Lecture 17: Fusion for Minimal Models

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References: [BPZ] Sections 5 and 6 and [FMS] Chapters 6, 7 and 8.

We will consider OPE of the primary field $\phi_h$ with other primary fields. We recall that the Virasoro generators act on a primary field $\phi_h$ of conformal weight $h$ by

$$[L_{-n}, \phi_h(z)] = (1 - n)hz^{-n}\phi_h(z) + z^{1-n}\frac{\partial}{\partial z}\phi_h(z).$$

Thus if $n > 0$

$$L_{-n}\phi_h(z)|0\rangle = (1 - n)hz^{-n}\phi_h(z)|0\rangle + z^{1-n}\frac{\partial}{\partial z}\phi_h(z)|0\rangle.$$

With this in mind we introduce the differential operator

$$\mathcal{L}_{-n} = \sum_{i=1}^{N} \left\{ \frac{(n - 1)h}{(w_i - z)^n} - \frac{1}{(w_i - z)^{n-1}} \frac{\partial}{\partial w_i} \right\}.$$
We claim that

\[ \langle L_{-k} \Phi_h(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) \rangle = \mathcal{L}_{-k} \langle \Phi_h(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) \rangle. \]

For example \( \mathcal{L}_{-1} = -\sum \partial_{w_i} \) which is equivalent to \( \partial_w \) by translation invariance. So

\[ \langle 0 | L_{-1} \Phi_h(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) | 0 \rangle = \partial_z \langle 0 | \Phi_h(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) | 0 \rangle \]

\[ = \mathcal{L}_{-1} \langle 0 | \Phi_h(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) | 0 \rangle, \]

etc. More generally if \( \Phi^k_h = L_{-k_n} \cdots L_{-k_1} \Phi_h \) is a general descendent field we have

\[ \langle \Phi^k_h(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) \rangle = \mathcal{L}_{-k_n} \cdots \mathcal{L}_{-k_1} \langle \Phi_h(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) \rangle. \]
Kac Determinant Again

In the last two lectures we gave two equivalent descriptions of the values $h_{r,s}$ that appear in the Kac determinant formula

$$\det_N(c, h) = \text{constant} \times \prod_{1 \leq r, s \leq n} (h - h_{r,s})^{p(n-rs)}.$$

The first was:

$$h_{r,s} = \frac{((m + 1)r - ms)^2 - 1}{4m(m + 1)}, \quad c = 1 - \frac{6}{m(m + 1)}.$$

The second, which looks different but is equivalent is that

$$h_{r,s} = \frac{(pr - p's)^2}{4pp'}, \quad c = 1 - \frac{6(p - p')^2}{pp'}.$$
Another formula for $h_{r,s}$

Yet another equivalent form that we will need today is that

$$h_{r,s} = h_0 + \frac{1}{4} (r\alpha_+ + s\alpha_-)^2, \quad h_0 = \frac{1}{24} (c - 1)$$

$$\alpha_\pm = \frac{\sqrt{1 - c} \pm \sqrt{25 - c}}{\sqrt{24}}.$$

With this in mind, we will use the following notation for a possibly nondegenerate primary field $\phi_\alpha$ where $\alpha$ is a real number chosen so that the $L_0$ eigenvalue of $\phi_\alpha$ is $h_0 + \frac{1}{4} \alpha^2$. 
**Degenerate primary fields** $h_{r,s}$

Suppose with $h = h_{r,s}$ for some $r, s$. Suppose we have a field $\phi(z)$ such that the corresponding vector

$$|h\rangle = \lim_{z \to 0} \phi(z)$$

has $L_0|h\rangle = h|h\rangle$. We will denote this field $\phi_{r,s}(z)$. (We are suppressing $\bar{z}$ for now.)

From the Kac determinant formula, we know that there is a singular vector $|\chi_{\text{null}}\rangle$ of level $h - rs$. It is orthogonal to the entire ambient Hilbert space.
The null vector is declared zero

We may set the null vector $|\chi_{null}\rangle$ to zero.

The physical justification for this is that in any $|\chi_{null}\rangle$ is orthogonal to the entire Hilbert space $\mathcal{H}$ and moreover is not detected by any correlation functions. That is,

$$\langle \chi_{null}(w) \phi_1(z_1) \cdots \phi_N(z_N) \rangle := \langle 0 | \chi(w) \phi_1(z_1) \cdots \phi_N(z_N) | 0 \rangle = 0.$$ 

These correlation functions are the physically measurable quantities associated with the theory, so the field $\chi_{null}(w)$ does not interact and may be disregarded.

With $\chi_{null}$ set to zero we will call the corresponding primary field $\phi_{r,s}$ degenerate.
Null vectors of level 2

For example consider the special case where the null vector is of level 2, that is, \( h_{r,s} = h_{1,2} \) or \( h_{2,1} \).

The Kac determinant

\[
\det_2(c, h) = 32(h - h_{1,1})(h - h_{2,1})(h - h_{1,2})
\]

where \( h_{1,1} = 0 \) and

\[
h_{1,2} = \frac{1}{16} \left( 5 - c - \sqrt{(1 - c)(25 - c)} \right),
\]

\[
h_{2,1} = \frac{1}{16} \left( 5 - c + \sqrt{(1 - c)(25 - c)} \right).
\]

Hence a necessary and sufficient condition for a null vector of level 2 is that \( h = h_{1,2} \) or \( h_{2,1} \).
Null vectors of level 2 (continued)

Assuming $h = h_{1,2}$ or $h = h_{2,1}$

\[ |\chi_{\text{null}}\rangle = \left[ L_{-2} + \frac{3}{2(2h + 1)} L^2_{-1} \right] |h\rangle, \]

for we compute $L_1 |\chi\rangle = L_2 |\chi\rangle = 0$. (See [DMS] Section 7.3.1 and [BPZ] Section 5.) So as we have explained this vector may we set to zero.
Setting the null vector to zero implies a differential equation for the correlation functions (vacuum expectation values)

\[
\left( \mathcal{L}_{-2} + \frac{3}{2(2h+1)} \mathcal{L}_{-1}^{2} \right) \langle \Phi_{r,s}(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) \rangle = 0
\]

or more explicitly

\[
\left\{ \sum_{i=1} \frac{1}{(z - w_i)} \frac{\partial}{\partial w_i} + \frac{h_i}{(z - w_i)^2} - \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} \right\} \\
\langle \Phi_{r,s}(z) \Phi_{h_1}(w_1) \cdots \Phi_{h_N}(w_N) \rangle = 0.
\]

We have used the fact that \( \mathcal{L}_{-1} = -\partial_z \) is equivalent to \( \sum \partial_{w_i} \) due to translation invariance.
Indeed conformal invariance shows that with $\phi_{r,s} = \phi_{2,1}$ or $\phi_{1,2}$ and $h = h_{2,1}$ or $h_{1,2}$ we have

$$\langle \phi_{r,s}(z) \phi_{h_1}(w_1) \phi_{h_2}(w_2) \rangle$$

$$= \frac{G}{(z - w_1)^{h_2 - h - h_1} (w_1 - w_2)^{h - h_1 - h_2} (z - w_2)^{h_1 - h - h_2}}$$

for some constant $G$ independent of $z, w_1, w_2$. This is because the global conformal group $SL(2, \mathbb{C})$ acts 3-transitively on the Riemann sphere so a conformally covariant function with three singularities has no degrees of freedom and is determined up to scalar by the conformal dimensions at $z, w_1, w_2$. 
The differential equation gives no new information when applied to the two-point functions but applied to the three point functions this differential equation tells us something new: it gives a constraint on $h_1$ and $h_2$ such that $\langle \phi_{r,s}(z) \phi_{h_1}(w_1) \phi_{h_2}(w_2) \rangle$ is nonzero. With $\langle \phi_{r,s}(z) \phi_{h_1}(w_1) \phi_{h_2}(w_2) \rangle$ as above the differential equation gives an identity

$$2(2h + 1)(h + 2h_2 - h_1) = 3(h - h_1 + h_2)(h - h_1 + h_2 + 1).$$
This constraint

\[ 2(2h + 1)(h + 2h_2 - h_1) = 3(h - h_1 + h_2)(h - h_1 + h_2 + 1). \]

can be formulated more elegantly using the notation \( \phi(\alpha) \) for the primary fields. We recall that \( \phi(\alpha) \) has \( h = h_0 + \frac{1}{4} \alpha^2 \) where \( h_0 = \frac{1}{24}(c - 1) \). We learn that \( \langle \phi_{r,s}(z)\phi(\alpha)(w_1)\phi(\beta)(w_2) \rangle \) can be nonzero only if \( \beta = \alpha \pm \alpha_+ \) if \( h = h_{2,1} \) or \( \beta = \alpha \pm \alpha_- \) if \( h = h_{1,2} \) where we recall

\[ \alpha_\pm = \frac{\sqrt{1 - c} \pm \sqrt{25 - c}}{\sqrt{24}}. \]

The conclusion remains true if \( \phi(\alpha) \) or \( \phi(\beta) \) is replaced by a descendent field, so it is a statement about any fields in the conformal families \([\phi(\alpha)]\) and \([\phi(\beta)]\).
It may be deduced by letting $w_2 \rightarrow w_1$ that $[\phi(\alpha-\alpha_+)]$ and $[\phi(\alpha+\alpha_+)]$ are the only conformal families that appear in the OPE of $\phi_{(2,1)}(z)\phi_{(\alpha)}(w)$. We use the notation

$$[\phi_{(2,1)}] \times [\phi_{(\alpha)}] = [\phi(\alpha-\alpha_+)] + [\phi(\alpha+\alpha_+)],$$

to represent this fact. Similarly

$$[\phi_{(1,2)}] \times [\phi_{(\alpha)}] = [\phi(\alpha-\alpha_-)] + [\phi(\alpha+\alpha_-)].$$

We wish to think of the conformal families as being the basis of a ring, with $\times$ being the multiplication. This operation is called fusion.
Fusion coefficients

To reiterate, let $[\phi_a]$ and $[\phi_b]$ be two conformal families, consisting of all descendents of primary fields $\phi_a$ and $\phi_b$. We write:

$$[\phi_a] \times [\phi_b] = \sum_c N_{ab}^c [\phi_c]$$

to indicate that $[\phi_c]$ are the conformal families that can appear in the OPE of $\phi_a(z) \phi_b(w)$ (where we may replace the primary fields $\phi_a$ and $\phi_b$ by descendent fields and the conclusion remains true).

The integers $N_{ab}^c \geq 0$ are multiplicities taking into account that $\phi_c$ might occur in more than one essentially different way. (For the problem at hand, the $N_{ab}^c$ are all 0 or 1.)
The Fusion Ring

The fusion operation

\[ [\phi_a] \times [\phi_b] = \sum_c N^c_{ab} [\phi_c] \]

makes the conformal families into an associative ring. Its identity element is \( \phi_{(1,1)} \). To see this note that since \( h_{1,1} = 0 \) we must have

\[ \frac{\partial}{\partial z} \langle \phi_{(1,1)}(z)X \rangle = 0. \]

Since \( \phi_{(1,1)} \) is holomorphic it follows that it is constant. We have

\[ \phi_{(1,1)} \times \phi(\alpha) = \phi(\alpha) \]

so indeed \( \phi_{(1,1)} \) is the identity element in the fusion ring.
The fusion rule for degenerate fields

The general fusion rule for degenerate fields is

\[
[\Phi_{(r,s)}] \times [\Phi_{(\alpha)}] = \sum_{k=1-r}^{k=r-1} \sum_{l=1-s}^{l=r-1} \Phi_{(\alpha+k\alpha_+ + l\alpha_-)}.
\]

k \equiv r + 1 \mod 2 \quad l \equiv s + 1 \mod 2
From this, we may imagine the BPZ minimal models that we will look at in our next lecture: we wish to construct theories in which all primary fields are degenerate (to limit the complexity of the OPE) and in which there are only a finite number of primary fields.

This can be accomplished if

\[ c = 1 - \frac{6(p - p')^2}{pp'} \]

with \( p, p' \) coprime integers and

\[ \mathcal{H} = \bigoplus_{1 \leq r < p'} M(c, h_r, s) \otimes M(c, h_r, s) \]

\[ 1 \leq s < p \]

as a \( \text{Vir} \otimes \text{Vir} \) module.