Lecture 17: Fusion for Minimal Models

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Differential Operators

References: [BPZ] Sections 5 and 6 and [FMS] Chapters 6, 7 and 8.

We will consider OPE of the primary field ϕ_h with other primary fields. We recall that the Virasoro generators act on a primary field ϕ_h of conformal weight h by

$$[L_{-n}, \phi_h(z)] = (1-n)hz^{-n}\phi_h(z) + z^{1-n}\frac{\partial}{\partial z}\phi_h(z).$$

Thus if $n > 0$

$$L_{-n}\phi_h(z)|0\rangle = (1-n)hz^{-n}\phi_h(z)|0\rangle + z^{1-n}\frac{\partial}{\partial z}\phi_h(z)|0\rangle.$$

With this in mind we introduce the differential operator

$$\mathcal{L}_{-n} = \sum_{i=1}^{N}\left\{\frac{(n-1)h}{(w_i-z)^n} - \frac{1}{(w_i-z)^{n-1}}\frac{\partial}{\partial w_i}\right\}.$$

Differential Equations

We claim that

$$\langle L_{-k}\phi_h(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)\rangle = \mathcal{L}_{-k}\langle\phi_h(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)\rangle.$$

For example $\mathcal{L}_{-1} = -\sum \partial_{w_i}$ which is equivalent to ∂_w by translation invariance. So

$$\langle 0|L_{-1}\phi_h(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)|0\rangle = \partial_z\langle 0|\phi_h(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)|0\rangle$$

$$= \mathcal{L}_{-1}\langle 0|\phi_h(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)|0\rangle,$$

etc. More generally if $\phi_h^{\mathbf{k}} = L_{-k_n}\cdots L_{-k_1}\phi_h$ is a general descendent field we have

$$\langle\phi_h^{\mathbf{k}}(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)\rangle = \mathcal{L}_{-k_n}\cdots\mathcal{L}_{-k_1}\langle\phi_h(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)\rangle.$$

Kac Determinant Again

In the last two lectures we gave two equivalent descriptions of the values $h_{r,s}$ that appear in the Kac determinant formula

$$\det_N(c,h) = \text{constant} \times \prod_{\substack{1 \leqslant r,s \\ rs \leqslant n}} (h - h_{r,s})^{p(n-rs)}.$$

The first was:

$$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \qquad c = 1 - \frac{6}{m(m+1)}.$$

The second, which looks different but is equivalent is that

$$h_{r,s} = \frac{(pr - p's)^2}{4pp'}, \qquad c = 1 - \frac{6(p-p')^2}{pp'}.$$

Another formula for $h_{r,s}$

Yet another equivalent form that we will need today is that

$$h_{r,s} = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2, \qquad h_0 = \frac{1}{24}(c-1)$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}.$$

With this in mind, we will use the following notation for a possibly nondegenerate primary field $\phi_{(\alpha)}$ where α is a real number chosen so that the L_0 eigenvalue of $\phi_{(\alpha)}$ is $h_0 + \frac{1}{4}\alpha^2$.

Degenerate primary fields $h_{r,s}$

Suppose with $h = h_{r,s}$ for some r, s. Suppose we have a field $\phi(z)$ such that the corresponding vector

$$|h\rangle = \lim_{z \longrightarrow 0} \phi(z)$$

has $L_0|h\rangle = h|h\rangle$. We will denote this field $\phi_{r,s}(z)$. (We are suppressing $\bar{z}$ for now.)

From the Kac determinant formula, we know that there is a singular vector $|\chi_{\text{null}}\rangle$ of level $h - rs$. It is orthogonal to the entire ambient Hilbert space.

The null vector is declared zero

We may set the null vector $|\chi_{\text{null}}\rangle$ to zero.

The physical justification for this is that in any $|\chi_{\text{null}}\rangle$ is orthogonal to the entire Hilbert space $\mathcal{H}$ and moreover is not detected by any correlation functions. That is,

$$\langle \chi_{\text{null}}(w)\phi_1(z_1)\cdots\phi_N(z_N)\rangle := \langle 0|\chi(w)\phi_1(z_1)\cdots\phi_N(z_N)|0\rangle = 0.$$

These correlation functions are the physically measurable quantities associated with the theory, so the field $\chi_{\text{null}}(w)$ does not interact and may be disregarded.

With χ_{null} set to zero we will call the corresponding primary field $\phi_{r,s}$ degenerate.

Null vectors of level 2

For example consider the special case where the null vector is of level 2, that is, $h_{r,s} = h_{1,2}$ or $h_{2,1}$.

The Kac determinant

$$\det_2(c,h) = 32(h - h_{1,1})(h - h_{2,1})(h - h_{1,2})$$

where $h_{1,1} = 0$ and

$$h_{1,2} = \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right),$$

$$h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right).$$

Hence a necessary and sufficient condition for a null vector of level 2 is that $h = h_{1,2}$ or $h_{2,1}$.

Null vectors of level 2 (continued)

Assuming $h = h_{1,2}$ or $h = h_{2,1}$

$$|\chi_{\text{null}}\rangle = \left[L_{-2} + \frac{3}{2(2h+1)} L_{-1}^2 \right] |h\rangle,$$

for we compute $L_1|\chi\rangle = L_2|\chi\rangle = 0$. (See [DMS] Section 7.3.1 and [BPZ] Section 5.) So as we have explained this vector may we set to zero.

Differential Equation

Setting the null vector to zero implies a differential equation for the correlation functions (vacuum expectation values)

$$\left(\mathcal{L}_{-2} + \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2\right)\langle\phi_{r,s}(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)\rangle = 0$$

or more explicitly

$$\left\{\sum_{i=1} \frac{1}{(z-w_i)}\frac{\partial}{\partial w_i} + \frac{h_i}{(z-w_i)^2} - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right\}$$

$$\langle\phi_{r,s}(z)\phi_{h_1}(w_1)\cdots\phi_{h_N}(w_N)\rangle = 0.$$

We have used the fact that $\mathcal{L}_{-1} = -\partial_z$ is equivalent to $\sum \partial_{w_i}$ due to translation invariance.

Three point function

Indeed conformal invariance shows that with $\phi_{r,s} = \phi_{2,1}$ or $\phi_{1,2}$ and $h = h_{2,1}$ or $h_{1,2}$ we have

$$\langle \phi_{r,s}(z)\phi_{h_1}(w_1)\phi_{h_2}(w_2)\rangle$$

$$= \frac{G}{(z-w_1)^{h_2-h-h_1}(w_1-w_2)^{h-h_1-h_2}(z-w_2)^{h_1-h-h_2}}$$

for some constant G independent of z, w_1, w_2. This is because the global conformal group $\mathrm{SL}(2,\mathbb{C})$ acts 3-transitively on the Riemann sphere so a conformally covariant function with three singularities has no degrees of freedom and is determined up to scalar by the conformal dimensions at z, w_1, w_2.

Information from the three point function

The differential equation gives no new information when applied to the two-point functions but applied to the three point functions this differential equation tells us something new: it gives a constraint on h_1 and h_2 such that $\langle \phi_{r,s}(z)\phi_{h_1}(w_1)\phi_{h_2}(w_2)\rangle$ is nonzero. With $\langle \phi_{r,s}(z)\phi_{h_1}(w_1)\phi_{h_2}(w_2)\rangle$ as above the differential equation gives an identity

$$2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1).$$

Deduction

This constraint

$$2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1).$$

can be formulated more elegantly using the notation $\phi_{(\alpha)}$ for the primary fields. We recall that $\phi_{(\alpha)}$ has $h = h_0 + \frac{1}{4}\alpha^2$ where $h_0 = \frac{1}{24}(c-1)$. We learn that $\langle \phi_{r,s}(z)\phi_{(\alpha)}(w_1)\phi_{(\beta)}(w_2)\rangle$ can be nonzero only if $\beta = \alpha \pm \alpha_+$ if $h = h_{2,1}$ or $\beta = \alpha \pm \alpha_-$ if $h = h_{1,2}$ where we recall

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}.$$

The conclusion remains true if $\phi_{(\alpha)}$ or $\phi_{(\beta)}$ is replaced by a descendent field, so it is a statement about any fields in the conformal families $[\phi_{(\alpha)}]$ and $[\phi_{(\beta)}]$.

Fusion

It may be deduced by letting $w_2 \longrightarrow w_1$ that $[\phi_{(\alpha-\alpha_+)}]$ and $[\phi_{(\alpha+\alpha_+)}]$ are the only conformal families that appear in the OPE of $\phi_{(2,1)}(z)\phi_{(\alpha)}(w)$. We use the notation

$$[\phi_{(2,1)}] \times [\phi_{(\alpha)}] = [\phi_{(\alpha-\alpha_+)}] + [\phi_{(\alpha+\alpha_+)}],$$

to represent this fact. Similarly

$$[\phi_{(1,2)}] \times [\phi_{(\alpha)}] = [\phi_{(\alpha-\alpha_-)}] + [\phi_{(\alpha+\alpha_-)}].$$

We wish to think of the conformal families as being the basis of a ring, with $\times$ being the multiplication. This operation is called fusion.

Fusion coefficients

To reiterate, let $[\phi_a]$ and $[\phi_b]$ be two conformal families, consisting of all descendents of primary fields ϕ_a and ϕ_b. We write:

$$[\phi_a] \times [\phi_b] = \sum_c \mathcal{N}_{ab}^c [\phi_c]$$

to indicate that $[\phi_c]$ are the conformal families that can appear in the OPE of $\phi_a(z)\phi_b(w)$ (where we may replace the primary fields ϕ_a and ϕ_b by descendent fields and the conclusion remains true).

The integers $\mathcal{N}_{ab}^c \geqslant 0$ are multiplicities taking into account that ϕ_c might occur in more than one essentially different way. (For the problem at hand, the $\mathcal{N}_{ab}^c$ are all 0 or 1.)

The Fusion Ring

The fusion operation

$$[\phi_a] \times [\phi_b] = \sum_c \mathcal{N}^c_{ab} [\phi_c]$$

makes the conformal families into an associative ring. Its identity element is $\phi_{(1,1)}$. To see this note that since $h_{1,1} = 0$ we must have

$$\frac{\partial}{\partial z} \langle \phi_{(1,1)}(z) X \rangle = 0.$$

Since $\phi_{(1,1)}$ is holomorphic it follows that it is constant. We have

$$\phi_{(1,1)} \times \phi_{(\alpha)} = \phi_{(\alpha)}$$

so indeed $\phi_{(1,1)}$ is the identity element in the fusion ring.

The fusion rule for degenerate fields

The general fusion rule for degenerate fields is

$$[\phi_{(r,s)}] \times [\phi_{(\alpha)}] = \sum_{\substack{k=1-r \\ k \equiv r+1 \text{ mod } 2}}^{k=r-1} \sum_{\substack{l=1-s \\ l \equiv s+1 \text{ mod } 2}}^{l=r-1} [\phi_{(\alpha+k\alpha_{+}+l\alpha_{-})}].$$

BPZ minimal models

From this, we may imagine the BPZ minimal models that we will look at in our next lecture: we wish to construct theories in which all primary fields are degenerate (to limit the complexity of the OPE) and in which there are only a finite number of primary fields.

This can be accomplished if

$$c = 1 - \frac{6(p-p')^2}{pp'}$$

with p, p' coprime integers and

$$\mathcal{H} = \bigoplus_{\substack{1 \leqslant r < p' \\ 1 \leqslant s < p}} M(c, h_{r,s}) \otimes M(c, h_{r,s})$$

as a $\mathbf{Vir} \otimes \mathbf{Vir}$ module.