

Lecture 13: Primary Fields and the Virasoro Algebra

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January 1, 2020

Overview

In the previous lecture we started with a two-dimensional Lorentzian CFT and produced two vertex algebras, following Kac, *Vertex Algebras for Beginners*, Chapter 1. We analytically continued the fields to the tube domain $\mathfrak{H} \times \mathfrak{H}$, then applied the Cayley transform to make them functions on $\mathcal{D} \times \mathcal{D}$, where \mathfrak{H} and \mathcal{D} are the Poincaré upper half plane and the unit disk. Taking the value of the resulting field at 0 gives the state-field correspondence. The algebraic structure of two vertex algebras emerge from this analytic picture.

In today's lecture we will heuristically motivate the assumption that there should be the further structure of a pair of representations of the Virasoro algebra \mathbf{Vir} on the fields of the theory.

Linear Fractional Transformations

The group $SL(2, \mathbb{C})$ acts on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ by linear fractional transformations: if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}),$$

then we define

$$\gamma(z) = \frac{az + b}{cz + d}.$$

It is understood that $\gamma(\infty) = a/c$ and $\gamma(-d/c) = \infty$. Let

$$j(\gamma, z) = \frac{d}{dz} \gamma(z) = (cz + d)^{-2}.$$

The chain rule for derivatives implies that

$$j(\gamma_1 \gamma_2, z) = j(\gamma_1, \gamma_2 z) j(\gamma_2, z).$$

We will call this the “cocycle condition.”

Why j is a cocycle

We digress to explain why j is a cocycle. Let Ω be a domain such as the upper half-plane \mathfrak{H} or the unit disk \mathfrak{D} , and let $\text{Hol}(\Omega)$ be the group of holomorphic transformations of Ω . We have $\text{Hol}(\mathfrak{H}) = \text{SL}(2, \mathbb{Z})$ acting by linear fractional transformations while

$$\text{Hol}(\mathfrak{D}) = \text{SU}(1, 1) = \left\{ \gamma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \mid |a|^2 - |b|^2 = 1 \right\}.$$

Now let $\mathcal{O}(\Omega)$ be the ring of holomorphic functions on Ω . This is a module for $\text{Hol}(\Omega)$.

The cocycle condition allows us to interpret j as a 1-cocycle representing an element of $H^1(\text{Hol}(\Omega), \mathcal{O}(\Omega)^\times)$, that is, a line bundle on Ω .

Conformal weights

Let Δ be a positive real number, $\gamma \in \text{SL}(2, \mathbb{C})$ and let Ω be a simply-connected domain in $\mathbb{P}^1(\mathbb{C})$ where $j(\gamma, z)$ is defined and never zero. We may define $j(\gamma, z)^\Delta$ to be $e^{\Delta \log(j, z)}$. This requires (for every γ) a choice of a branch of \log , introducing an ambiguity. However if $\Delta \in \frac{1}{2}\mathbb{Z}$ there is no ambiguity, since $j(\gamma, z) = (cz + d)^{-2}$ is raised to an integer power. The apparent ambiguity can be removed more generally though we will not discuss this point in detail now. Suffice it to say that Δ will appear with another positive real number $\bar{\Delta}$ and what we need is for $\Delta - \bar{\Delta} \in \frac{1}{2}\mathbb{Z}$.

The quantities Δ and $\bar{\Delta}$ are called **conformal weights**.

Two dimensional CFT

As in Lecture 12, start with a two-dimensional Lorentzian CFT acting on a Hilbert space \mathcal{H} with a representation of the conformal group

$$\mathcal{C} = \text{SO}(2, 2)^\circ \cong (\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})) / \{\pm(I, I)\}.$$

After switching to light-cone coordinates, points in Minkowski space are parametrized by pairs $(t, \bar{t}) \in \mathbb{R} \times \mathbb{R}$, where $t = x_0 - x_1$ and $\bar{t} = x_0 + x_1$. The conformal group acts by linear fractional transformations: if $\gamma \in \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ then

$$(\gamma, \bar{\gamma}) : (t, \bar{t}) \mapsto \left(\frac{at + b}{ct + d}, \frac{\bar{a}\bar{t} + \bar{b}}{\bar{c}\bar{t} + \bar{d}} \right) = (t', \bar{t}'),$$

$$(\gamma, \bar{\gamma}) = \left(\left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \left(\begin{array}{cc} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{array} \right) \right).$$

Quasi-Primary Fields

A field Φ_a is **quasi-primary** if there exist positive constants Δ_a and $\bar{\Delta}_a$ such that for γ as above then

$$U(\gamma)\Phi_a(t, \bar{t})U(\gamma)^{-1} = j(\gamma, t)^{\Delta_a} \bar{j}(\gamma, \bar{t})^{\bar{\Delta}_a} \Phi(\gamma(t, \bar{t})).$$

The field Φ_a has analytic continuation to the interior of the tube domain $\mathcal{H} \times \mathcal{H}$.

We will use the **Cayley transform**

$$\mathfrak{c} = \frac{1}{\sqrt{-2i}} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$$

to transfer the fields to $\mathfrak{D} \times \mathfrak{D}$. This is the linear fractional transformation that maps the upper half-plane \mathfrak{H} to the unit disk \mathfrak{D} . We have

$$\mathfrak{c} \mathrm{SL}(2, \mathbb{R}) \mathfrak{c}^{-1} = \mathrm{SU}(1, 1).$$

Quasi-Primary Fields on $\mathcal{D} \times \mathcal{D}$

It follows from the cocycle relation that if we define

$$\tilde{\Phi}_a(z, \bar{z}) = (*) j(\mathfrak{c}^{-1}, t)^{\Delta_a} j(\mathfrak{c}^{-1}, \bar{t})^{\bar{\Delta}_a} \Phi_a(t, \bar{t})$$

where $(*)$ is a constant to be chosen at our convenience, then $\tilde{\Phi}_a$ satisfies

$$\tilde{U}((\gamma, \bar{\gamma})) \tilde{\Phi}_a(z, \bar{z}) \tilde{U}((\gamma, \bar{\gamma}))^{-1} = j(\gamma, z)^{\Delta_a} j(\bar{\gamma}, \bar{z})^{\bar{\Delta}_a} \tilde{\Phi}_a(\gamma(z, \bar{z})),$$

when $(\gamma, \bar{\gamma}) \in \mathrm{SU}(1, 1) \times \mathrm{SU}(1, 1)$, and \tilde{U} is the representation U conjugated by $(\mathfrak{c}, \mathfrak{c})$ to give a representation of $\mathrm{SU}(1, 1) \times \mathrm{SU}(1, 1)$ or by linearity, its complexified Lie algebra $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

The Virasoro Algebra

Recall that the **Virasoro algebra** \mathbf{Vir} is spanned by L_n ($n \in \mathbb{Z}$) together with a central element C , subject to

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} \cdot C.$$

The Witt Lie algebra is the quotient of \mathbf{Vir} by $\mathbb{C}C$, and is the Lie algebra of holomorphic vector fields. The span of L_{-1}, L_0, L_1 is a 3-dimensional Lie subalgebra and is identified with $\mathfrak{sl}(2, \mathbb{C})$, for these are the vector fields that exponential to global conformal automorphisms of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$.

We may therefore ask that the representation of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ on \mathcal{H} extends to a representation of $\mathbf{Vir} \times \mathbf{Vir}$, and that the quasi-primary fields satisfy a more precise identity that we will state presently.

Quasi-Primary Fields on $\mathfrak{D} \otimes \mathfrak{D}$

Remember if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ then

$$j(\gamma, z) = \frac{d\gamma(z)}{dz} = (cz + d)^{-2}.$$

Up to constant $j(c^{-1}, t) = (t + 1)^{-2}$ so

$$\tilde{\Phi}_a(z, \bar{z}) = (1 + t)^{-2\Delta_a} (1 + \bar{t})^{-2\bar{\Delta}_a} \Phi_a(t, \bar{t}).$$

We see that the quasi primary fields, analytically continued to \mathfrak{H} , then transferred to the disk by the Cayley transform, have transformation properties with respect to $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. We want to argue that it is plausible to extend this representation to $\mathbf{Vir} \times \mathbf{Vir}$.

Decoupling the two theories

Remember if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ then

$$j(\gamma, z) = \frac{d\gamma(z)}{dz} = (cz + d)^{-2}.$$

Up to constant $j(c^{-1}, t) = (t + 1)^{-2}$ so

$$\tilde{\Phi}_a(z, \bar{z}) = (1 + t)^{-2\Delta_a} (1 + \bar{t})^{-2\bar{\Delta}_a} \Phi_a(t, \bar{t}).$$

We will omit the tilde from the notation and denote fields on the disk as $\Phi_a(z, \bar{z})$. The theories for z and \bar{z} are decoupled, so we may consider them separately and then impose the condition that z and \bar{z} are complex conjugates at the end, to obtain a Euclidean CFT.

Primary Fields

Assuming that the representation of $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$ or its complexification $\mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})$ on \mathcal{H} has been extended to the Virasoro algebra, we call the field Φ_a **primary** if

$$[L_n, \Phi_a(z, \bar{z})] = (n+1)\Delta_a z^n \Phi_a(z, \bar{z}) + z^{n+1} \frac{\partial}{\partial z} \Phi_a(z, \bar{z}).$$

It will follow from our discussion that a primary field is quasi-primary, so this is a strengthening of our hypotheses that is satisfied in all important examples.

The Left Regular Representation

We digress to consider the left and right regular representations of a Lie algebra on $C^\infty(G)$ where G is a Lie group. If $X \in \mathfrak{g} = \text{Lie}(G)$, then the right regular representation is defined by

$$\rho(X)f(g) = \left. \frac{d}{dt} f(ge^{tX}) \right|_{t=0}.$$

For the left regular representation, we use

$$\lambda(X)f(g) = \left. \frac{d}{dt} f(e^{-tX}g) \right|_{t=0}.$$

Note the $-$ sign which is necessary to obtain a representation. If f is a function on the upper half-plane we may regard f as a function on $\text{SL}(2, \mathbb{R})$ that is right invariant by $\text{SO}(2)$. We see that the left regular representation is the action on functions on \mathfrak{H} by

$$Xf(z) = \left. \frac{d}{dt} f(e^{-tX}z) \right|_{t=0}$$

Quasi-primary fields and the Lie algebra

Returning to our field $\tilde{\Phi}_a$, hold \bar{z} fixed and consider the identity

$$\begin{aligned} \tilde{U}((e^{tX}, 1))\tilde{\Phi}_a(z, \bar{z})\tilde{U}((e^{tX}, 1))^{-1} &= j(e^{tX}, z)^{\Delta_a}\tilde{\Phi}_a((e^{tX}z, \bar{z})) \\ &= \left(\frac{d}{dz}e^{tX}z\right)^{\Delta_a}\tilde{\Phi}_a((e^{tX}z, \bar{z})). \end{aligned}$$

We wish to differentiate with respect to t , then set $t = 0$. The left-hand side produces $[X, \Phi_a(z, \bar{z})]$. As for the right-hand side, this is $-X$ applied to the function

$$g \longmapsto \left(\frac{d}{dz}g(z)\right)^{\Delta_a}\tilde{\Phi}_a((g(z), \bar{z})), \quad g \in \text{SL}(2, \mathbb{C})$$

evaluated at $g = 1$. The negative sign came from the definition of the left regular representation.

Taking the derivative

Now we assume that this action extends to the Witt Lie algebra \mathcal{W} consisting of holomorphic vector fields on \mathbb{C} , with basis $L_n = -z^{n+1} \frac{d}{dz}$. Because we are using the left regular representation, $L_n f(z)$ must be interpreted as $\left. \frac{d}{dt} f(z + tz^{n+1}) \right|_{t=0}$. We obtain (heuristically) the formula

$$\begin{aligned} [L_n, \Phi_a(z, \bar{z})] &= \frac{d}{dt} \left(\frac{d}{dz}(z + tz^{n+1}) \right)^{\Delta_a} \tilde{\Phi}_a((z + tz^{n+1}, \bar{z})|_{t=0}) \\ &= \frac{d}{dt} (1 + t(n+1)z^n)^{\Delta_a} \tilde{\Phi}_a((z + tz^{n+1}, \bar{z})|_{t=0}) \end{aligned}$$

Conclusion: the Virasoro action

Therefore

$$[L_n, \Phi_a(z, \bar{z})] = (n+1)\Delta_a z^n \Phi(z, \bar{z}) + z^{n+1} \frac{\partial}{\partial z} \Phi_a(z, \bar{z}).$$

Note that may be a **projective** representation of the Witt Lie algebra, because states of the field correspond to elements of the projective space over the ambient Hilbert space \mathcal{H} . Thus it is a representation of the unique central extension \mathbf{Vir} of \mathfrak{W} by \mathbb{C} .

There is of course a similar action for the other \mathbf{Vir} , applying the same considerations to \bar{z} . We will denote the corresponding Virasoro generators \bar{L}_n .

The Energy Momentum Tensor in Classical Mechanics

In classical mechanics, the **energy-momentum tensor**, also called the **stress-energy tensor** describes the motion of masses in space. It plays an important role in general relativity, where it appears in the field equation.

If we think of the masses being a collection of particles in motion, each has an energy-momentum vector attached to it. If these are smoothly distributed, the energy-momentum vectors form a vector field. The energy-momentum tensor is the symmetric square of the energy-momentum vector field. It thus has $\frac{1}{2}d(d+1)$ independent components.

In a CFT the energy-momentum tensor is traceless, so there is one fewer component. If $d = 2$ there are only two.

The Energy Momentum Tensor in CFT

In our situation the two components of the energy-momentum tensor are the fields

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}.$$

The following operator product expansions may be checked. It is assumed that the eigenvalue c of the Virasoro central element C is constant on all fields.

$$T(z)T(w) \sim \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{dT}{dw}(w) \frac{1}{(z-w)},$$

and for a primary field

$$T(z)\Phi_a(w) \sim \frac{\Delta_a}{(z-w)^2} \Phi_a(w) + \frac{1}{z-w} \frac{\partial}{\partial w} \Phi_a(w).$$