,1)$ case (continued)

This leads to the result that the Lie algebra of local conformal transformations is $\mathfrak{d}_{\mathbb{R}} \oplus \mathfrak{d}_{\mathbb{R}}$, vector fields that integrate to left-moving and right-moving conformal maps.

As in the holomorphic case, there is a finite-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$. This is the Lie algebra of

$$SO(2,2)^\circ \cong (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\{\pm(I,I)\},$$

which is the global conformal group in this case.