Lecture 15: Virasoro Discrete Series

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Primitive Vectors

Let us consider the general cas of a Lie algebra

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

with triangular decomposition, let V be a highest weight module. Reference: Kac, *Infinite-dimensional Lie algebras,* Chapter 9. Let V be a $\mathfrak{g}$-module with a highest weight decomposition. A vector $v \in V$ is called *primitive* if there exists a submodule U such that $v \notin U$ but $\mathfrak{n}_+ v \subseteq U$. An important special case is that $U = 0$. Then $v \neq 0$ but $\mathfrak{n}_+ v = 0$; in this case v generates a highest weight representation with highest weight λ. If this is true, we say that v is a *singular vector*.

Example: $\mathfrak{sl}(2,\mathbb{C})$

A necessary and sufficient condition for a module in Category $\mathcal{O}$ to be irreducible is that it has a unique (up to scalar) primitive vector. This vector will be a highest weight vector. If $v \in V_\lambda$ then $V \cong L(\lambda)$.

Let us illustrate these examples with the example $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ spanned by

$$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H.$$

Let λ be the linear functional on $\mathfrak{h} = \mathbb{C}H$ defined by $\lambda(H) = k$, where $k \in \mathbb{C}$. Let v_λ be the highest weight vector, so $Hv_\lambda = kv_\lambda$ and $Ev_\lambda = 0$.

Example, continued

Because the map $U(\mathfrak{n}_-) = \mathbb{C}[F]$ to $M(\lambda)$ sending ξ to ξv_λ is an isomorphism, a basis of $M(\lambda)$ consists of $F^m v$ with $m = 0, 1, 2, \cdots$. An induction using $[H, F] = -2F$ shows that $H \cdot F^m v_\lambda = (k - 2m) v_\lambda$. Then another induction using $[E, F] = H$ shows that $EF^m v_\lambda = m(k - m + 1) v_\lambda$. Thus assuming this for some m,

$$EF^{m+1} v_\lambda = (EF - FE) F^m v_\lambda + FEF^m v_\lambda = HF^m v_\lambda + FEF^m v_\lambda$$

and by induction this equals

$$(k - 2m) v_\lambda + m(k - m + 1) v_\lambda = (m + 1)(k - m) v_\lambda .$$

This completes the induction.

Example, concluded

Since $EF^m v_\lambda = m(k-m+1)v_\lambda$, we see that $F^m v_\lambda$ is a singular vector if $m = k+1$. This means that k is a nonnegative integer, or equivalently, λ is a dominant weight. The singular vector $F^m v_\lambda$ is a highest weight vector for a submodule isomorphic to $M(\lambda-(k+1)\alpha)$. Then $L(\lambda) = M(\lambda)/M(\lambda-(k+1)\alpha)$ is finite-dimensional.

If we regard $\mathfrak{g}$ as the complexification of $\mathfrak{su}(2)$ then $L(\lambda)$ is unitary as an $\mathfrak{su}(2)$-module in this case where λ is dominant. For general λ a highest weight module for λ will contain vectors of negative norm (“ghosts”) but not for $L(\lambda)$ when λ is dominant.

The inner product for $\mathbf{Vir}$

Let V be a highest weight representation of $\mathbf{Vir}$ with highest weight (c, h), meaning that $Cv = cv$ for all $v \in V$, and $L_0 v_\lambda = h v_\lambda$ if v_λ is a highest weight vector. We will fix a highest weight vector and denote $v_\lambda = |h\rangle$.

In a unitary representation that comes from a conformal field theory, L_n must be the adjoint of L_{-n}. See Ginsparg, Applied CFT (arXiv:hep-th/9108028) Section 3.4 for justification of this. It is proved in Kac and Raina, Proposition 2.2 that if V is a highest weight representation of $\mathbf{Vir}$ that there is a unique Hermitian inner product on V in which L_n and L_{-n} are adjoints. However this inner product may not be positive definite.

Solvable lattice models

Determining whether this inner product on the irreducible highest weight module $L(c, h)$ is positive definite is a problem solved by the *Kac determinant*, which we now describe, following [FMS] Section 7.2.1 and Kac-Raina, Chapters 8 and 12.

When $c < 1$ the representations $L(c, h)$ when $M(c, h)$ contains a singular vector are used in constructing the two-dimensional *minimal models* of [BPZ], which important in statistical mechanics since they often model two-dimensional solvable lattice models such as the Ising model at the critical temperature.

The Verma module

The Verma module $M(c,h)$ is graded as follows. A basis consists of vectors

$$|\mathbf{k}\rangle = L_{-k_1}\cdots L_{-k_m}|h\rangle, \qquad 1 \leqslant k_1 \leqslant \cdots \leqslant k_m.$$

We call $\sum k_i = N$ the level of the vector. Let $\mathbf{k} = (k_1, \cdots, k_n)$ be the corresponding partition (written backwards since traditionally partitions are written in descending order). If $\mathbf{k}$ and $\mathbf{l}$ are two such partitions of the same level l, then the inner product $\langle \mathbf{l}|\mathbf{k}\rangle$ equals

$$\langle h|L_{k_m}\cdots L_{k_1}L_{-l_1}\cdots L_{-k_n}|h\rangle.$$

(If $\mathbf{l}$ and $\mathbf{k}$ have different level then $|\mathbf{l}\rangle$ and $|\mathbf{k}\rangle$ are orthogonal.)

Inner products

The number of partitions of level N is denoted $p(N)$. The $p(N) \times p(N)$ matrix of inner products $\langle \mathbf{l} | \mathbf{k} \rangle$ is denoted $\det_N(c, h)$.

Let us compute some inner products. To compute $\langle h | L_1 L_{-1} | h \rangle$ we use the identity $[L_1, L_{-1}] = 2L_0$ and we see that $\langle h | L_1 L_{-1} | h \rangle = 2h \langle h | h \rangle = 2h$. Again, let us compute $\langle h | L_1^2 L_{-2} | h \rangle$. For this we use $L_1 L_{-2} = L_{-2} L_1 + 3L_{-1}$. Remembering that $L_1 | h \rangle = 0$ we get

$$\langle h | L_1^2 L_{-2} | h \rangle = \langle h | 3L_{-1} | h \rangle = 6h.$$

Again, using the cocycle $\frac{k^3 - k}{12} = \frac{1}{2}$ when $k = 2$,

$$\langle h | L_2 L_{-2} | h \rangle = \left\langle h | 4L_0 + \frac{C}{2} | h \right\rangle = 4h + \frac{c}{2}, \qquad \text{etc.}$$

Determinants

The determinant of the $p(N) \times p(N)$ matrix of inner products is denoted $\det_N(c, h)$ and we compute

$$\det_1(c, h) = 2h,$$

and

$$\det_2(c, h) = \det \begin{pmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_2L_{-1}L_{-1}|h\rangle \\ \langle h|L_2L_{-1}L_{-1}|h\rangle & \langle h|L_1L_1L_{-1}L_{-1}|h\rangle \end{pmatrix} =$$

$$= \begin{vmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 8h^2 + 4h \end{vmatrix} = 2h(16h^2 + 2hc - 10h + c).$$

These determinants must be non-negative if the module $L(c, h)$ is unitary. (We allow the inner product to be semidefinite but not indefinite.) Thus we need $h \geqslant 0$ and

$$0 \leqslant c < 1 - (4h-1)^2/(2h+1).$$

The numbers $h_{r,s}$

As another application, we may now see when $M(c,h)$ has a singular vector of level 2. From the above, we must have $0 = \det_2(c,h)$ and so $16h^2 + (2c-10)h + c$. Solving the quadratic equation for h we must have

$$h == \frac{1}{16}\left(c - 5 \pm \sqrt{(c-1)(c-25)}\right).$$

To proceed further the higher Kac determinants are needed. A formula for these was found by Kac (1978). Let

$$h_{r,s}(c) = \frac{1}{48}\big[(13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}(r^2-s^2)$$

$$-24rs - 2 + 2c\big].$$

The Kac Determinant Formula

The Kac determinant formula is

$$\det_n(c,h) = K \prod_{\substack{r,s\in\mathbb{N} \\ 1\leqslant rs\leqslant n}} (h - h_{r,s}(c))^{p(n-rs)}$$

where K is an explicit positive constant. The proof is somewhat difficult and may be found in Kac-Raina Chapters 8 and 12.
A first consequence is that the Verma module $M(c,h)$ is irreducible and unitary if $c > 1$ and $h > 0$, the key step being the positivity of all the Kac determinants. If $c = 1$ then $M(c,h)$ is unitary unless $4h$ is a square in $\mathbb{Z}$, and it is always (weakly) unitary. The case $c = 1$ is relevant to some interesting conformal field theories, including the free boson. See Ginsparg Figure 14 for a survey of CFT when $c = 1$. If $c = 0$ only the trival representation $L(0,0)$ is unitary.

Unitary Representations

It is better to revise the notation and write $h_{r,s}$ as a function of a parameter m chosen so that

$$c(m) = 1 - \frac{6}{m(m+1)}$$

and then

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}.$$

Theorem (Friedan, Shenkar, Qiu)

The module $L(c,h)$ is unitary if and only if $c = c(m)$ with m an integer $\geqslant 2$ and $h = h_{r,s}(m)$ for some r, s with $1 \leqslant s \leqslant r < m$.

Proofs of this deep result were also given by Kac-Wakimoto (independently) and Langlands (later). The Kac-Wakimoto proof is described in Kac-Raina Chapter 12.