Lecture 12: From CFT to VA

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References

This lecture is a report on Kac, Vertex Algebras for Beginners, Chapter 1. Unfortunately this book is not available on-line. Nevertheless it is possible to read the first chapter on-line either through the AMS bookstore or Google Books.

- ▸ Google Books Kac, Vertex Algebras for Beginners.

The rest of the book has a different character than the first chapter, which starts with the Wightman axioms and produces a VA (actually two). In the remainder of the book, the algebraic theory of VA is developed and is independent of (and perhaps easier than) the first chapter.

Kac Chapter 1

In the first Chapter, Kac starts with the (Lorentzian) Wightman axioms for QFT and CFT. He then specializes to $d = 2$ CFT and produces the operator-state correspondence in the form of a field $Y(a, z)$ for a in a subset of the Hilbert space $V = \mathcal{H}$. This gives a a vertex algebra that is called a chiral algebra.

There are actually two chiral algebras, one corresponding to holomorphic (“right moving”) fields, the other corresponding to antiholomorphic (“left moving”) fields.

The appearance of two vertex algebras is explained by the fact that the conformal group

$$SO(2,2)^{\circ} \cong (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\{\pm(I,I)\}.$$

Wightman Axioms following Kac

Let us review the Wightman axioms as described by Kac. I will follow his notation except for elements of the Poincaré group.

Let M be d-dimensional Minkowski space with the norm $|x|^2 = x_0^2 - x_1^2 - \ldots - x_{d-1}^2$. The Poincaré group $\mathcal{P}$ is the semidirect product of the Lorentz group L and the translation group which we may identify with M.

The Wightman QFT depends on the following data. A complex Hilbert space $\mathcal{H}$, containing a vacuum vector $|0\rangle$, with a unitary representation U of the Poincaré group $\mathcal{P}$ on $\mathcal{H}$, and a collection of fields Φ_a. (Here a is just an index.) These are operator-valued tempered distributions on M. We will abuse notation by writing the field as $\Phi_a(x)$ like a function of $x \in M$, even though it is actually a distribution.

Wightman Axiom 1: Covariance

W1. $U(\gamma)\Phi_a(z)U(\gamma)^{-1} = \Phi_a(\gamma^{-1}z)$ for $\gamma \in \mathcal{P}$.

Applying this to translations gives a 1-parameter family of unitary operators in every translation direction. That is, if $q \in M \subset P$ we have

$$U(q)\Phi_a(z)U(q)^{-1} = \Phi_a(z+q).$$

Thus there exist Hermitian operators P_k such that

$$U(q) = \exp\left(i\sum_{k=0}^{d-1} q_k P_k\right)$$

Differentiating we obtain

$$i[P_k, \Phi_a] = \partial_{x_k}\Phi_a.$$

Wightman Axioms 2,3,4

W2. The vacuum $|0\rangle$ is fixed by all $U(\gamma)$. The joint spectrum of the P_k lies in the forward cone.

W3. The vacuum vector $|0\rangle$ is in the domain of any polynomial in the $\Phi_a(f)$ with $f \in \mathcal{S}(M)$ and the linear subspace $\mathcal{D}$ of $\mathcal{H}$ spanned by $\Phi_a(f)|0\rangle$ is dense in $\mathcal{H}$.

Remark: Wightman and Streater do not assume that the dense subspace D is generated by the fields but mention this as an open problem.

W4. (Locality) $\Phi_a(f)$ and $\Phi_b(h)$ commute if the supports of f and h are spacelike-separated.

Superalgebra

To accomodate fermions, the fields should be graded by $\mathbb{Z}/2\mathbb{Z}$. If p denotes the grading, then commutativity is in the superalgebra sense:

$$\Phi_a(f)\Phi_b(h) = (-1)^{p(a)q(b)}\Phi_b(h)\Phi_a(f).$$

Remark. I think that the above foundations describe only scalar fields and that the foundations should include an action of the Lorentz group $spin(1, d-1)$ on the fields as in Wightman and Streater. By the spin-statistics theorem, a scalar field is necessarily bosonic. That is, if Φ_a is fixed by the Lorentz group then $p(a) = 0$.

Special Conformal Transformations

To obtain a conformal field theory we enlarge the Poincaré group $\mathcal{P}$ to the larger conformal group $\mathcal{C} = SO(2,d)^{\circ}$. This contains the inversion $x \mapsto -x/|x|^2$. Conjugating $x \mapsto x - b$ with $b \in M$ by the inversion gives the special conformal transformation

$$s_b(x) = \frac{x + |x|^2 b}{1 + 2\langle x, b\rangle + |x|^2|b|^2}.$$

The denominator here is

$$\varphi(b,x) = 1 + 2\langle x, b\rangle + |x|^2|b|^2.$$

Special conformal transformations preserve space-like separatedness. The conformal group is generated by the translations and special conformal transformations. It also contains the dilations $x \mapsto \lambda x$ with $\lambda \in \mathbb{R}_+^{\times}$.

Conformal Weight

We assume that the representation U of $\mathcal{P}$ extends to the conformal group. The field Φ_a is quasi-primary if for a special conformal transformation there exists a real number Δ_a called the conformal weight such that

$$U(s_b)\Phi_a(x)U(s_b)^{-1} = \varphi(b,x)^{-\Delta_a}\Phi(s_b(x)).$$

It may be shown that $\Delta_a > 0$. Here $\varphi(b,x)^{-d}$ is the Jacobian $j(s_b,x)$. Recall the cocycle property for f_1 and f_2 smooth

$$J(f_1 f_2, x) = J(f_1, f_2 x)J(f_2, x).$$

It follows that

$$U(f)\Phi_a(x)U(f)^{-1} = j(f,x)^{\Delta_a/d}\Phi_a(fx)$$

for conformal maps f and in particular if $\lambda \in \mathbb{R}_+^\times$ we have

$$U(\lambda)\Phi_a(x)U(\lambda)^{-1} = \lambda^{\Delta_a}\Phi_a(\lambda x).$$

Infinitesimal generators of the special conformal maps

Conformal invariance for the s_b implies that there are Hermitian operators Q_k such that

$$U(s_b) = \exp\left(i\sum_{k=0}^{d-1} b_k Q_k\right).$$

Kac gives the formula

$$i[Q_k, \Phi_a(x)] = (|x|^2\partial_{x_k} - 2\eta_k x_k E - 2\Delta_a\eta_k x_k)\Phi_a(a)$$

where $E = \sum_{m=0}^{d-1} x_m\partial_{x_k}$ is the Euler operator and $\eta_0 = 1$, $\eta_k = -1$ for $k \geqslant 1$.

Overview

In this section we specialize to $d = 2$ and show how to produce the data of a vertex algebra from a CFT. This means a vector space V, fields $Y(a, z)$ and a translation operator. It turns out that there are two such algebras, a space of “right-moving” fields and a space of “left-moving” ones. These may be different.

For example, there are five main superstring theories, believed to be related by dualities. (There are 6 if we count 11d SUGRA.) Two of the theories are “heterotic” meaning that the left and right moving theories are different.

- Heterotic String Theory Wikipedia link.

Light cone coordinates

Now we take $d = 2$. We introduce light cone coordinates $t = x_0 - x_1$ and $\bar{t} = x_0 + x_1$. The bar does not mean complex conjugation since t and $\bar{t}$ are real. Let

$$P = \frac{1}{2}(P_0 - P_1), \qquad P = \frac{1}{2}(P_0 + P_1).$$

Recall that if $q \in M \subset P$ we have

$$U(q)\Phi_a(z)U(q)^{-1} = \Phi_a(z+q),$$

$$U(q) = \exp\left(i\sum_{k=0}^{d-1} q_k P_k\right).$$

Since the vacuum is invariant under $U(q)$,

$$\Phi_a(z+q)|0\rangle = \exp\left(i\sum_{k=0}^{d-1} q_k P_k\right)\Phi_a(z)|0\rangle.$$

Energy-momentum operators in light cone coordinates

In light-cone coordintates, this identity

$$\Phi_a(z+q)|0\rangle = \exp\left(i\sum_{k=0}^{d-1} q_k P_k\right)\Phi_a(z)|0\rangle$$

can be rewritten written

$$\Phi_a\left(t+q,\bar{t}+\overline{q}\right) = e^{i\left(qP+\overline{q}\,\overline{P}\right)}\Phi_a\left(t,\bar{t}\right)|0\rangle.$$

The upper half-plane

By Axiom W2, the joint spectrum of P and $\overline{P}$ lines in the domain $t, \bar{t} \geqslant 0$ and therefore (as explained in Lecture 7) $\Phi_a(t, \bar{t})|0\rangle$ has analytic continuation to the tube domain $\mathfrak{H} \times \mathfrak{H}$ where $\mathfrak{H}$ is the Poincaré upper half plane $\mathrm{im}(t) > 0$.

We have mentioned that the conformal group

$$\mathrm{spin}(2,2)^\circ \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}).$$

The upper half-plane is a homogeneous space of $SL(2,\mathbb{R})$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}.$$

This accounts for the fact that fields live on $\mathfrak{H} \times \mathfrak{H}$. The theories on the two copies of $\mathfrak{H}$ are largely decoupled and we can try to separate them.

Decoupling the conformal weights

Let γ be the element

$$\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \right).$$

Then in light-frame coordinates our earlier equation

$$U(f)\Phi_a(x)U(f)^{-1} = j(f,x)^{\Delta_a/d}\Phi_a(fx)$$

becomes

$$U(f)\Phi_a(x)U(f)^{-1} = (ct+d)^{-2\Delta_a}\left(\bar{c}\,\bar{t}+\bar{d}\right)^{-2\overline{\Delta}_a}\Phi\left(\gamma\left(t,\bar{t}\right)\right)$$

with $\overline{\Delta_a} = \Delta_a$. However due to the decoupling of the t and $\bar{t}$ dependencies it is not necessary to assume that $\overline{\Delta_a} = \Delta_a$.

P's and Q's

Using the operators P, $\overline{P}$ and (for the infinitesimal special conformal operators

$$Q = -\frac{1}{2}(Q_0 + Q_1), \qquad \overline{Q} = \frac{1}{2}(Q_1 - Q_0)$$

We have

$$i\left[P, \Phi_a(t,\bar{t})\right] = \partial_t \Phi_a(t,\bar{t}), \qquad i\left[\overline{P}, \Phi_a(t,\bar{t})\right] = \partial_{\bar{t}} \Phi_a(t,\bar{t}),$$

$$i\left[Q, \Phi_a(t,\bar{t})\right] = (t^2\partial_t + 2\Delta_a t)\Phi_a(t,\bar{t}),$$

$$i\left[\overline{Q}, \Phi_a(t,\bar{t})\right] = \left(\bar{t}^2\partial_{\bar{t}} + 2\overline{\Delta}_a\bar{t}\right)\Phi_a(t,\bar{t}).$$

The Cayley Transform

The Cayley transform is a conformal bijection of the upper half-plane $\mathfrak{H}$ onto the unit disk $\mathfrak{D}$.

If $t \in \mathfrak{H}$ then t is closer to i than to $-i$, so $\frac{t-i}{t+i} \in \mathfrak{H}$. We multiply this by -1 and denote

$$z = \frac{1+it}{1-it}, \qquad \bar{z} = \frac{1+i\bar{t}}{1-i\bar{t}}.$$

Using this we may transfer the fields from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{D} \times \mathfrak{D}$. Let

$$Y\left(a, z, \bar{z}\right) = \left(1+z\right)^{-2\Delta_a} \left(1+\bar{z}\right)^{-2\overline{\Delta}_a} \Phi\left(t, \bar{t}\right).$$

The State-Field Correspondence

So far a has just been a vector, but now we define a to be the value of $Y(a, z, \bar{z})|0\rangle$ at $z = \bar{z} = 0$. We are using the state-field correspondence to turn a into an element of the Hilbert space $\mathcal{H}$.

The translation operator T and friends

Let

$$T = \frac{1}{2}(P + [P, Q] - Q),$$

$$T^* = \frac{1}{2}(P - [P, Q] - Q),$$

$$H = \frac{1}{2}(P + Q).$$

(There are similar operators with bars, which we omit.) Our earlier relations can be rewritten

$$[T, Y(a, z, \bar{z})] = \partial_z Y(a, z, \bar{z}) \; ,$$

$$[H, Y(a, z, \bar{z})] = (z\partial_z + \Delta_a) Y(a, z, \bar{z}) \; ,$$

$$[T^*, Y(a, z, \bar{z})] = (z^2\partial_z + 2\Delta_a z) Y(a, z, \bar{z}) \; .$$

Positivity of the conformal weights

The operator T annihilates the vacuum vector. We have

$$[H,T] = T, \qquad [H,T^*] = -T^*, \qquad [T^*,T] = 2H$$

and

$$Ha = \Delta_a a, \qquad T^* a = 0.$$

Kac proves that $\Delta_a \geqslant 0$ at this point, noting that P and $\overline{P}$ are positive as well as Q and $\overline{Q}$; hence H is positive semidefinite and $\Delta_a \geqslant 0$.

Right chiral fields

We may now further decouple the left and right fields, arriving at the ingredients for two VA called chiral algebras.

Consider right chiral fields, for which $\partial_{\bar{t}}\Phi_a = 0$. If $t \neq t'$ for $t, t' \in \mathfrak{H}$ then t, t' are spacelike separated and so $\Phi_a(t)$, $\Phi_b(t')$ commute. Therefore $[\Phi_a(t), \Phi_b(t')]$ is a distribution concentrated on the diagonal and we may write

$$[\Phi_a(t), \Phi_b(t')] = \sum_{j=0}^{\infty} \delta^{(j)}(t-t')\Psi^j(t')$$

for some fields Ψ^j. The Wightman applies to these fields and we may add them to our CFT to obtain

$$[Y(a,z), Y(b,w)] = \sum_{j\geqslant 0} \delta^{(j)}(z-w)Y(c_j,w).$$

Locality

We may check that the fields $Y(a,z)$ are local in the VA sense. Using

$$[H, Y(a,z)] = (z\partial_z + \Delta_a)Y(a,z),$$

we may compute the commutator of H with both sides and set $z = 0$ to see that $Y(c_j, w)$ has conformal weight $\Delta_a + \Delta_b - j - 1$. Due to the positivity $Y(c_j, w)$ is zero for all but finitely many j and

$$[Y(a,z), Y(b,w)] = \sum_{j=0}^{N-1} \delta^{(j)}(z-w)Y(c_j, w)$$

is a finite sum. It follows that for large N

$$(z-w)^N [Y(a,z), Y(b,w)] = 0.$$

Chiral algebras

If we take V to be the vector space of a as $Y(a, z)$ runs through the right chiral fields, we see that we have the ingredients of a VA, and have checked many of the axioms.

Although the left and right chiral algebras have been separated out, they still have to play together. We may specialize to the domain where z and $\bar{z}$ are complex conjugates to obtain a Euclidean CFT.