Lecture 14: Virasoro Vertex Algebras

Daniel Bump

January 1, 2020

Reminder of Lecture 13

We recall from Lecture 13 that a field $\Phi_a(z,\bar{z})$ is primary if there exist positive constants Δ_a and $\bar{\Delta}_a$ such that

$$[L_n, \Phi_a(z,\bar{z})] = (n+1)\Delta_a z^n \Phi_a(z,\bar{z}) + z^{n+1}\frac{\partial}{\partial z}\Phi_a(z,\bar{z}),$$

$$\left[\overline{L}_n, \Phi_a(z,\bar{z})\right] = (n+1)\overline{\Delta}_a \bar{z}^n \Phi_a(z,\bar{z}) + \bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}\Phi_a(z,\bar{z}).$$

Remember from Lecture 13 the two components of the energy-momentum tensor are

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad \overline{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \overline{L}_n \bar{z}^{-n-2}.$$

These are fields (not primary).

Normal Order

We define the normal order for two field

$$A(z) = \sum A_n z^{-n}, \qquad B(w) = \sum B_m w^{-m}$$

to be

$$\sum_{n,m} :A_n z^{-n} B_m w^{-m}:$$

where

$$:A_n z^{-n} B_m w^{-m}: = \begin{cases} B_m A_n z^{-n} w^{-m} & \text{if } -n \geqslant 0, \\ A_n B_m z^{-n} w^{-m} & \text{otherwise.} \end{cases}$$

(Since the field might be written $A_n z^{-n-1}$ we note that it is the exponent z^{-n} that determines the two cases, not the subscript of the operator A_n.)

The operator product expansion

With Φ_a a primary field, we will prove the operator product expansion

$$T(z)\Phi_a(w,\overline{w}) = \frac{\Delta_a}{(z-w)^2}\Phi_a(w) + \frac{1}{z-w}\frac{\partial}{\partial w}\Phi_a(w) + :T(z)\Phi_a(w,\overline{w}): .$$

We have

$$T(z)\Phi_a(w) - :T(z)\Phi_a(w): = \sum_{n\leqslant -2}[L_n, \Phi_a(w)]z^{-n-2} =$$

$$\sum_{n\leqslant -2}(n+1)\Delta_a w^n z^{-n-2}\Phi_a(w) + w^{n+1}z^{-n-2}\frac{\partial}{\partial z}\Phi_a(w).$$

We regard this as an expansion at z. Let $k=-2-n$. This equals

$$-\Delta_a \sum_{k=0}^{\infty}(k+1)w^{-k-2}z^k + \sum_{k=0}^{\infty} w^{-1-k}z^k \frac{\partial}{\partial z}\Phi_a(w)$$

Proof: OPE

Both series are convergent when $|z| < |w|$ this equals

$$\frac{\Delta_a}{(z-w)^2}\Phi_a(w) + \frac{1}{z-w}\frac{\partial}{\partial w}\Phi_a(w).$$

Note that although we proved this expansion by summing a power series in z, the operator product expansion gives information at w. That is, since $:T(z)\Phi_a(w,\overline{w}):$ is analytic when $z = w$ we may write

$$T(z)\Phi_a(w,\overline{w}) \sim \frac{\Delta_a}{(z-w)^2}\Phi_a(w) + \frac{1}{z-w}\frac{\partial}{\partial w}\Phi_a(w)$$

and this expands the product of the two operators in terms of local fields at w.

OPE for the Virasoro field

Assume that the Virasoro generator C acts by the scalar c on all fields, including $T(z)$ itself. Then we say that the CFT has central charge c. In this case a slightly more difficult computation shows that

$$T(z)T(w) = \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + :T(z)T(w):,$$

or

$$T(z)T(w) \sim \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} .$$

This can be deduced from the Virasoro commutation rules

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}c,$$

and indeed this OPE encodes this identity: see Kac Vertex Algebras for Beginners, Theorems 2.6 and 4.10.

Conformal vectors

This leads to the notion (due to Borcherds) of a conformal vertex algebra. In addition to Kac, see [FBZ] for this notion.

Let V be a vertex algebra, $\omega \in V$ a vector. Usually we use the notation $Y(v,z) = \sum v_{(n)} z^{-n-1}$ but shift the indices and write

$$Y(v,z) = \sum_{n \in \mathbb{Z}} L_n v^{-n-2}.$$

Denote $T(z) = Y(v,z)$ and assume that we have the OPE

$$T(z)T(w) \sim \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \ .$$

This implies that the L_n generate a Virasoro algebra with central charge c. Assume also that $L_{-1} = T$, the translation operator, and that L_0 diagonalizable. Then ω is called a conformal vector and V is called a conformal vertex algebra.

Alternative formulation of the OPE

For a vertex algebra, the identity

$$T(z)T(w) = \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + :T(z)T(w):,$$

is equivalent to

$$[T(z), T(w)] = \frac{c}{12}\partial_w^3\delta(z-w) + 2T(w)\partial_w\delta(z-w) + T'(w)\delta(z-w).$$

See [FBZ] Lemma 2.5.4, Proposition 3.3.1 and (3.4.2).

Kac's Lemma on the OPE

In the context of a vertex algebra, the equivalence of the two statements is a lemma due to Kac, that for fields ϕ, ψ and γ_j

$$[\phi(z), \psi(w)] = \sum_{j=0}^{N} \frac{1}{j!} \gamma_j(w) \delta_w^{(j)} \delta(z-w)$$

if and only if

$$\phi(z)\psi(w) = \sum_{j=0}^{N-1} \frac{\gamma_j(w)}{(z-w)^{j+1}} + :\psi(z)\phi(w):.$$

For the proof, see [FBZ] Proposition 3.3.1.

Return to Heisenberg

Let us return to the example of the Heisenberg vertex algebra $\mathfrak{H}$ from Lectures 8 and 9. The Heisenberg Lie algebra is spanned by elements b_n ($n \in \mathbb{Z}$) and $\mathbb{1}$ such that

$$[b_m, b_n] = m\delta_{m,-n}\mathbb{1}.$$

The Bosonic Fock space is $B = \mathbb{C}[b_{-1}, b_{-2}, \cdots]$, a subspace of the universal enveloping algebra $U(\mathfrak{H})$. It is a $\mathfrak{H}$-module:

If $n > 0$ then b_{-n} acts by multiplication and b_n acts by $n\partial/\partial b_{-1}$. We let b_0 act by 0 and $\mathbb{1}$ act by 1 on B. The element $1 \in B$ (not to be confused with $\mathbb{1}$) is the vacuum vector.

Review: Heisenberg VA

Then B has the structure of a vertex algebra, and we had partially proved this. Today we will talk a bit more about the proof. We begin by defining the translation operator T by

$$T(b_{j_1}\cdots b_{j_k}) = -\sum_{i=1}^{k} j_i b_{j_1}\cdots b_{j_i-1}\cdots b_{j_k}.$$

We also require that $Y(b_{-k},z) = b(z)$ where

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}.$$

More generally, we want

$$Y(b_{-k},z) = \frac{1}{(k-1)!}\partial^{k-1}b(z).$$

Review: Locality of the Heisenberg fields

The fields $b(z)$, $b(w)$ are local due to the operator product expansion

$$b(z)b(w) = \frac{1}{(z-w)^2} + :b(z)b(w):$$

or equivalently

$$[b(z), b(w)] = \partial_w \delta(z-w).$$

It follows that their derivatives $\partial^{k-1}b(z)$ and $\partial^{l-1}b(w)$ are local by differentiating the identity

$$(z-w)^2[b(z), b(w)] = 0$$

to obtain

$$(z-w)^{2+k+l}[\partial_z^k b(z), \partial_w^l b(w)] = 0.$$

At this point, one may complete the construction by invoking a reconstruction theorem which is Theorem 2.3.11 or Theorem 4.4.1 of [FBZ].

Hypotheses of the Reconstruction Theorem

Assume that we have a vector space V with a nonzero vector $|0\rangle$ and a finite or countable collection of vectors with fields

$$a^{\alpha}(z) = \sum_{n \in \mathbb{Z}} a^{\alpha}_{(n)} z^{-n-1}$$

such that for all α, $a^{\alpha}(z)|0\rangle \in \mathrm{End}(V)[[z]]$ and

$$a^{\alpha}(z)|0\rangle|_{z=0} = a^{\alpha}.$$

It is further assumed that $[T, a^k(z)] = \partial_z a^k(z)$, that the fields $a^k(z)$, $a^l(w)$ are mutually local. Finally we assume the index set $\{\alpha\}$ to be ordered such that V has a basis of vectors

$$a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)} |0\rangle$$

with $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m < 0$ and such that if $j_i = j_{i+1}$ then $\alpha_i \leqslant \alpha_{i+1}$.

Reconstruction Theorem

Then the reconstruction theorem asserts that V may be made into a vertex algebra with

$$Y(a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)}|0\rangle, z) =$$

$$\frac{1}{(-j_1-1)! \cdots (-j_m-1)!} : \partial_z^{-j_1-1} a^{\alpha_1}(z) \cdots \partial_z^{-j_m-1} a^{\alpha_m}(z) : .$$

For the Heisenberg Lie algebra, we need only one α, and $a^{\alpha} = b_{-1}$,

$$a^{\alpha}(z) = b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}.$$

Review of Lecture 4: Lie algebras with triangular decomposition

Let $\mathfrak{g}$ be a complex Lie algebra that can be written as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, where $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$ are Lie subalgebras with $\mathfrak{h}$ abelian, such that

$$[\mathfrak{h}, \mathfrak{n}_+] \subseteq \mathfrak{n}_+, \qquad [\mathfrak{h}, \mathfrak{n}_-] \subseteq \mathfrak{n}_-.$$

We require that

$$[\mathfrak{h}, \mathfrak{n}_+] \subset \mathfrak{n}_+, \qquad [\mathfrak{h}, \mathfrak{n}_0] \subset \mathfrak{n}_-.$$

This implies that $\mathfrak{n}_\mu \oplus \mathfrak{h}$ are Lie algebras, denoted $\mathfrak{b}$ and $\mathfrak{b}_-$.

We assume that $\mathfrak{n}_\pm$ have weight space decompositions with respect to the adjoint representation under $\mathfrak{h}$ and that 0 is not a weight. Moreover we assume there is a closed convex cone $D \subset \mathfrak{h}^*$ such that D (resp. $-D$) contains the weights of $\mathfrak{n}_+$ ($\mathfrak{n}_-$) and that $D \cap (-D) = \{0\}$.

Review: BGG Category $\mathcal{O}$

Let Φ_- be the set of weights in $\mathfrak{n}_-$ which is an $\mathfrak{h}$-module under the adjoint representation. Let Q_- be the set of finite sums of elements of Φ_- (with repetitions allowed). This is a discrete subset of $-D$.

The Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ of modules can be defined for any Lie algebra with triangular decomposition. A module V in this category is assumed to have a weight space decomposition with finite-dimensional weight spaces. Furthermore, it is assumed that there is a finite set of weights $\lambda_1, \cdots, \lambda_N$ such that the weights of V lie in the set

$$\bigcup_i (\lambda_i + Q_-).$$

Review: Highest weight modules

A module V is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there is a vector $v \in V(\lambda)$ such that $X \cdot v = 0$ for $X \in \mathfrak{n}_+$, and such that $V = U(\mathfrak{g}) \cdot v$. Since

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \otimes U(\mathfrak{n}_-)$$

this is equivalent to $V = U(\mathfrak{n}_n) \cdot v$.

Any highest weight module is in Category $\mathcal{O}$.

Review: Verma modules

There exists a unique highest weight module M_λ such that if V is a highest weight module with highest weight λ then V is isomorphic to a quotient of M_λ. To construct M_λ, note that λ extends to a character of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ by letting $\mathfrak{n}_+$ act by zero. Let $\mathbb{C}_\lambda$ be $\mathbb{C}$ with this $\mathfrak{b}$-module structure, with generator 1_λ. Define

$$M_\lambda = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda.$$

In view of

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \otimes U(\mathfrak{n}_-),$$

the map $\xi \to \xi \otimes 1_\lambda$ is a vector space isomorphism $U(\mathfrak{n}_- \to M_\lambda$. Let $v_\lambda = 1 \otimes 1_\lambda$ be the highest weight element of M_λ, unique up to scalar.

Lemma

Let $\mathfrak{h}$ be an abelian Lie algebra and let V be a $\mathfrak{h}$-module. We say that V has an weight space decomposition with respect to $\mathfrak{h}$ if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \qquad V_\lambda = \{v | X \cdot v = \lambda(X) v, \qquad X \in \mathfrak{h}\}$$

and the spaces V_λ are finite-dimensional.

Lemma

If V has a weight-space decomposition and U is any submodule then U also has a weight space decomposition.

For a proof see Kac-Raina, Corollary 1.1.

Review: Irreducibles

Let V be a highest weight module, for example a Verma module, with highest weight vector u_λ.

Lemma

V has a unique maximal proper submodule.

To prove this, note that a submodule U of V is proper if and only if $u_\lambda \notin V$. Indeed, if U is not proper, then $u_\lambda \in U$, and conversely if U is proper, then u_λ cannot be in U because u_λ generates V. Now every proper submodule U has a weight space decomposition

$$U = \bigoplus_{\mu \neq \lambda} U_\mu.$$

So the sum of the proper submodules has a weight space decomposition not involving λ, and is therefore proper.

Two triangular decompositions of $\mathbf{Vir}$

There are two noteworthy triangular decompositions of $\mathbf{Vir}$. The one we usually use is $\mathfrak{h} = \mathbb{C}C \oplus \mathbb{C}L_0$,

$$\mathfrak{n}_+ = \bigoplus_{n>0} \mathbb{C}L_n, \qquad \mathfrak{n}_- = \bigoplus_{n<0} \mathbb{C}L_n.$$

The characters of $\mathfrak{h}$ are determined by the eigenvalues c and h of C and L_0.

The other triangular decomposition has $\mathfrak{h}' = \mathbb{C}C$,

$$\mathfrak{n}'_+ = \bigoplus_{n\geqslant -1} \mathbb{C}L_n, \qquad \mathfrak{n}'_- = \bigoplus_{n<-1} \mathbb{C}L_n.$$

To check that $\mathfrak{n}'_+$ is a Lie algebra, note that $[L_m, L_n] = (m-n)L_{m+n}$ because $\delta_{m,-n}\frac{m^3-m}{12} = 0$ if $m, n \geqslant -1$.

Virasoro Vertex Algebras

Fix $c \in \mathbb{C}^{\times}$. We take the Verma module $M(c)$ for the alternative triangular decomposition with the character $C \to c$ of $\mathfrak{h}'$. Following [FBZ] we want to make this a vertex algebra. The highest weight element becomes the vacuum $|0\rangle$ and the translation operator $T = L_{-1}$, so $T|0\rangle = 0$. We require

$$Y(L_{-2}|0\rangle) = T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

Because

$$[T(z), T(w)] = \frac{c}{12}\partial_w^3\delta(z-w) + 2T(w)\partial_w\delta_w(z-w) + T'(w)\delta(z-w),$$

we have $(z-w)^4[T(z), T(w)] = 0$ and therefore $T(z)$ is local.

The construction of the vertex algebra is concluded using the reconstruction theorem.