Lecture 10: Operator Product Expansions

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January 1, 2020

A note on complex variables

In the $d = 2$ Euclidean case, we may identify $\mathbb{R}^{(2,0)} = \mathbb{C}$ and now conformal maps are either holomorphic or antiholomorphic maps with nonvanishing derivative. It is useful to maintain the fiction that $z = x + iy$ and $\bar{z} = x - iy$ can be treated as independent variables. The formalism of the multivariable chain rule works if we define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{dx} - y\frac{\partial}{dy}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{dx} + y\frac{\partial}{dy}\right).$$

With these operators the Cauchy-Riemann equations may be conveniently expressed: f is holomorphic if $\partial f/\partial \bar{z} = 0$.

Euclidean QFT

We consider a Euclidean QFT of dimension $d = 2$. We will identify $\mathbb{R}^{(2,0)} = \mathbb{C}$. As in the Lorenzian case, operators act on a Hilbert space $\mathcal{H}$. The unitary representation U of the Lorentz group becomes a representation of the Euclidean group $SO(2) \ltimes \mathbb{C}$; in a conformal field theory this is enlarged to the conformal group $SO(3,1)$ or its double cover spin$(3,1) \cong SL(2,\mathbb{C})$. As before a field is an operator-valued distribution.

We also require an action $\phi \to {}^{\gamma}\phi$ of $SL(2,\mathbb{C})$ on fields that can be extended to the Euclidean group by letting $\mathbb{C}$ act trivially. We are interested in fields that transform by finite-dimensional subgroups of $SL(2,\mathbb{C})$ with the compatibility

$$U(\gamma)\phi(z)U(\gamma)^{-1} = {}^{\gamma}\phi(\gamma z).$$

Quasi-primary fields

A field ϕ_i is called quasi-primary if for a conformal map $w(z)$ we have

$$\phi_i(w,\overline{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h_i} \left(\frac{\partial \overline{w}}{\partial \overline{z}}\right)^{-\overline{h}_i} \phi_i(z,\overline{z})$$

where h_i and $\overline{h}_i$ are real constants. Despite the notation, $\overline{h}_i$ is not the complex conjugate of h_i. We write $\phi_i(z) = \phi_i(z,\overline{z})$ to emphasize the independence of z and $\overline{z}$.

We show now that the two-point function $\langle \phi_i(z)\phi_j(w)\rangle$ (i.e. $\langle 0|\phi_i(z)\phi_j(w)|0\rangle$ vanishes unless $h_i = h_j$ and $\overline{h}_i = \overline{h}_j$, and if this is true, denoting these h and $\overline{h}$

$$\langle \phi_i(z)\phi_j(w)\rangle = \frac{C_{ij}}{(z-w)^{2h}(\overline{z}-\overline{w})^{2\overline{h}}}.$$

Proof

First we use translation invariance to take $w = 0$. Then using invariance under the dilation $z \to \lambda z$ we have

$$\phi_i(\lambda_z) = \lambda^{-h_i - \bar{h}_i}\phi_i(z).$$

Therefore for some constant

$$\langle \phi_i(z)\phi_j(w)\rangle = \frac{C_{ij}}{(z-w)^{h_i+h_j}(\bar{z}-\bar{w})^{\bar{h_i}+\bar{h_j}}}.$$

Taking $w = 0$ and using the invariance under rotations shows that this vanishes unless $h_i = h_j$.

Reference: See Di Francesco, Mathieu and Senechal (DMS) equation (5.25).

Three point functions

Similarly it may be shown that for some constants C_{ijk}

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle =$$

$$\frac{C_{ijk}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2} \cdot \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2}}$$

where we are denoting $z_{ij} = z_i - z_j$.

There is no similar expression for the four-point functions, which carry more information than just the h_i and $\bar{h}_i$. Rather the four-point function involves an undetermined function of the cross ratios

$$\frac{z_{14}z_{23}}{z_{13}z_{24}}, \qquad \frac{z_{12}z_{34}}{z_{23}z_{14}}.$$

Operator Product Expansion

Operator Product Expansions were introduced by K. Wilson (1969) in QFT. Belavin, Polyakov and Zamolodchikov called attention to the associativity of the OPE in CFT. We expand two primary fields at distinct points z and w, and the expansion has the form

$$\phi_i(z)\phi_j(w) = \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j}(\overline{z}-\overline{w})^{\overline{h}_k-\overline{h}_i-\overline{h}_j}\phi_k(w).$$

The “associativity” refers to the fact that the expansion of $\phi_i(z)\phi_j(w)\phi_k(u)$ in terms at fields at u may be done in two different ways, leading to a relation among the C_{ijk} that is like the relations for the structure constants of an associative ring, except that it will also involve powers of $(z-w)$, etc.

OPE in Vertex Algebras

Operator Product Expansions (OPE) have turned out to be an essential technique in CFT. They are also a fundamental tool in Vertex Algebras. Our expansion from Lecture 9

$$b(z)b(w) = \frac{1}{(z-w)^2} + : b(z)b(w) :,$$

where $: b(z)b(w) :$ is holomorphic when $z \to w$. Here recall that

$$b(z) = \sum_{j \in \mathbb{Z}} b_n z^{-n-1}$$

where b_n are the Heisenberg generators. This is a kind of OPE.

Translation

We will follow [FBZ] Section 3.2.

Let us prove in a vertex algebra

$$e^{wT}Y(A,z)e^{-wT} = Y(A,z+w).$$

Indeed by the Baker-Campbell-Hausdorff formula

$$e^{wT}Y(A,z)e^{-wT} = \sum_{n=0}^{\infty} (\text{ad } wT)^n \cdot Y(A,z),$$

where $\text{ad}(X)Y = XY - YX$. Because $[T, Y(A,z)] = \partial_z Y(A,z)$, this equals

$$\sum_{n=0}^{\infty} w^n \partial_z^n Y(A,z),$$

and the statement follows from Taylor's theorem.

Associativity

Our goal is to show:

$$Y(Y(Z,z-w)B,w)C = Y(A,z)Y(B,w)C$$

in the sense that both are expansions of the same element of $\mathbb{C}[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$. This will be done in Lecture 11, following Frenkel and Ben-Zvi, Chapter 3. For the moment we note that this identity is the analog of the associativity of the OPE in the context of CFT mentioned above.