Weyl Group
Multiple Dirichet Series

Available Preprints

Weyl Group Multiple Dirichlet Series are multiple Dirichlet series involving Gauss sums. These functions and their residues unify and generalize a number of striking examples which have previously been treated individually, often with applications to analytic number theory. The following papers are available at

`http://math.stanford.edu/~bump/`

- **On Kubota's Dirichlet Series** by Brubaker and Bump.
- **Weyl Group Multiple Dirichlet Series I.** by Brubaker, Bump, Chinta, Friedberg, Hoffstein.
- **Weyl Group Multiple Dirichlet Series II: The Stable Case** by Brubaker, Bump and Friedberg.
- These slides.
- Later this year two more papers in preparation, on Eisenstein series, and on a connection with the work of Friedberg-Hoffstein-Lieman.

Prehistory: Metaplectic groups

Many multiple Dirichlet series were originally studied by applying Rankin-Selberg constructions to Eisenstein series on metaplectic groups.

$F =$ global field, $\mathbb{A} =$ adele ring
$G =$ reductive group over F.
$\mu_n = n$-th roots of unity. Assume $\mu_n \subset F$.

- May need more roots of unity if G is not simply-connected. If $G = O(k)$, n even, assume $\mu_{2n} \subset F$.

Weil, Kubota, Matsumoto constructed a cover

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G}_{\mathbb{A}} \longrightarrow G_{\mathbb{A}} \longrightarrow 1$$

The cover **splits** over G_F which means that there is a map $G_F \longrightarrow \tilde{G}_{\mathbb{A}}$ that makes the diagram commute:

$$\begin{tikzcd} & G_F \arrow[dl] \arrow[d] \\ \tilde{G}_{\mathbb{A}} \arrow[r] & G_{\mathbb{A}} \end{tikzcd}$$

So we can consider G_F as a subgroup of $\tilde{G}_{\mathbb{A}}$ and consider automorphic forms on $G_F \backslash \tilde{G}_{\mathbb{A}}$.

- Half-integral weight modular forms are naturally automorphic forms on the double cover of SL_2.
- Theta functions (Jacobi, Siegel, Weil, ...) and generalied theta functions (Kubota, Kazhdan-Patterson, ...) are residues of Eisenstein series on the metaplectic group.

Metaplectic forms and MDS

Multiple Dirichlet series (MDS) arise from metaplectic forms in two different ways.

- Whittaker coefficients of metaplectic Eisenstein series of non-maximal parabolic type are MDS.
- Rankin-Selberg constructions applied to metaplectic forms can produce important MDS.

These constructions, applied to ordinary automorphic forms typically produce Euler products. In both cases, the groups of functional equations are often Weyl groups.

- Whittaker coefficients of ordinary Eisenstein series are Dirichlet polynomials based on Schur functions (Casselman-Shalika formula).
- Rankin-Selberg constructions based on unique periods applied to nonmetaplectic forms typically give Euler products.

In many cases the **same** Rankin-Selberg constructions that give Euler products in the nonmetaplectic case give important MDS in the metaplectic case.

- The Weyl group multiple Dirichlet series that we will consider are conjecturally the same as Whittaker coefficients of metaplectic Eisenstein series.

Prehistory: $\sum L(s, f, \chi_D)|D|^{-w}$

Let f be an automorphic form on $\mathrm{GL}(r)$. One may form a Dirichlet series

$$Z(s,w) = L(s, f, \chi_D)|D|^{-w}.$$

- We may work over any field, but for simplicity, assume the ground field $F = \mathbb{Q}$.
- Here D runs through fundamental discriminants. $\chi_D(n) = \left(\frac{D}{n}\right)$ is the quadratic Dirichlet character modulo D.

The precise definition of the Dirichlet series requires more care. The form we have given is only "philosophically correct."

Eventually it became clear that:

- The function $Z(s, w)$ has meromorphic continuation to all s, w if $r \leqslant 3$. It does **not** if $r > 3$.
- The group of functional equations is a Weyl group of type A_2, B_2 or G_2 if $r = 1, 2$ or 3. These are dihedral groups of orders $6, 8$ or 12.
- The earliest attacks on these multiple Dirichlet series were based on the Rankin-Selberg method.
- The scope and limitations of the Rankin-Selberg method are unclear, so the limitation to $r \leqslant 3$ was unclear until another approach was taken.

Metaplectic Eisenstein series

The metaplectic double cover of $\mathrm{GSp}(2r,\,\mathbb{A})$ splits over the subgroup $\mathrm{GL}(r,\,\mathbb{A})$. This means that one may take the **nonmetaplectic** automorphic form f on $\mathrm{GL}(r)$ and form a **metaplectic** Eisenstein series $E_f(g,\quad s)$ on $\mathrm{GSp}(2r)$. Then one may form the Whittaker coefficient

$$\int_{U(F)\backslash U(\mathbb{A})} E_f(ug,s)\,\psi_D(u)\,du,$$

where U is the standard maximal unipotent subgroup of the and ψ_D is the character

$$\begin{pmatrix} U & UX \\ & {}^tU^{-1} \end{pmatrix} \longmapsto \sum \psi(u_{12}+u_{23}+\ldots u_{n-1,n}+Dx_{n-1,1})$$

$$U=\begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ & 1 & & u_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}, \qquad X=(x_{ij}),\ X={}^tX.$$

Here $\psi\colon \mathbb{A}/F \longrightarrow \mathbb{C}^\times$ is an additive character.

Essentially this Whittaker coefficient is $L(s,\,f,\,\chi_D)$. For $r=1$, this is a very old result of Maass. For $r>1$, this fact is due to Bump, Friedberg and Hoffstein.

The Spin L-function connection

- GSp($2r$) has an L-function, an Euler product of degree 2^r. It is conjectured to have analytic continuation and functional equation (Langlands).
- Rankin-Selberg constructions prove the functional equations when $r = 1$ (Hecke), 2 (Novodvorsky) and $3, 4, 5$ (Bump-Ginzburg).
- Rankin-Selberg constructions for the spin L-functions on GSp($2r$), applied to the metaplectic Eisenstein series potentially give $Z(s,\ w)$. The cases $r = 4, 5$ fail due to cover matching problems.
- For $r = 1$ this is just the Mellin transform (Hecke integral). Construction of $\sum L(s,\ f,\ \chi_D)|D|^{-w}$ was observed by Siegel and exploited by Goldfeld and Hoffstein to obtain mean value results for quadratic L-functions.
- For $r = 2$, the Novodvorsky construction was used by Bump, Friedberg and Hoffstein to give nonvanishing results for $L(s,\ f,\ \chi_D)$ and for $L'(s,\ f,\ \chi_D)$ when $L(s, f, \chi_D) = 0$.
- For $r = 3$, the Bump-Ginzburg construction for the spin L-function of degree 8 of GSp(6) can be applied to give $\sum L(s,\ f,\ \chi_D)|D|^{-w}$ though the details have not been carried out or published.

Other examples

- Friedberg, Hoffstein and Lieman considered multiple Dirichlet series with the philosophical form

$$Z(s,w) = \sum_D L\left(2s - \frac{1}{2}, \chi_D\right) \mathbb{N}(D)^{\frac{1}{2}-2w},$$

  where now χ_D is in terms of the n-th order symbol; these were applied to mean values of $L(s, \chi_D)$.

- Brubaker, Friedberg and Hoffstein exhibited a multiple Dirichlet series having the symmetric cube of a GL(2) cusp form as its residue. This fascinating example becomes exceedingly complex and interesting if (Brubaker) the cusp form is replaced by an Eisenstein series, related to the Weyl group E_6.

- Chinta studied an intriguing example whose group of functional equations is the Weyl group of type A_5. He used these to give mean values of zeta functions of biquadratic fields.

Clearly there is a need for a unifying principle that will explain and predict such examples.

- Whittaker coefficients of metaplectic Eisenstein series, and partial residues of such Eisenstein series, offer the hope of such a unifying principle.

The Old Approach

- Kubota, Patterson and Toshiaki Suzuki considered Fourier coefficients of Eisenstein series on metaplectic covers of GL(2). Patterson and Heath-Brown applied these to the distribution of cubic Gauss sums.

- For GL(r), Kazhdan and Patterson considered Eisenstein series on the metaplectic covers. The principal aim of there investigation was the study of the residues (generalized theta functions), but such Eisenstein series and their analogs on other groups are potentially the great unifying principal.

- The Whittaker coefficients of these Eisenstein series are multiple Dirichlet series whose groups of functional equations are Weyl groups.

- Bump, Friedberg and Hoffstein investigated the Whittaker coefficients by Plücker parametrization of the big Bruhat cell. Good results were obtained by this method, such as the nonvanishing of derivatives of quadratic twists of GL(2) L-functions, needed by Kolyvagin. Neverthess the approach has its difficulties.

Minimal Parabolic Eisenstein Series

Whittaker coefficients of non-maximal parabolic Eisenstein series are themselves multiple Dirichlet series. A difficulty is in evaluating these coefficients, which leads to difficult computations. To see the complication, consider the GL_2 and Sp_4 cases. For GL_2, the multiple Dirichlet series has the form

$$\sum_{C_1, C_2} H(C_1, C_2)\, \mathbb{N}C_1^{-s_1}\, \mathbb{N}C_2^{-s_2},$$

where $H(C_1, C_2) =$

$$\sum_{\substack{A_1, B_1 \bmod C_1 \\ A_2, B_2 \bmod C_2 \\ A_1C_2 + B_1B_2 + C_1A_2 \equiv 0 \bmod C_1C_2}} \kappa(\gamma)\, \psi\left(\frac{B_1}{C_1} + \frac{B_2}{C_2}\right),$$

where $\kappa(\gamma)$ is the symbol of Kubota and Bass-Milnor-Serre; for example if $\gcd(C_1, C_2) = 1$ then

$$\kappa(\gamma) = \left(\frac{B_1}{C_1}\right)\left(\frac{B_2}{C_2}\right)\left(\frac{C_1}{C_2}\right)^{-1}.$$

The coefficients $H(C_1,\ C_2)$ can be worked out using Gauss and Jacobi sums.

- Due to a twisted multiplicativity (described later) it is enough to describe $H(p^{k_1}, p^{k_2})$.
- Only 6 values of (k_1, k_2) give nonzero $H(p^{k_1}, p^{k_2})$, which can be expressed in terms of Gauss sums.

Yuck!

For Sp_4 the story is worse: $H(C_1, C_2)$ is worse. Denoting $C_1 = A_1$ and $C_2 = A_{12}$ for notational reasons, the double Dirichlet series is

$$\sum_{A_1} \sum_{A_{12}} H(A_1, A_{12})\, \mathbb{N}(A_1)^{-s} \mathbb{N}(A_{12})^{-w},$$

where $H(A_1, A_{12}) =$

$$\sum_{\substack{A_2, A_3, A_4 \bmod A_1 \\ A_{13}, A_{14}, A_{24}, A_{34} \bmod A_{12} \\ A_{12}A_{34} - A_{13}A_{24} - A_{14}^2 = 0 \\ \gcd(A_i) = \gcd(A_{ij}) = 1 \\ \begin{pmatrix} 0 & -A_{34} & A_{24} & A_{14} \\ A_{34} & 0 & -A_{14} & A_{13} \\ -A_{24} & A_{14} & 0 & -A_{12} \\ -A_{14} & -A_{13} & A_{12} & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = 0}} \kappa(\gamma)\, e\left(\frac{A_2}{A_1} + \frac{A_{13}}{A_{12}}\right),$$

where again $\kappa(\gamma)$ is a Kubota symbol.

- Such multiple Dirichlet series (Whittaker coefficients of metaplectic Eisenstein series) may be a unifying construction that organizes a maze of confusing examples.
- The difficulty of such sums impeded progress for 15 years.
- Fortunately, there is a good alternative approach to these important multiple Dirichlet series.

A Better Way

Given the difficulties of directly treating Weyl group multiple Dirichlet series from metaplectic forms, another approach was suggested by Bump, Friedberg and Hoffstein (1996). This may be outlined as follows.

- Given an MDS $Z(\boldsymbol{s}) = Z(s_1, \cdots, s_r)$, defined on a region $\Omega \subset \mathbb{C}^r$, define for each s_i an analytic continuation to a larger region Ω_i and a functional equation $Z(\boldsymbol{s}) = Z(\sigma_i \boldsymbol{s})$, where σ_i is an affine-linear transformation of order two and Ω_i is the convex hull of $\Omega \cup \sigma_i \Omega$.

- The i-th component of $\sigma_i \boldsymbol{s}$ is $1 - s_i$ though the other components depend on the example.

- Check that the linear maps σ_i generate a **finite** reflection group.

- Then an argument based on Hartogs' theorem shows $Z(\boldsymbol{s})$ has analytic continuation to all $\mathbb{C}^r$.

- The argument shows at last $\sum L(s, f, \chi_D)|D|^{-w}$ has continuation when f is an automorphic form on $\mathrm{GL}(r)$ and $r \leqslant 3$ but **not** when $r > 3$.

- A problem remains of describing the “correction factors.” Until this year, this was always handled on a case-by-case basis.

Root Systems

If $0 \neq \alpha \in \mathbb{R}^r$ let $\sigma_\alpha \colon \mathbb{R}^r \longrightarrow \mathbb{R}^r$ be the reflection in the hyperplane through 0 perpendicular to α. A finite subset Φ of $\mathbb{R}^r - \{0\}$ is a **root system** if

- when $\alpha \in \Phi$ we have $\sigma_\alpha(\Phi) = \Phi$;
- when $\alpha, \beta \in \Phi$, $\beta - \sigma_\alpha(\beta)$ is an integer multiple of α.

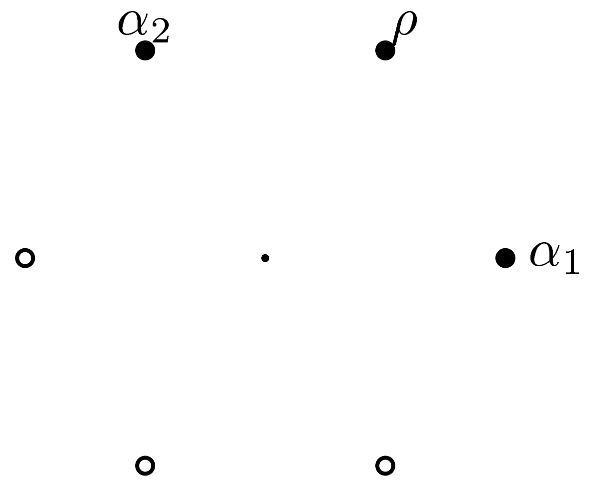
The root system is called **reduced** if α and 2α are not both in Φ, and **simply-laced** if all roots have the same length. It is **irreducible** if it cannot be decomposed as a union of two smaller root systems.

- The root system Φ may be partitioned into **positive and negative roots**, denoted Φ^+ and Φ^-.
- If $\alpha \in \Phi^+$ cannot be decomposed into a sum of other positive roots it is called **simple**. Let Σ be the set of simple positive roots.
- The group generated by the σ_α $(\alpha \in \Phi)$ is called the **Weyl Group**. It is generated by σ_α $(\alpha \in \Sigma)$.
- A special role is played by

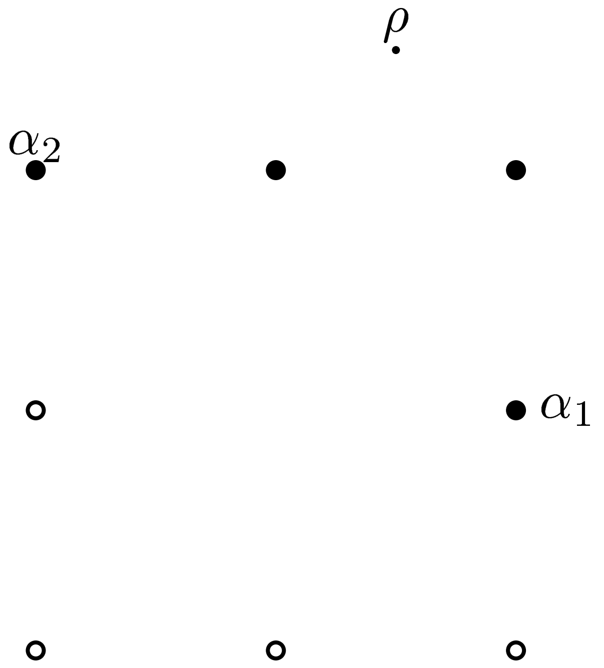
$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

Examples of Root Systems

The reduced irreducible root systems are A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7 and E_8 in Cartan's classification.



Above: $\boldsymbol{A_2}$; Below: $\boldsymbol{C_2}$. $(\Phi^+ = \bullet, \Phi^- = \circ)$.



The Weyl groups are dihedral of orders 6 and 8.

Weyl groups and MDS

Let Φ be a root system. The group generated by the σ_α with $\alpha \in \Phi$ is also generated by the σ_α where α is a simple positive root. This is the Weyl group $W(\Phi)$.

- A root system and Weyl group can be associated with any reductive algebraic group.
- The functional equations of Eisenstein series are Weyl groups.
- Thus Whittaker coefficients of metaplectic Eisenstein series are multiple Dirichlet series whose functional equations are Weyl groups.
- However they have the defect of being difficult to compute.
- After the 1996 paper of Bump, Friedberg and Hoffstein, a gradual realization dawned that one could associate an MDS with Φ **directly** thus avoiding the need to work with the metaplectic group.
- Now there is a completely satisfactory theory when n is large enough.

Data for Weyl Group MDS

- An integer $n > 1$.
- A reduced root system Φ in $\mathbb{R}^r$.
- A global field $F \supset \mu_n$, the group of n-th roots of unity; also assume -1 is an n-th power. These assumptions imply F is totally complex.
- A sufficiently large finite set S of places of F containing the archimedean ones and all those ramified over $\mathbb{Q}$ (including those dividing n). Write $S = S_\infty \cup S_{\text{fin}}$ where S_∞ and S_{fin} are the archimedean and nonarchimedan places in S.
- Additional data Ψ and Ω to be described later.

Let $F_S = \prod_{v \in S} F_v$. Embed the ring of ***S*-integers**

$$\mathfrak{o}_S = \{a \in F \,|\, a_v \in \mathfrak{o}_v \text{ for all } v \notin S\}$$

in F_S. It is discrete and cocompact. We assume that S is large enough that **$\mathfrak{o}_S$ is a principal ideal domain**.

- Let $\psi\colon F_S \longrightarrow \mathbb{C}$ be an additive character whose conductor is precisely $\mathfrak{o}_S$; thus if $\alpha \in F_S$

  $$\psi(\alpha \mathfrak{o}_S) = 1 \quad \Longleftrightarrow \quad \alpha \in \mathfrak{o}_S.$$

  It may be arranged that if $v \in S_{\text{fin}}$ then the restriction ψ_v to $F_v = \mathbb{C}$ is $z \longmapsto e^{4\pi i \, \mathrm{re}(z)}$.

Reciprocity Law

The **Hilbert symbol** is a skew-symmetric bilinear pairing $F_S^\times \times F_S^\times \longrightarrow \mu_n$. It is denoted $(\alpha, \beta) \in \mu_n$ for α, $\beta \in F_S^\times$, and can be written $\prod_{v \in S} (\alpha_v, \beta_v)_v$ in terms of "local symbols" defined in the local class field theory.

We will say that $\Omega \subset F_S^\times$ is **isotropic** if $(\alpha, \beta) = 1$ for α, $\beta \in F_S$. It follows from the Hilbert reciprocity law that

$$\Omega_0 = \mathfrak{o}_S^\times (F_S^\times)^n$$

is isotropic. Let Ω be an isotropic subgroup of $F_S^\times$ containing Ω_0. It is of finite index in $F_S^\times$.

The **power residue symbol** $\left(\frac{\alpha}{\beta}\right)$ is defined for α, $\beta \in \mathfrak{o}_S$, $\beta \neq 0$. It depends only on the ideal $\mathfrak{b}$ generated by β. If $\beta = \pi$ is prime, then

$$\left(\frac{a}{\pi}\right) \equiv a^{(\mathbb{N}\pi - 1)/n} \bmod \pi$$

where $\mathbb{N}\colon \mathfrak{o}_S \longrightarrow \mathbb{Z}$ is the norm. The **reciprocity law** (Gauss, Eisenstein, Jacobi, Kummer, Hilbert)

$$\left(\frac{\alpha}{\beta}\right) = (\beta, \alpha)\left(\frac{\beta}{\alpha}\right), \qquad 0 \neq \alpha, \beta \in \mathfrak{o}_S.$$

“Philosophical” WMD’s

Let

$$\Sigma = \{\alpha_1, \cdots, \alpha_r\}$$

be the set of simple positive roots, and denote $\sigma_i = \sigma_{\alpha_i}$. Then W has a presentation consisting of the relations

$$\sigma_i^2 = 1, \qquad (\sigma_i \sigma_j)^{r(\alpha_i, \alpha_j)} = 1,$$

where

$$r(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha, \beta \text{ are orthogonal.} \\ 1 & \text{if } \theta = \frac{2\pi}{3}; \\ 2 & \text{if } \theta = \frac{3\pi}{4}; \\ 3 & \text{if } \theta = \frac{5\pi}{6}. \end{cases}$$

The rough shape of the WMD is $Z(\boldsymbol{s}) =$

$$\sum_{\substack{c_\alpha \in \mathfrak{o}_S / \mathfrak{o}_S^\times \\ (\alpha \in \Sigma)}} g_\alpha(c_\alpha) \left[\prod_{\alpha, \beta} \left(\frac{c_\alpha}{c_\beta} \right)^{-r(\alpha, \beta)} \right] \prod_{\alpha \in \Sigma} \mathbb{N}(c_\alpha)^{-2s_\alpha}.$$

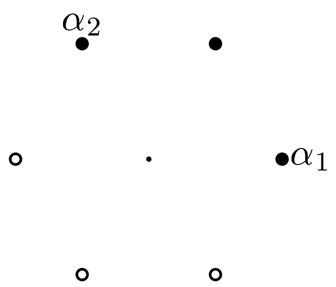
where the product is over pairs of simple roots α and β. There is one s_α variable for each simple root. (-) is the power residue symbol and the Gauss sum

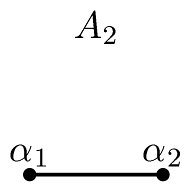
$$g_\alpha(m) = \begin{cases} g_1(1, m) & \text{if } \alpha \text{ is a long root,} \\ g_2(1, m) & \text{if } \alpha \text{ is a short root and } \Phi \neq G_2; \\ g_3(1, m) & \text{if } \alpha \text{ is a short root and } \Phi = G_2., \end{cases}$$

Dynkin Diagrams

If Φ is a root system, we associate with it a graph whose vertices are the simple roots.

- Two nodes are adjacent if the corresponding roots are not orthogonal.
- If the roots make an angle of $\frac{2\pi}{3}$ they are joined by a single bond.
- If they make an angle of $\frac{3\pi}{4}$ or $\frac{5\pi}{6}$ they are joined by double or triple bonds (with an arrow pointing from the long to the short root).
- If only single bonds occur the diagram is called **simply-laced.**





If the diagram is simply-laced the philosophical form of the WMD can be expressed as follows:

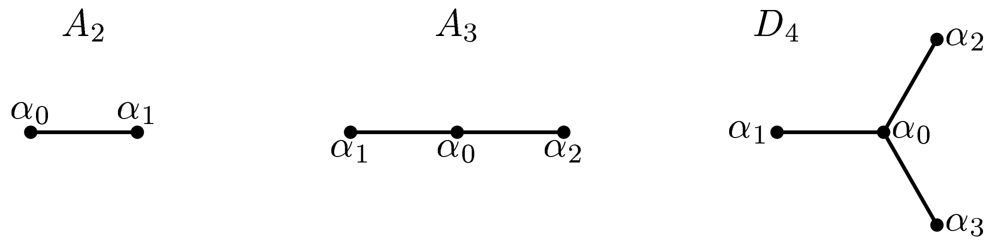
$$\sum_{\substack{c_\alpha \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (\alpha \in \Sigma)}} g_\alpha(c_\alpha) \left[\prod_{\alpha,\beta \text{ adjacent}} \left(\frac{c_\alpha}{c_\beta} \right)^{-1} \right] \prod_{\alpha \in \Sigma} \mathbb{N}(c_\alpha)^{-2s_\alpha}.$$

Some $n=2$ Examples

If $n=2$ there is little harm in replacing the Gauss sum $g_\alpha(c_\alpha)$ by $\sqrt{\mathbb{N}(c_\alpha)}$. Thus we get

$$\sum_{\substack{c_\alpha \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (\alpha\in\Sigma)}} \left[\prod_{\alpha,\beta \text{ adjacent}} \left(\frac{c_\alpha}{c_\beta}\right)^{-1} \right] \prod_{\alpha\in\Sigma} \mathbb{N}(c_\alpha)^{\frac{1}{2}-2s_\alpha}.$$

In this form, the three Dynkin diagrams A_2, A_3 and D_4 are "star-shaped" with a central node α_0.



Let χ_D be the Hecke character which maps the principal ideal generated by α to $\left(\frac{D}{\alpha}\right)$. Writing $D=c_{\alpha_0}$ and collecting the c_{α_0} coefficient, $Z(s)=$

$$\sum_D L\left(2s_1-\frac{1}{2},\chi_D\right)\cdots L\left(2s_r-\frac{1}{2},\chi_D\right)\mathbb{N}(D)^{\frac{1}{2}-2s_0}.$$

- One may replace the product of L-functions by $L(s,f,\chi_D)$ with f a cusp form on $\mathrm{GL}(r)$, $r\leqslant 3$.
- There are no other star-shaped Dynkin diagrams.
- A striking non-star-shaped example with an application to L-functions is A_5 (Chinta).

Philosophy 101

In the years following 1996, the philosophical WMD was surmised by Bump when $n = 2$, and when $n > 2$ by the group at Brown, consisting of Brubaker, Chinta, Friedberg and Hoffstein.

- The philosophical form is not sufficient basis for a complete theory, but is very suggestive.
- If we write $Z(\boldsymbol{s})$ in the form

  $$\sum_{c_{\alpha_i},\ \alpha_i \text{ simple}} H(c_{\alpha_1}, \cdots, c_{\alpha_r}) \prod_{\alpha_i \text{ simple}} \mathbb{N}(c_{\alpha_i})^{-s}$$

  we have only specified the coefficients when the c_{α_i} are coprime.
- The pre-2005 correction polynomial technology could only suggest a case-by-case handling of the problem of making this description correct.
- Still the philosophical form is enough to predict the functional equations and multiplicativity of the coefficients H.
- It is a step towards the fully satisfactory solution in the preprint of Brubaker, Bump and Friedberg (2005).

Overview

Given the data we define a **Weyl Group MDS**

$$Z_\Psi(s_1, \cdots, s_r).$$

- Z_Ψ has analytic continuation to all $s = (s_1, \cdots, s_r)$ and has W as its group of functional equations.
- The coefficients of the Weyl group multiple Dirichlet series are multiplicative, but the multiplicativity is **twisted**, so the Dirichlet series is not an Euler product.
- Due to the multiplicativity, description of the coefficients reduces to the case where the parameters are powers of a single prime p; the p-part coefficients form a Dirichlet polynomial.
- In the "stable case" where n is sufficiently large (depending on Φ), the number of nonzero coefficients in the p-part is equal to the order of the Weyl group. Indeed, these nonzero coefficients are parametrized in a natural way by the Weyl group elements.
- The p-part coefficient parametrized by a Weyl group element w is a product of $l(w)$ Gauss sums, where l is the well-known "length function" on the Weyl group.

The Nature of Z_Ψ

The function $Z_\Psi(s_1, \cdots, s_r)$ is of the form

$$\sum_{\substack{0 \neq c_1 \in \mathfrak{o}_S/(\mathfrak{o}_S^\times) \\ \vdots \\ 0 \neq c_r \in \mathfrak{o}_S/(\mathfrak{o}_S^\times)}} \Psi(c_1, \cdots, c_r) H(c_1, \cdots, c_r) \prod_{i=1}^{r} \mathbb{N}c_i^{-2s_i}.$$

- Since $\mathfrak{o}_S$ is a principal ideal domain the sum is actually over **ideals** $c_i \mathfrak{o}_S$.
- The function H is defined only for $c_i \in \mathfrak{o}_S$. It is important and combinatorially interesting.
- The function Ψ is defined on all of $(F_S^\times)^r$. It is less important.

Specification of the function H requires two points.

- Describe the multiplicativity of H;
- Describe the p-part $H(p^{k_1}, \cdots, p^{k_r})$ when p is prime.
- The p-part shows a beautiful combinatorial structure that we will describe fully if n is sufficiently large. This is the **stable case**.
- The simplest root system is A_1. The corresponding Dirichlet series were studied by Kubota; they play a role in the proofs of the general case so we treat them first.

Kubota Dirichlet Series

If m, $c \in \mathfrak{o}_S$ are nonzero and $t \in \mathbb{Z}$ define the **Gauss sum**

$$g_t(m,c) = \sum_{d \bmod c} \left(\frac{d}{c}\right)^t \psi\left(\frac{md}{c}\right).$$

If $t = 1$ we write $g_t(m, c) = g(m, c)$. Kubota defined Dirichlet series formed with Gauss sums and proved their functional equations using Eisenstein series on the higher metaplectic covers of SL_2, which he invented.

Kubota Dirichlet series were applied, for example, by Patterson and Heath-Brown to prove the equidistribution of cubic Gauss sums, settling Kummer's conjecture.

We describe a refinement of Kubota's functional equations by Brubaker and Bump that is required for the construction of Weyl group multiple Dirichlet series. Let $\mathcal{M}(\Omega)$ be the vector space of functions Ψ on $F_{\text{fin}}^\times$ that satisfy

$$\Psi(\varepsilon c) = (c, \varepsilon)^{-1} \Psi(c),$$

when $\varepsilon \in \Omega$. If $\Psi \in \mathcal{M}(\Omega)$ and $\eta \in F_S^\times$ denote

$$\tilde{\Psi}_\eta(c) = (\eta, c) \Psi(c^{-1} \eta^{-1}).$$

Then $\tilde{\Psi}_\eta \in \mathcal{M}(\Omega)$. It depends only on η in $F_S^\times / (F_S^\times)^n$.

Kubota Functional Equations

The **Kubota Dirichlet Series** is

$$\mathcal{D}(s,\Psi,\alpha)=\sum_{0\neq c\in\mathfrak{o}_S/\mathfrak{o}_S^\times} g(\alpha,c)\,\Psi(c)\,\mathbb{N}(c)^{-2s}\,.$$

Normalized:

$$\mathcal{D}^*(s,\Psi,\alpha)=\boldsymbol{G}_n(s)^{\frac{1}{2}[F:\mathbb{Q}]}\zeta_F(2ns-n+1)\mathcal{D}(s,\Psi,\alpha),$$

$$\boldsymbol{G}_n(s)=(2\pi)^{-(n-1)(2s-1)}\frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

where ζ_F is the Dedekind zeta function of F. By an ***S*-Dirichlet polynomial** we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}.

Theorem. (Brubaker and Bump) *Let $\Psi\in\mathcal{M}(\Omega)$, and let $\alpha\in\mathfrak{o}_S$. Then $\mathcal{D}^*(s,\Psi,\alpha)$ has meromorphic continuation to all s, analytic except possibly at $s=\frac{1}{2}\pm\frac{1}{2n}$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta(s)$ that depend only on the image of η in $F_S^\times/(F_S^\times)^n$ such that*

$$\mathcal{D}^*(s,\Psi,\alpha)=\mathbb{N}(\alpha)^{1-2s}\sum_{\eta\in F_S^\times/(F_S^\times)^n}P_{\alpha\eta}(s)\mathcal{D}^*(1-s,\tilde{\Psi}_\eta,\alpha).$$

This is a refinement of Kubota's results. **The simplicity of the α-dependence is very important, as is the fact that P_η is independent of Ψ.**

Proof

The Brubaker-Bump theorem on Kubota Dirichlet series is the sole metaplectic vestige of in the WMD papers. We sketch the proof.

Let $G = \mathrm{SL}_2$. A map $X: G(F_S) \longrightarrow F_S^{\times}$ is defined by

$$X\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0; \\ d & \text{otherwise.} \end{cases}$$

Let $\sigma: G(F_S) \times G(F_S) \longrightarrow \mu_n$ be the map

$$\sigma(g_1, g_2) = \left(\frac{X(g_1 g_2)}{X(g_1)}, \frac{X(g_1 g_2)}{X(g_2)} \right).$$

Theorem 1. (Kubota) *We have*

$$\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3).$$

Thus σ is a cocycle defining a cohomology class in $H^2(G(F_S), \mu_n)$.

The metaplectic double cover $\tilde{G}(F_S) = \widetilde{\mathrm{SL}}_2(F_S)$ consists of ordered pairs $[a, \zeta]$ with $a \in G(F_S)$ and $\zeta \in \mu_n$, with group law

$$[a, \zeta][b, \zeta'] = [a\,b, \zeta\zeta'\sigma(a, b)].$$

We will denote the standard section by $\boldsymbol{i}$: $G(F_S) \longrightarrow \tilde{G}(F_S)$, so $\boldsymbol{i}(g) = [g, 1]$.

Kubota Symbol, Eisenstein Series

Theorem 2. *Define* $\kappa\colon \mathrm{SL}_2(\mathfrak{o}_S) \longrightarrow \mu_n$ *by*

$$\kappa\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{c}\right) & \textit{if } c \neq 0; \\ 1 & \textit{if } c = 0. \end{cases}$$

Then

$$\kappa(\gamma\gamma') = \sigma(\gamma, \gamma')\kappa(\gamma)\kappa(\gamma').$$

Let $I(s)$ be the space of K-finite $f\colon \tilde{G}_2(F_S) \longrightarrow \mathbb{C}$ with

$$f\left(\left[\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix}, \zeta\right]\tilde{g}\right) = |a|^{2s}\zeta f(\tilde{g}), a \in \Omega, x \in F_S, \zeta \in \mu_n.$$

Let $f_s \in I(s)$. The Eisenstein series is

$$E(\tilde{g}, s, f_s) = \sum_{\gamma \in B(\mathfrak{o}_S)\backslash\Gamma} \kappa(\gamma) f_s(\boldsymbol{i}(\gamma)\tilde{g}), \qquad \tilde{g} \in \widetilde{\mathrm{SL}}(2, F_S),$$

where

$$B(\mathfrak{o}_S) = \left\{\begin{pmatrix} a & u \\ & a^{-1} \end{pmatrix} \middle| a \in \mathfrak{o}_S^\times, u \in \mathfrak{o}_S\right\}.$$

- Converges if $\mathrm{re}(s) > 1$, continues to $s \neq \frac{1}{2} \pm \frac{1}{2n}$.
- Fourier coefficients are Kubota DS.

If $\Psi \in \Omega$ and $\mathfrak{a}$ is a large ideal let $f_s^{\mathfrak{a}} = f_{s,\infty}^{\circ} \otimes f_{s,\mathrm{fin}}^{\mathfrak{a}}$, where $f_{s,\infty}^{\circ}$ is the $K = \prod_{v \in S_\infty}$ SU(2)-fixed vector and

$$f_{s,\mathrm{fin}}^{\mathfrak{a}}\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta\right]\right) = \begin{cases} \mathrm{vol}(\mathfrak{a})^{-1}\, \zeta\, |c|_{\mathrm{fin}}^{-2s}\, \Psi(c) & \text{if } d/c \in \mathfrak{a}, \\ 0 & \text{otherwise.} \end{cases}$$

- Thus at the nonarchimedean places we choose f_s to have small support in the big Bruhat cell.
- We've built Ψ into the definition, so the α-th Fourier coefficient involves

$$\int_{F_S/\mathfrak{o}_S} E^*\left(\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s, f_s^{\mathfrak{a}}\right) \psi(\alpha x)\, dx = C\, \boldsymbol{G}(s)^r\, \zeta_S(2n\, s - n + 1)$$
$$\times\, \mathcal{D}(s, \Psi, \alpha)\, |D|^{\frac{1}{2}-s} |\alpha|_\infty^{s-1/2} K_{2s-1}(4\pi\sqrt{|\alpha_v|_v}).$$

- The proof uses a "local functional equation" proved using a trick from the thesis of Banks.
- The polynomials

$$p_\eta(s) = \int_{y \in \mathfrak{M} \cap \eta(F_S^\times)^n} |y|_{\mathrm{fin}}^{2s-2} \psi_{\mathrm{fin}}(y)\, dy,$$

  where $\mathfrak{M}$ is a certain fractional ideal.

Multiplicativity: An example

Consider the A_2 root system. If $\gcd(c_1c_2, c_1'c_2') = 1$ then

$$H(c_1c_1', c_2c_2') = H(c_1,c_2)H(c_1',c_2')$$
$$\times \left(\frac{c_1}{c_1'}\right)\left(\frac{c_1'}{c_1}\right)\left(\frac{c_2}{c_2'}\right)\left(\frac{c_2'}{c_2}\right)\left(\frac{c_1}{c_2'}\right)^{-1}\left(\frac{c_1'}{c_2}\right)^{-1}$$

- Due to the "twisting" by power residue symbols the Weyl group multiple Dirichlet Series is not an Euler product.
- The multiplicativity is sufficient to reduce specification of H to the case c_1 and c_2 are prime powers. We call $H(p^{k_1}, p^{k_2})$ the ***p*-part**.

The function Ψ on $(F_S^\times)^2$ is drawn from the finite dimensional space $\mathcal{M}(\Omega^2)$ of functions satisfying

$$\Psi(\varepsilon_1 c_1, \varepsilon_2 c_2) = (\varepsilon_1, c_1)(\varepsilon_1^{-1}\varepsilon_2, c_2)\,\Psi(c_1, c_2)$$

when ε_1 and ε_2 are in Ω.

- See **Weyl Group Multiple Dirichlet Series II** for the general case of these definitions.
- Though we only describe the multiplicativity of H in a special case we will describe the p-part in full generality.

Stability

Our aim is to describe the beautiful combinatorial structure of the the function $H(p^{k_1}, \cdots, p^{k_r})$.

- The description we will give is only correct if n is sufficiently large. We describe this assumption as **stability**.
- If Φ is simply-laced, let α be the longest positive root, and write $\alpha = \sum_i d_i \alpha_i$ where, we recall, the α_i are the simple positive roots. In this case, if $n \geqslant \sum_i d_i$ then we are in the stable case.
- If Φ is not simply-laced, we omit the condition for stability.

Let $\mathrm{Supp}(H) = \{(k_1, \cdots, k_r) \mid H(p^{k_1}, \cdots, p^{k_r}) \neq 0\}$. This is the **support** of H.

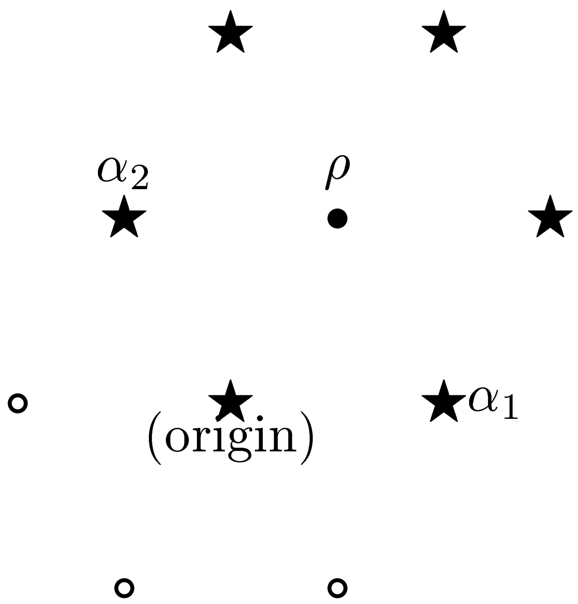
- The support of H is finite. In the stable case, its cardinality is the order $|W|$ of the Weyl group.
- We have $(k_1, \cdots, k_r) \in \mathrm{Supp}(H)$ if and only if

  $$\sum k_i \alpha_i = \rho - w(\rho)$$

  for some $w \in W$ (recall $\alpha_i =$ the simple roots).
- In this case $H(p^{k_1}, \cdots, p^{k_r})$ is a product of $l(w)$ Gauss sums (l is the **length function** on W).

Example: A_2

The points $\rho - w(\rho)$ are marked by stars:



Here,re the nonzero values of $H(p^{k_1}, p^{k_2})$:

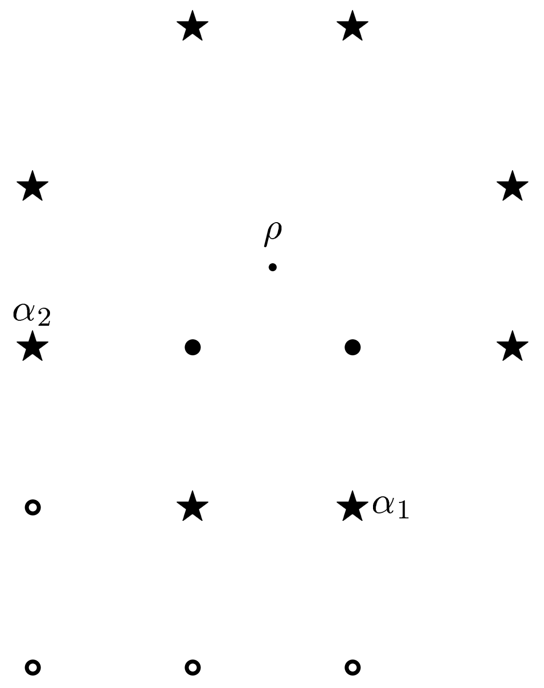
(k_1, k_2)	$H(p^{k_1}, p^{k_2})$
$(0, 0)$	1
$(1, 0)$	$g_1(1, p)$
$(0, 1)$	$g_1(1, p)$
$(1, 2)$	$g_1(1, p)\, g_1(p, p^2)$
$(2, 1)$	$g_1(1, p)\, g_1(p, p^2)$
$(2, 2)$	$g_1(1, p)^2\, g_1(p, p^2)$

(1)

Thus

$$\mathrm{supp}(H) = \{(0,0), (1,0), (0,1), (1,2), (2,1), (2,2)\} = \{\, (k_1, k_2) \mid k_1\alpha_1 + k_2\alpha_2 = \rho - w(\rho) \text{ for some } w \in W\}.$$

A Not-Simply-Laced Example: C_2



Again $\rho - w(\rho) = k_1\rho_1 + k_2\rho_2$ when $(k_1, k_2) \in \mathrm{Supp}(H)$.

$k_2 \backslash k_1$	0	1	2	3
0	1	$g_2(1,p)$		
1	$g_1(1,p)$		$g_1(1,p)$ $g_2(p,p^2)$	
2				
3		$g_2(1,p)$ $g_1(p^2,p^3)$		$g_2(1,p)\,g_2(p,p^2)$ $g_1(p^2,p^3)$
4			$g_1(1,p)$ $g_2(p,p^2)$ $g_1(p^2,p^3)$	$g_2(1,p)\,g_1(1,p)$ $g_2(p,p^2)\,g_1(p^2,p^3)$

The General Case

We describe $H(p^{k_1}, \cdots, p^{k_r})$ in the stable case. Let

$$r(\alpha,\beta)=\begin{cases} 0 & \text{if } \alpha,\beta \text{ are orthogonal;} \\ 1 & \text{if } \theta=\frac{2\pi}{3}; \\ 2 & \text{if } \theta=\frac{3\pi}{4}; \\ 3 & \text{if } \theta=\frac{5\pi}{6}. \end{cases}$$

These appear in the **braid relation** $(\sigma_\alpha\sigma_\beta)^{r(\alpha,\beta)} = 1$. Let

$$g_\alpha(m)=\begin{cases} g_1(1,m) & \text{if } \alpha \text{ is a long root,} \\ g_2(1,m) & \text{if } \alpha \text{ is a short root and } \Phi\neq G_2; \\ g_3(1,m) & \text{if } \alpha \text{ is a short root and } \Phi=G_2., \end{cases}$$

$$d(\alpha)=\sum_i c_i, \text{ where } \alpha=\sum_{i=1}^r c_i\alpha.$$

Then

$$H(p^{k_1}, \cdots, p^{k_r}) = \prod_{\substack{\alpha\in\Phi^+ \\ w(\alpha)\in\Phi^-}} g_\alpha(p^{d(\alpha)-1}, p^{d(\alpha)}),$$

$$\sum_{i=1}^r k_i\alpha_i = \rho - w(\rho).$$

- The number of Gauss sums in this product is the number of simple reflections σ_i in a minimal decomposition of w – this is the **length** $l(w)$.

The Functional Equations

The normalizing factor involves both zeta functions and Gamma functions. We describe the zeta functions only. For each simple positive root α decompose

$$\alpha = \sum_{i=1}^{r} k_i \alpha_i$$

where α_i are the simple roots. Let $\zeta_\alpha(s_1, \cdots, s_r)$ be:

$$\zeta\Big(1 + 2n \textstyle\sum_{\alpha_i \text{ long}} k_i(s_i - \frac{1}{2}) + n \sum_{\alpha_i \text{ short}} k_i(s_i - \frac{1}{2})\Big)$$

if n is even and $\Phi \neq G_2$;

$$\zeta\Big(1 + 2n \textstyle\sum_{\alpha_i \text{ long}} k_i(s_i - \frac{1}{2}) + \frac{2n}{3} \sum_{\alpha_i \text{ short}} k_i(s_i - \frac{1}{2})\Big)$$

if $3|n$ and $\Phi = G_2$;

$$\zeta\Big(1 + 2n \textstyle\sum_{\text{all } \alpha_i} k_i(s_i - \frac{1}{2})\Big) \qquad \text{otherwise.}$$

The product of the ζ_α (with some Gamma functions) is the normalizing factor. We will denote the normalized multiple Dirichlet series by Z_Ψ^*.

Theorem. (Brubaker, Bump, Friedberg) Z_Ψ^* *has meromorphic continuation to all* $s_1, \cdots, s_r$ *and a group of functional equations isomorphic to the Weyl group of* Φ*, acting on* $s_1, \cdots, s_r$ *and* Ψ*.*

Example: A_2

We recall that Ψ is drawn from the finite-dimensional vector space of functions $\mathcal{M}(\Omega)$ satisfying

$$\Psi(\varepsilon_1 c_1, \varepsilon_2 c_2) = (\varepsilon_1, c_1)(\varepsilon_1^{-1}\varepsilon_2, c_2)\,\Psi(c_1, c_2).$$

Let $\mathfrak{M}$ be the space of

$$\Psi_s(c_1, c_2), \qquad s = (s_1, s_2) \in \mathbb{C}^2,\ c_1, c_2 \in F_S^\times$$

such that for all $(s_1, s_2) \in \mathbb{C}^2$ the function

$$(c_1, c_2) \longmapsto \Psi_s(c_1, c_2)$$

is in $\mathcal{M}(\Omega^2)$, while for all $(c_1, c_2) \in (F_S^\times)^2$, the function

$$(s_1, s_2) \longmapsto \Psi_s(c_1, c_2)$$

is a polynomial in $q_v^{\pm 2s_1}$, $q_v^{\pm 2s_2}$ where v runs through the finite set S_{fin}. Let P_η be as in the Kubota Dirichlet series functional equations. Define

$$\begin{aligned}(\sigma_1\Psi_s)(c_1, c_2) = \sum_{\eta \in F_S^\times/(F_S^\times)^n} & (\eta, c_1 c_2^{-1})\, P_{c_2 c_1^{-2}\eta}(s_1) \\ & \times \Psi\left(1 - s_1, s_1 + s_2 - \frac{1}{2}, \eta^{-1}c_1, c_2\right),\end{aligned}$$

$$\begin{aligned}(\sigma_2\Psi_s)(c_1, c_2) = \sum_{\eta \in F_S^\times/(F_S^\times)^n} & (\eta, c_2)\, P_{c_1 c_2^{-2}\eta}(s_2) \\ & \times \Psi\left(s_1 + s_2 - \frac{1}{2}, 1 - s_2, c_1, \eta^{-1}c_2\right).\end{aligned}$$

Proposition. *If* $\Psi \in \mathfrak{M}$*, so are* $\sigma_1\Psi$ *and* $\sigma_2\Psi$*.*

Let $\mathfrak{W}$ be the group of automorphisms of $\mathfrak{M}$ generated by σ_1 and σ_2. We have used the same notation σ_1 and σ_2 to denote the simple reflections generating W; this reflects a homomorphism $\mathfrak{W} \longrightarrow W$.

The operators σ_1 and σ_2 on $\mathfrak{M}$ correspond to two transformations of $\mathbb{C}^2$, also denoted σ_1 and σ_2:

$$\sigma_1 = \begin{cases} s_1 \longmapsto 1 - s_1, \\ s_2 \longmapsto s_1 + s_2 - \frac{1}{2} \end{cases}$$

and

$$\sigma_2 = \begin{cases} s_1 \longmapsto s_1 + s_2 - \frac{1}{2}, \\ s_2 \longmapsto 1 - s_2. \end{cases}$$

These transformations of $\mathbb{C}^2$ satisfy the braid relation $(\sigma_1\sigma_2)^3 = 1$, hence generate a group of transformations isomorphic to the Weyl group W. We have, correspondingly, a homomorphism $\mathfrak{W} \longrightarrow W$. We recall

$$Z_\Psi(s_1, s_2) = \sum_{\substack{0 \neq c_1 \in \mathfrak{o}_S / \mathfrak{o}_S^\times \\ 0 \neq c_2 \in \mathfrak{o}_S / \mathfrak{o}_S^\times}} \Psi(c_1, c_2) H(c_1, c_2) \mathbb{N}(c_1)^{-2s_1} \mathbb{N}(c_1)^{-2s_1}.$$

This is convergent if $\mathrm{re}(s_1), \mathrm{re}(s_2) > \frac{3}{4}$.

Theorem. *Let $\Psi_s \in \mathfrak{M}$. The function $Z_{\Psi_s}(s_1,\ s_2)$ has meromorphic continuation to all s_1 and s_2; it is analytic except where s_1, s_2 or $\frac{3}{2} - s_1 - s_2$ equals $\frac{1}{2} \pm \frac{1}{2n}$, and it satisfies*

$$Z^*_{\sigma\Psi_s}(\sigma s) = Z^*_{\Psi_s}(s). \tag{2}$$

for all $\sigma \in \mathfrak{W}$.

Proof (sketch): If c_1 is cubefree and

$$(c_1, c_2) = (\gamma_1 p^{k_1},\ \gamma_2 p^{k_2}),$$

where $p \nmid \gamma_i$ and $k_1 \leqslant 2$ we say c_2 is **c_1-reduced at p** if (k_1, k_2) occur in the following table:

k_1	k_2
0	0
1	0
2	1

We say that c_2 is **c_1-reduced** if it is c_1-reduced at p for all p. Here is the key Lemma:

Lemma. *Suppose that c_1 is cube-free and that c_2 is c_1-reduced. Then $c_0 = c_1 c_2^{-2} \in \mathfrak{o}_S$ and for every $\alpha \in \mathfrak{o}_S$ we have*

$$\frac{H(c_1, \alpha c_2)}{H(c_1, c_2)}(\alpha, c_2) = g(c_0, \alpha).$$

(The proof is an exercise in careful bookkeeping.)

The nonzero terms are of the form

$$H(c_1, dc_2)\Psi(c_1, dc_2)\mathbb{N}(c_1)^{-2s_1}\mathbb{N}(dc_2)^{-2s_2},$$

where c_1 is cube-free and c_2 is c_1-reduced. Collecting these together and using the Lemma to produce a Kubota Dirichlet series, we obtain

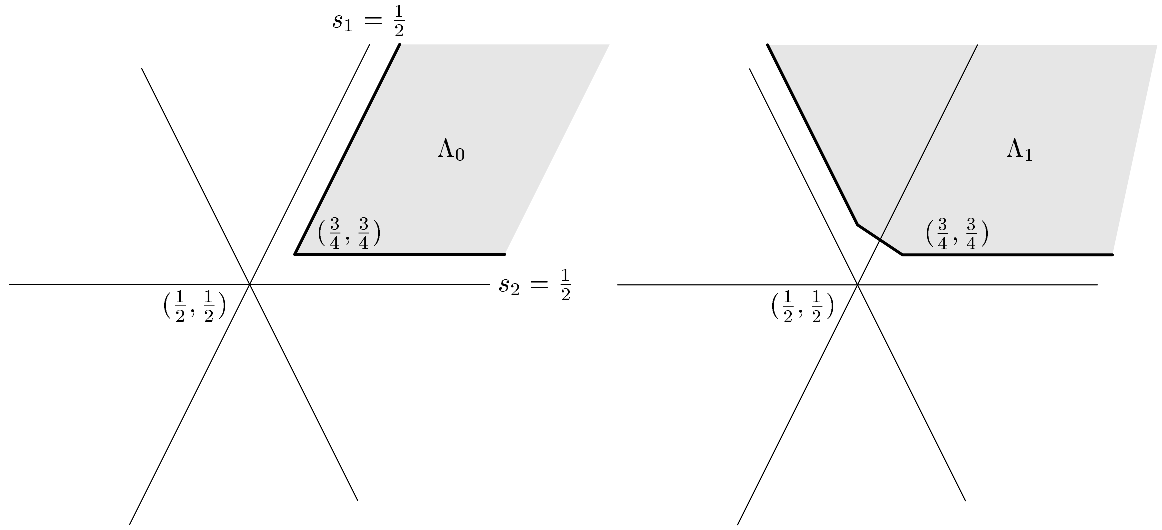
$$Z_\Psi(s_1, s_2) =$$

$$\sum_{\substack{0 \neq c_1, c_2 \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S \\ c_1 \text{ cube-free} \\ c_2\ c_1\text{-reduced}}} \mathbb{N}c_1^{-2s_1}\mathbb{N}c_2^{-2s_2} H(c_1, c_2)\mathcal{D}(s_2, \Psi^{c_1,c_2}, c_1 c_2^{-2}),$$

where given c_1 and c_2, $\Psi^{c_1,c_2} \in \mathcal{M}(\Omega)$ is

$$\Psi^{c_1,c_2}(\alpha) = \Psi(c_1, \alpha c_2)(\alpha, c_2)^{-1}.$$

The Kubota functional equation now applies.



We have analytic continuation to a bigger region Λ_1 and a functional equation for σ_2. The functional equation for σ_1 is similar.

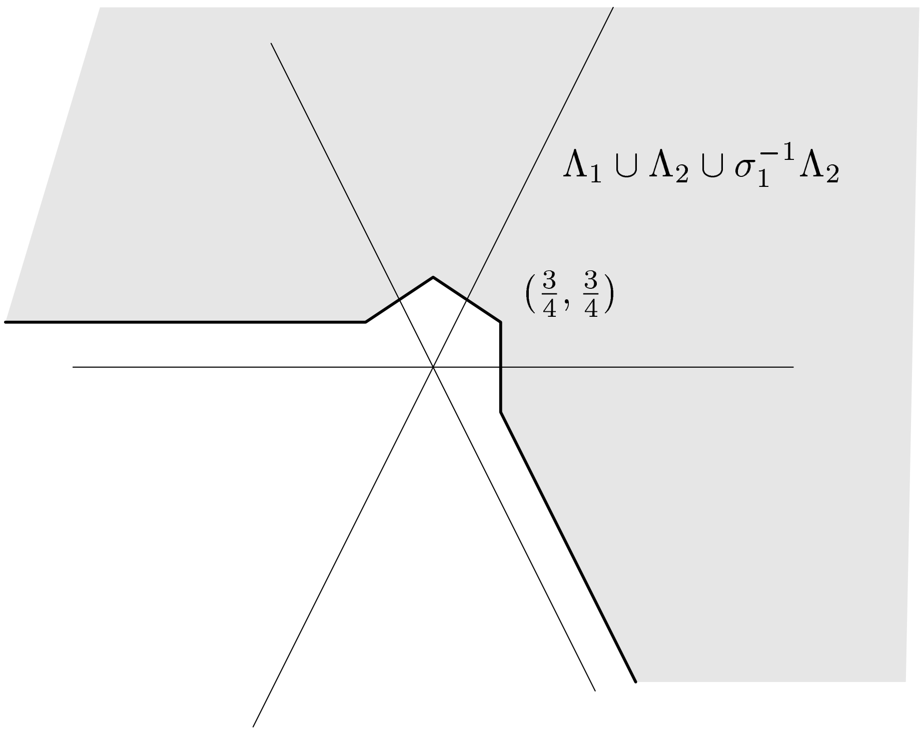
We may now prove the global meromorphic continuation of $Z^*(s)$. First we obtain continuation of Z_Ψ to the region

$$\Lambda_1 \cup \Lambda_2 \cup \sigma_1^{-1}\Lambda_2.$$

On Λ_2, we have already noted the analytic continuationand the continuation to Λ_1 is given by the same method. Of course, the two continuations agree on the overlap $\Lambda_1 \cap \Lambda_2 = \Lambda_0$. Now the continuation to $\sigma_1\Lambda_2$ is obtained by the formula

$$Z^*_\Psi(\sigma_1^{-1}s) = Z^*_{\sigma_1\Psi}(s), \qquad s \in \Lambda_2.$$

The two continuations agree on $\Lambda_1 \cap \Lambda_2 \cap \sigma_1^{-1}\Lambda_2 = \sigma_1\Lambda_0$.



The meromorphic continuation to all $\mathbb{C}^2$ follows from Hartogs' theorem. (The general case is similar.)