Lecture 16: Minimal Models, I

Daniel Bump

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Review: Kac determinant

References: Kac-Raina, Chapters 3, 8 and 12, and [DMS] Chapters 6,7,8, and Belavin-Polyakov-Zamolodchikov, Infinite conformal symmetry in two-dimensional QFT (1984) [BPZ].

Let $M(c,h)$ be the Virasoro Verma module with highest weight element $|h\rangle$. Since L_0 is self-adjoint, its eigenspaces are orthogonal. The vectors of eigenvalue $h+N$ comprise the level N eigenspace. Let $\mathbf{k}=(k_1,\cdots,k_n)$ be a partition of N, $k_1 \geqslant k_2 \geqslant \cdots$. Denote $|\mathbf{k}\rangle = L_{-k_n}\cdots L_{-k_1}|h\rangle$. The Kac determinant $\det_N(c,h)$ is the $p(N)\times p(N)$ matrix of inner products

$$(L_{-k_n}\cdots L_{-k_1}|h\rangle, L_{-l_m}\cdots L_{-l_1}|h\rangle) =$$

$$\langle h|L_{k_1}\cdots L_{k_n}L_{-l_m}\cdots L_{-k_1}|h\rangle = \langle \mathbf{k}|\mathbf{l}\rangle.$$

Review: Kac determinant

We will show that if $\det_N(c,h)=0$, and if $\det_{N-1}(c,h)\neq 0$, then $M(c,h)$ has a singular vector of level N. Indeed, there is a vector v of level N that is orthogonal to all vectors of level N, hence to all of $M(c,h)$. Then if $k>0$ the vector $L_k v$ is of lower level and is also orthogonal to $M(c,h)$, hence $L_k v=0$ so v is singular.

Thus we may use the Kac determinant formula

$$\det_N(c,h)=K\prod_{\substack{r,s\geqslant 0\\ 1\leqslant rs\leqslant n}}(h-h_{r,s})^{p(n-rs)}$$

to detect singular vectors. We will define $h_{r,s}$ later. The constant K is nonnegative. We see that if $h=h_{r,s}$ for some r,s, then $M(c,h)$ has a singular vector of level rs.

The singular values $h_{r,s}$

Last time we defined

$$h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}$$

where m is chosen so that

$$q = 1 - \frac{6}{m(m+2)}.$$

In this connection, we mentioned the theorem of Qiu-Friedan-Shenkar that if m is an integer and $h = h_{r,s}$ then $L(q,h)$ is unitary.

An alternative formula for $h_{r,s}$

Now we are interested in more general values of q where m is not an integer. Then other formulas for $h_{r,s}$ may be more convenient. For example, there is the following formula when

$$q = 1 - \frac{6(p-p')^2}{pp'}.$$

Then

$$h_{r,s} = \frac{(pr-p's)^2 - (p-p')}{4pp'}.$$

Proof of the alternative formula for $h_{r,s}$

To check this, we must show that if p, p' and m satisfy $0 < p < p'$, $0 < m$ and

$$\frac{pp'}{(p-p')^2} = m(m+1)$$

then

$$\frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'}.$$

To see this, note that both

$$\frac{pp'}{(p-p')^2}, \qquad \frac{(pr - p's)^2 - (p-p')^2}{4pp'}$$

are invariant under scaling $(p, p') \longrightarrow (\lambda p, \lambda p')$ so we may arrange that $p - p' = 1$ and then we must have $p = m+1$, $p' = m$.

Consequences of the alternative formula for $h_{r,s}$

Now assume that p, p' are coprime integers and $p > p'$. We have checked that if

$$c = 1 - 6\frac{(p-p')^2}{pp'}$$

where p, p' are coprime integers and $p > p'$ then

$$h_{r,s} = \frac{(pr - p's)^2 - (p-p')}{4pp'}.$$

We now assume that p, p' are integers. We note the symmetry

$$h_{r,s} = h_{p'-r,p-s}.$$

Consequences, continued

From the Kac determinant formula, the Verma module $M(c, h_{r,s})$ has a singular vector of level rs. This generates a highest weight representation that is a quotient of $M(c, h_{r,s} + rs)$.
Similarly since $h_{r,s} = h_{p'-r,p-s}$ it has a singular vector of level $(p'-r)(p-s)$, generating a quotient of
$M(c, h_{r,s} + (p'-r)(p-s))$.
Now we have the symmetry properties

$$h_{r,s} + rs = h_{p'+r,p-s} = h_{p'-r,p+s}$$

$$h_{rs} + \left(p'-r\right)(p-s) = h_{r,2p-s} = h_{2p'-r,s}$$

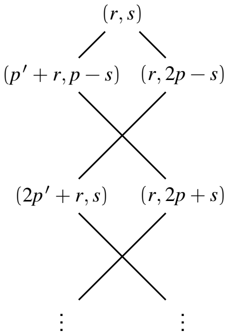
so these Verma modules $M(c, h_{r,s} + rs)$ and
$M(c, h_{r,s} + (p'-r)(p-s))$ are themselves singular.

The lattice of submodules

Assuming still that p, p' are coprime integers and $p > p'$,

$$c = 1 - 6\frac{(p - p')^2}{4pp'}$$

we get a lattice of subgroups of $M(c, h_{r,s})$:



Virasoro modules in a CFT

We turn now to a conformal field theory.

The vacuum $|0\rangle$ is invariant under the subalgebra $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ which contains the Virasoro generators L_{-1}, L_0, L_1 and $\bar{L}_{-1}, \bar{L}_0, \bar{L}_1$ and. Therfore the vacuum is annihilated by L_n and $\bar{L}_n$ when $n \geqslant -1$. We expect that $\mathcal{H}$ will be a direct sum of modules of the form $L(c,h) \otimes L(c,\bar{h})$.

The central charge c will be the same for all of these, but we may have various h, $\bar{h}$. So let us write

$$\mathcal{H} = \bigoplus_a L(c,h_a) \otimes L(c,\bar{h}_a),$$

summing over primary fields Φ_a with conformal weights $h_a, \bar{h}_a$.

Operator Product Expansion

Our goal for this lecture and the next is to determine the fields that can occur in the OPE. We may write

$$\Phi_a(z,\bar{z})\Phi(w,\bar{w}) =$$

$$\sum_c \sum_{\mathbf{k},\bar{\mathbf{k}}} C_{ab}^{c,\mathbf{k},\bar{\mathbf{k}}}(z-w)^{h_c-h_a-h_b+\sum k_i}(\bar{z}-\bar{w})^{\bar{h}_c-\bar{h}_a-\bar{h}_b+\sum \bar{k}_i}\Psi_c^{\mathbf{k},\bar{\mathbf{k}}}(w,\bar{w})$$

where $\Psi_c^{\mathbf{k},\bar{\mathbf{k}}}$ are descendent fields $L_{-k_n}\cdots L_{-k_1}\bar{L}_{-\bar{k}_m}\cdots\bar{L}_{-\bar{k}_1}\Phi_c$. The question is when $C_{ab}^{c,\mathbf{k},\bar{\mathbf{k}}}$ is nonzero. Less precisely

$$\Psi_a(z)\Psi_b(w) = \sum_c C_{ab}^c(z-w)\Psi_c(w).$$

Fusion

Each component $L(c, h_a) \otimes L(c, \bar{h}_a)$ contains a unique primary field Φ_a and the remaining fields, called *descendents* of Φ_a are those that may be obtained from Φ_a by applying the operators L_{-k} and $\bar{L}_{-k}$. The fields in $L(c, h_a) \otimes L(c, \bar{h}_a)$ are said to lie in the same *conformal family*, denoted $\{\Phi_a\}$.

Given fields Ψ_a and Ψ_b from the conformal families $\{\Phi_a\}$ and $\{\Phi_b\}$, the operator product expansion will have the form

$$\Psi_a(z)\Psi_b(w) = \sum_c C^c_{ab}(z-w)\Psi_c(w)$$

Informally write

$$\{\Phi_a\} \times \{\Phi_b\} = \sum_c \mathcal{N}^c_{ab}\{\Phi_c\},$$

Fusion, continued

In the “fusion expansion”

$$\{\Phi_a\} \times \{\Phi_b\} = \sum_c \mathcal{N}_{ab}^c \{\Phi_c\},$$

$\mathcal{N}_{ab}^c$ is the number of essentially different ways the conformal family $\{\Phi_c\}$ appears in the OPE of $\Phi_a(z)\Phi_b(w)$. For the problem at hand these multiplicities will all be 0 or 1. The complex span of the $\{\Phi_a\}$ is a ring, called the *fusion ring* and the multiplication $\times$ is called *fusion*.

Minimal models

It is a special case when there are only a finite number of conformal families and the sum

$$\mathcal{H} = \bigoplus_a L(c, h_a) \otimes L(c, \bar{h}_a)$$

is finite. What [BPZ] proved is that if $c = 1 - \frac{6(p-p')}{pp'}$ where p, p' are coprime integers with $p > p'$ then we may take

$$\mathcal{H} = \bigoplus_{\substack{1 \leqslant r < p' \\ 1 \leqslant s < p}} L(c, h_{r,s}) \otimes L(c, \bar{h}_{r,s}).$$

The resulting *minimal model* will be denote $\mathcal{M}(p, p')$.

Statistical Mechanics

For the minimal models, what we need is for

$$\frac{\sqrt{c-1}-\sqrt{c-25}}{\sqrt{c-1}+\sqrt{c-25}}$$

to be rational.

In a second 1984 paper [BPZ] showed that certain models from statistical physics such as the two-dimensional Ising model at the critical temperature are described by models of this type. For the Ising model the relevant CFT is $\mathcal{M}(4,3)$, and there are other similar examples.

Highest weight vectors

Let $\Phi\,(z,\bar{z})$ be a primary field of conformal dimensional $h,\bar{h}$ in a conformal field theory. This generates a module for $\mathbf{Vir} \oplus \mathbf{Vir}$ but for the time being we will discuss only the holomorphic first component and write $\Phi = \Phi(z)$. The primary field satisfies

$$[L_n, \Phi(z)] = z^{n+1}\partial_z\Phi(z) + h(n+1)z^n\Phi(z).$$

for all n. We consider the state $|h\rangle = \Phi(z)|0\rangle$ where $|0\rangle$ is the vacuum in $\mathcal{H}$. Since the vacuum is invariant under the global conformal group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, which is generated by L_{-1}, L_0, L_1 and $\bar{L}_{-1}, \bar{L}_0, \bar{L}_1$ we have $L_n|0\rangle = \bar{L}_n|0\rangle = 0$ if $n \geqslant -1$. It follows that

$$L_0\Phi(z)|0\rangle = [L_0, \Phi(z)] = z\partial_z\Phi(z)|0\rangle + h\Phi(z)|0\rangle$$

and taking $z = 0$ we get $L_0|h\rangle = h|0\rangle$. Similarly if $n > 0$ we have $L_n|h\rangle = 0$ so $|h\rangle$ generates a highest weight module.

Highest weight vectors

The basic idea is that if the Verma module $M(c,h)$ has singular vectors then terms can be omitted from the fusion expansion

$$\{\Phi_a\} \times \{\Phi_b\} = \sum_c \mathcal{N}_{ab}^c \{\Phi_c\}$$

that would ordinarily be there. making it easier to keep the number of primary fields in the decomposition

$$\mathcal{H} = \bigoplus_a L(c,h_a) \otimes L(c,\bar{h}_a)$$

finite.