Lecture 11: OPE and associativity for VA

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Operator Product Expansions

Before Wilson (1968) it was understood that if z and w are nearby points, then local fields in QFT can be expanded in terms of local fields

$$[A(z), B(w)] = \sum f_i(z-w)C_i(w).$$

Motivated by problems in particle physics Wilson proposed that instead one may try to expand the product of the operators themselves:

$$A(z)B(w) = \sum g_i(z-w)C_i(w).$$

This proved to be a powerful idea. The key point is to understand better the divergences as $z \to w$.

Reminder: Locality

Let $\phi(z)$ and $\psi(w)$ be fields and $v \in V$. Then $\phi(z)\psi(w)v \in V((z))((w))$. We define $\phi(z)$, $\psi(w)$ to be mutually local if for all $v \in V$

$$\phi(z)\phi(w)v, \qquad \psi(w)\phi(z)v$$

are expansions the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ in $V((z))((w))$ and $V((w))((z))$ respectively.

An equivalent condition is that

$$(z-w)^N\phi(z)\psi(w) = (z-w)^N\psi(w)\phi(z)$$

as an identity in $\mathrm{End}(V)[[z,w]][z^{-1},w^{-1}]$ for sufficiently large N. See [FBZ] Proposition 1.2.5.

Associativity in Vertex Algebras

If we regard the vertex algebra V as an analog of a commutative and associative ring, the analog of commutativity $AB = BA$ is locality: the commutator $[A(z), B(w)]$ is not zero, but the next best thing, it is a distribution concentrated on the diagonal $z = w$.

Associativity $A(BC) = (AB)C$ asserts imprecisely that

$$Y(A,z)Y(B,w)C = Y(Y(A,z-w)B,w)C.$$

To be precisely, both sides represent the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ coerced into $V((z))((w))$ and $V((w))((z-w))$, respectively. We will prove this today.

Where they live

We claim $Y(A,z)Y(B,w)C \in V((z))((w))$. Indeed, write this

$$\sum_m \left(\sum_n A_{(n)} B_{(m)} C z^{-n-1} \right) w^{-m-1}.$$

Note that $B_{(m)}C = 0$ for m sufficiently large. Moreover for fixed m, The expression in brackets is in $V((z))$. So $Y(A,z)Y(B,w)C \in V((z))((w))$.

On the other hand

$$Y(A, z-w)B = \sum_n A_{(n)} B (z-w)^{-n-1},$$

$$Y(Y(A, z-w)B, w)C = \sum_n Y(A_{(n)}B, w)C(z-w)^{-n-1}.$$

This is in $V((w))((z-w))$.

Caveat

To say that $Y(A,z)$ and $Y(B,w)$ are local is not quite the same as saying that $Y(A,z)Y(B,w) = Y(B,w)Y(A,z)$. It only means that $Y(A,z)Y(B,w) - Y(B,w)Y(A,z)$ is a distribution concentrated on the diagonal; instead it can be a linear combination of $\delta(z-w)$ and its derivatives. Similarly we will prove that

$$Y(A,z)Y(B,w)C, \qquad Y(Y(A,z-w)B,w)C$$

are the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ coerced into $V((z))((w))$ and $V((w))((z-w))$ does not mean they are equal. So associativity is in a weak sense similar to locality.

Associativity and the OPE

The meaning of the statement

$$Y(A,z)Y(B,w)C = Y(Y(A,z-w)B)C$$

is that both elements of (respectively) $V((z))((w))$ and $V((w))((z-w))$ correspond to the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$.

Formally we may write

$$Y(A,z)Y(B,w) = Y(Y(A,z-w)B) = \sum_n Y(A_{(n)}B,w)(z-w)^{-n-1}.$$

This is the operator product expansion expanding the product of two fields $Y(A,z)Y(B,w)$ in terms of fields at w, with coefficients at worst polar in $z-w$, of bounded order, since $A_{(n)}B = 0$ for sufficiently large n.

Review: Translation Identity

We will follow [FBZ] Section 3.2.

Let us prove in a vertex algebra the Translation Identity

$$\boxed{e^{wT}Y(A,z)e^{-wT} = Y(A,z+w). \qquad (TI)}$$

Indeed by the Baker-Campbell-Hausdorff formula

$$e^{wT}Y(A,z)e^{-wT} = \sum_{n=0}^{\infty} \frac{1}{n!}(\text{ad } wT)^n \cdot Y(A,z),$$

where $\text{ad}(X)Y = XY - YX$. Using $[T, Y(A,z)] = \partial_z Y(A,z)$,

$$e^{wT}Y(A,z)e^{-wT} = \sum_{n=0}^{\infty} \frac{1}{n!} w^n \partial_z^n Y(A,z),$$

and the statement follows from Taylor's theorem.

Translation and the Vacuum

Next we prove for any state A:

$$\boxed{e^{zT}A = Y(A,z)|0\rangle.}$$

Switching z and w in the last identity

$$e^{zT}Y(A,w)|0\rangle = Y(A,z+w)e^{zT}|0\rangle.$$

By the vacuum axiom

$$A = Y(A,w)|0\rangle|_{w=0}.$$

Furthermore $e^{tZ}|0\rangle = |0\rangle$ since $T|0\rangle = 0$. Hence taking $w = 0$ gives the required formula.

Skew-Symmetry

Next we prove

$$\boxed{Y(A,z)B = e^{zT}Y(B,-z)A.}$$

By locality

$$(z-w)^N Y(A,z)Y(B,w)|0\rangle = (z-w)^N Y(B,w)Y(A,z)|0\rangle,$$

for sufficiently large N. From the previous slide and (TI)

$$(z-w)^N Y(A,z)e^{wT}B = (z-w)^N Y(B,w)e^{zT}A = (z-w)^N e^{zT}Y(B,w-z)A.$$

The negative powers of $w-z$ that can appear on the right-hand side are bounded, so if we take N large this is an identity in $V((z))[w]$ and we can take $w=0$, then multiply by z^{-N} to obtain the advertised identity.

The left-hand side

Our goal is to prove that

$$Y(A,z)Y(B,w)C = Y(Y(A,z-w)B,w)C$$

in the weak sense that both sides represent the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ coerced into $V((z))((w))$ and $V((w))((z-w))$, respectively.

Using skew-symmetry and $e^{-Tw}Y(A,z)e^{Tw} = Y(A,z-w)$ we have

$$Y(A,z)Y(B,w)C = Y(A,z)e^{Tw}Y(C,-w)B = e^{Tw}Y(A,z-w)Y(C,-w)B.$$

That is, $Y(A,z)Y(B,w)C$ and $e^{Tw}Y(A,z-w)Y(C,-w)B$ are the expansions in $V((z))((w))$ and $V((z-w))((w))$ respectively of the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$.

The right-hand side

On the other hand

$$Y(Y(A,z-w)B,w)C = \sum_n (z-w)^{-n-1} Y(A_{(n)}B,w)C$$

and using skew-symmetry this equals

$$e^{wT} \sum_n (z-w)^{-n-1} Y(C,-w)A_{(n)}B = e^{wT} Y(C,-w)Y(A,z-w)B.$$

This is an element of $V((w))((z-w))$. By locality, this and

$$e^{Tw} Y(A,z-w)Y(C,-w)B$$

represent the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ coerced into $V((w))((z-w))$ and $V((z-w))((z))$, respectively.

Comparison

Using the binomial theorem we may identify the power series spaces $V[[z,w]]$ and $V[[z-w,w]]$ We have shown $Y(A,z)Y(B,w)C$ and $Y(Y(A,z-w)B,w)C$ represent the same element of

$$V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}] = V[[z-w,w]][z^{-1},w^{-1},(z-w)^{-1}]$$

coerced into $V((z))((w))$ and $V((w))((z-w))$, respectively, and we are done.