Weyl Group Multiple Dirichlet Series IV: The Stable Twisted Case *

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Abstract

Weyl group multiple Dirichlet series were associated with a root system Φ and a number field F containing the n-th roots of unity by Brubaker, Bump, Chinta, Friedberg and Hoffstein [3] and Brubaker, Bump and Friedberg [4] provided n is sufficiently large; their coefficients involve n-th order Gauss sums and reflect the combinatorics of the root system. Conjecturally, these functions coincide with Whittaker coefficients of metaplectic Eisenstein series, but they are studied in these papers by a method that is independent of this fact. The assumption that n is large is called *stability* and allows a simple description of the Dirichlet series. "Twisted" Dirichet series were introduced in Brubaker, Bump, Friedberg and Hoffstein [5] without the stability assumption, but only for root systems of type A_r. Their description is given differently, in terms of Gauss sums associated to Gelfand-Tsetlin patterns. In this paper, we reimpose the stability assumption and study the twisted multiple Dirichlet series for general Φ by introducing a description of the coefficients in terms of the root system similar to that given in the untwisted case in [4]. We prove the analytic continuation and functional equation of these series, and when $\Phi = A_r$ we also relate the two different descriptions of multiple Dirichlet series given here and in [5] in the stable case.

1 Introduction

Fourier-Whittaker coefficients of Eisenstein series on reductive algebraic groups G contain Dirichlet series in several complex variables with arithmetic interest. Metaplectic groups, certain central extensions of split G by n-th roots of unity, have Whittaker coefficients that contain Dirichlet series that are "twisted" by n-th order characters. For example, nonvanishing of twists of GL(2) automorphic forms by quadratic or cubic characters may be proved in this way. (See Bump, Friedberg and Hoffstein [8] and Brubaker, Friedberg and Hoffstein [6].) Unfortunately, computing these Whittaker coefficients on higher rank metaplectic groups yields intractable exponential sums. So even though the resulting Dirichlet series inherits a Weyl group of functional equations, it is extremely difficult to directly realize it as explicitly consisting of recognizable arithmetic functions.

Motivated by the theory of metaplectic Eisenstein series, one may attempt to construct Dirichlet series in several complex variables with similar properties. In [3] and [4], a family of "Weyl group

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multiple Dirichlet series" are described using data consisting of a fixed positive integer n and a number field F containing the group μ_{2n} of $2n$-th roots of unity, together with a reduced root system Φ. The group of functional equations of these multiple Dirichlet series is similarly isomorphic to the Weyl group W of Φ.

Conjecturally, the Weyl group multiple Dirichlet series are Whittaker coefficients of metaplectic Eisenstein series. To be precise, let G be a split simply-connected semisimple algebraic group whose root system is the dual root system $\hat{\Phi}$, and let $\tilde{G}(\mathbb{A})$ be the n-fold metaplectic cover of $G(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of F, constructed by Kubota [11] and Matsumoto [13]. Let U be the unipotent radical of the standard Borel subgroup of G. The metaplectic cover splits over U, and we identify $U(\mathbb{A})$ with its image in $\tilde{G}(\mathbb{A})$. If α is a root of Φ, let $i_\alpha : \mathrm{SL}_2 \longrightarrow G$ be the embedding corresponding to a Chevalley basis of Lie(G). We consider the additive character $\psi_U : U(\mathbb{A})/U(F) \longrightarrow \mathbb{C}$ such that for each simple positive root α, the composite $\psi_U \circ i_\alpha \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ is a fixed additive character ψ of $\mathbb{A}/F$; it is assumed that the conductor of ψ is $\mathfrak{o}_v$ for any prime $v \notin S$, where S is a finite set of places to be described in further detail below.

Now let $E(g, s_1, \cdots, s_r)$ be an Eisenstein series of Borel type on $\tilde{G}(\mathbb{A})$. The coefficient

$$Z(s_1, \cdots, s_r) = \int_{U(\mathbb{A})/U(F)} E(u, s_1, \cdots, s_r)\, \psi_U(u)\, du$$

is a multiple Dirichlet series whose group of functional equations is isomorphic to W. Conjecturally, this is the same as the multiple Dirichlet series described in [3], [4], and [5].

This paper builds on the results of [4] and [5] as we will now describe. In [4], Weyl group multiple Dirichlet series are defined for any reduced root system Φ and shown to possess a Weyl group of functional equations. However, the setting is specialized in two ways.

- We require that the integer n is "large enough," depending on Φ. We refer to this condition as the "stability assumption."
- The Dirichlet series is "untwisted" in a sense that will be made precise below upon comparison with other examples.

With these assumptions, the Dirichlet series admits a simple description. We will denote the coefficients of the Dirichlet series by $H(C_1, \cdots, C_r)$, where the C_i are elements of the ring $\mathfrak{o}_S$ of S-integers, with S a finite set of places containing the archimedean ones and enough others that $\mathfrak{o}_S$ is a principal ideal domain.

- The coefficients exhibit a twisted multiplicativity. This means that the Dirichlet series is not an Euler product, but specification of the coefficients is reduced to the specification of $H(p^{k_1}, \cdots, p^{k_r})$, where p is a fixed prime of $\mathfrak{o}_S$.
- Given $(k_1, \cdots, k_r)$, the coefficient $H(p^{k_1}, \cdots, p^{k_r})$ is zero unless there exists a Weyl group element $w \in W$ such that $\rho - w(\rho) = \sum k_i \alpha_i$, where ρ is half the sum of the positive roots in Φ, and $\alpha_1, \cdots, \alpha_r$ are the simple positive roots. If this is true, then $H(p^{k_1}, \cdots, p^{k_r})$ is a product of $l(w)$ n-th order Gauss sums, where $l : W \longrightarrow \mathbb{Z}$ is the length function.

We call the associated coefficients of the multiple Dirichlet series "untwisted, stable" coefficients owing to the special restrictions above.

In [5], "twisted" Weyl group multiple Dirichlet series are studied using a rather different perspective. The twisted Dirichlet series involve coefficients that we will denote $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$. Roughly, these are twists of the original multiple Dirichlet series by a set of n-th order characters. More specifically, if $\gcd(C_1 \cdots C_r, m_1 \cdots m_r) = 1$ we have

$$H(C_1, \cdots, C_r; m_1, \cdots, m_r) = \left(\frac{m_1}{C_1}\right)^{-||\alpha_1||^2} \cdots \left(\frac{m_r}{C_r}\right)^{-||\alpha_r||^2} H(C_1, \cdots, C_r), \tag{1}$$

where $\|\cdot\|$ is a fixed W-invariant inner product on V and $\left(\frac{\cdot}{\cdot}\right)$ is the n-th order power residue symbol.

Although the coefficients $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$ are thus *roughly* twists of the original coefficients, this is only approximately true, since (1) *fails* when the m_i are not coprime to the C_i. It does, however, together with the twisted multiplicativity of the coefficients C_i, to be described below, allow us to reduce the specification of the coefficients to the case where the C_i and the m_i are all powers of the same prime p. In [5] this was only accomplished when Φ is of type A_r, and in that case, the description is only conjectural unless $r \leqslant 2$, or $n = 2$ and $r \leqslant 5$, or $n = 1$. In that case, the following is observed, in [5] and in this paper.

- The existence of stable coefficients in correspondence with Weyl group elements described above for $H(p^{k_1}, \cdots, p^{k_r})$ persist, but the support in terms of k_i is changed. That is, with $m_i = p^{l_i}$ fixed and n sufficiently large, there are still $|W|$ distinct values $(k_1, \cdots, k_r)$ such that $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ is nonzero, and the coefficient corresponding to $w \in W$ is still a product of $l(w)$ Gauss sums. But when $l_i > 0$ the locations of the $(k_1, \cdots, k_r)$ parametrizing these stable coefficients form the vertices of a larger polytope than in the untwisted case. These coefficients will be called *twisted, stable* coefficients.

- If n is not sufficiently large, further nonzero coefficients appear inside the polytope whose vertices are spanned by the stable coefficients. In [5], these coefficients are described as products of Gauss sums parametrized by strict Gelfand-Tsetlin patterns. These coefficients are given a uniform description for all n, but due to the properties of Gauss sums they can vanish, and if n is sufficiently large, only the $|W|$ stable coefficients remain.

In the paper at hand, we will generalize the theory of [4] and prove a special case of a conjecture of [5] by studying twisted, stable multiple Dirichlet series. More specifically, we explain the modifications of [4] that are needed for the statements and the proofs in the stable twisted case. Moreover, we will verify the consistency of this description with that in [5] by showing that the $|W|$ stable coefficients do agree with the Gauss sums of the "stable" strict Gelfand-Tsetlin patterns with prescribed top row, depending on $l_1, \cdots, l_r$. The proof amounts to a combinatorial exercise. We note that a general solution to the Gelfand-Tsetlin conjecture, that is, a proof that these Dirichlet series possess functional equations in the case where n is not necessarily sufficiently large, remains open.

In [4], the theory of Eisenstein series is suppressed, except for rank one Eisenstein series that underlie the proofs. A direct definition of the series $Z(s_1, \cdots, s_r)$ is given, though it was arrived at by considerations connected with Eisenstein series, including early versions of computations that are included in [5], where the Whittaker coefficients of metaplectic Eisenstein series on GL_3 were worked out.

The relationship of the Eisenstein series with the twisted Dirichlet series may now be explained. Given $m_1, \cdots, m_r \in \mathfrak{o}_S$, let $\psi_{U,m} : U(\mathbb{A})/U(F) \longrightarrow \mathbb{C}$ be such that $\psi_{U,m}\, i_{\alpha_i} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ has conductor $p^{\mathrm{ord}_p(m_i)}$ for each prime p of $\mathfrak{o}_S$. Then (conjecturally)

$$Z(s_1, \cdots, s_r; m_1, \cdots, m_r) = \int_{U(\mathbb{A})/U(F)} E(u, s_1, \cdots, s_r)\, \psi_{U,m}(u)\, du.$$

Evidence for this description may be found in the GL_3 computations in [5], but in this paper, we follow [4] in giving an axiomatic description of the Dirichlet series, and prove its analytic continuation and functional equation without explicit reference to Eisenstein series. Again, a direct approach to these Dirichlet series based on Eisenstein series leads to combinatorial complications that we are able to avoid by the present approach. It should be understood, however, that the Eisenstein series on the n-fold metaplectic cover of SL_2, whose functional equations were originally proved by Kubota [12] following the methods of Selberg and Langlands, underlie the proofs.

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2 Preliminaries

2.1 Weyl group action

Let V be a real vector space of dimension r containing the rank r root system Φ. Any $\alpha \in V$ may be expressed as $\alpha = \sum_{i=1}^r b_i \alpha_i$ for a basis of simple positive roots α_i with $b_i \in \mathbb{R}$. Then we define the pairing $B(\alpha, \mathbf{s}) : V \times \mathbb{C}^r \longrightarrow \mathbb{C}$ for $\alpha \in V$ and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ by

$$B(\alpha, \boldsymbol{s}) = \sum b_i s_i. \tag{2}$$

Note that B is just the complexification of the usual dual pairing $V \times V^\vee \longrightarrow \mathbb{R}$, but we prefer the definition above for the explicit computations of subsequent sections.

The Weyl group W of Φ has a natural action on V in terms of the pairing. For a simple reflection σ_α in a hyperplane perpendicular to α we have $\sigma_\alpha : V \longrightarrow V$ given by

$$\sigma_\alpha(x) = x - B(x, \alpha^\vee)\, \alpha$$

where $\alpha^\vee$ is the corresponding element of the dual root system $\Phi^\vee$. In particular, the effect of σ_i on roots $\alpha \in \Phi$ is

$$\sigma_i : \alpha \mapsto \alpha - \frac{2\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i, \tag{3}$$

where $\langle \quad , \quad \rangle$ is the standard Euclidean inner product.

We now define a Weyl group action on $\mathbf{s} \in \mathbb{C}^r$. We will denote the image under this action by $w(\mathbf{s})$. Let $\rho^\vee$ be the Weyl vector for the dual root system, i.e. half the sum of the positive coroots. Identifying $V_{\mathbb{C}}^\vee$ with $\mathbb{C}^r$ we may take

$$\rho^\vee = (1, 1, \ldots, 1). \tag{4}$$

The action of W on $\mathbb{C}^r$ is defined implicitly according to the identification

$$B\left(w\alpha, w(\boldsymbol{s}) - \frac{1}{2}\rho^\vee\right) = B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right). \tag{5}$$

For simple reflections, we have the following result ([4], Prop. 3.1).

Proposition 1 *The action of σ_i on $\boldsymbol{s} = (s_1, \ldots, s_r)$ according to (5) is given by:*

$$s_j \longmapsto s_j - \frac{2\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\left(s_i - \frac{1}{2}\right), \qquad j = 1, \cdots, r. \tag{6}$$

In particular, $s_i \longmapsto 1 - s_i$. Note also that

$$-\frac{2\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \geqslant 0 \quad \text{if } j \neq i.$$

2.2 Two Lemmas Using Root Systems

In this section, we give two lemmas concerning root systems which will be used in proving local functional equations. Let Φ be a reduced root system of rank r. Recall that $\lambda \in V$ is a *weight* if $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$, and the weight is *dominant* if $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \geqslant 0$ for all $\alpha \in \Phi^+$. It is well-known that ρ is a dominant weight; in fact it is the sum of the fundamental dominant weights ([7], Proposition 21.16).

Let $\varepsilon_1, \ldots, \varepsilon_r$ be the fundamental dominant weights, which satisfy

$$\frac{2\langle \varepsilon_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij} \quad (\delta_{ij} = \text{Kronecker delta.}) \tag{7}$$

Let Λ_{weight} be the weight lattice, generated by the ε_i. It contains the root lattice Λ_{root} generated by the α_i.

We will fix non-negative integers $l_1, \cdots, l_r$ and let $\lambda = \sum l_i \varepsilon_i$ be the corresponding weight.

Lemma 1 *Let $w \in W$.*
(i) The cardinality of Φ_w is the length $l(w)$ of w.
(ii) Express $\rho + \lambda - w(\rho + \lambda)$ as a linear combination of the simple positive roots:

$$\rho + \lambda - w(\rho + \lambda) = \sum_{i=1}^{r} k_i \alpha_i. \tag{8}$$

Then the k_i are nonnegative integers.
(iii) If $w, w' \in W$ such that $\rho + \lambda - w(\rho + \lambda) = \rho + \lambda - w'(\rho + \lambda)$ then $w = w'$.

Proof Part (i) follows from Proposition 21.2 of [7]. For (ii), note that the expression (8) as an integral linear combination is valid by Proposition 21.14 of [7]. To show that this is a non-negative linear combination, note that $\rho + \lambda$ lies inside positive Weyl chamber, as the l_i used to define λ are non-negative. Hence, in the partial ordering, $\rho + \lambda \succ w(\rho + \lambda)$ for all $w \in W$, and the claim follows.

For (iii), we again use the fact that $\rho + \lambda$ is in the interior of the positive Weyl chamber, so $w(\rho + \lambda) = w'(\rho + \lambda)$ means that the positive Weyl chamber is fixed by $w^{-1}w'$ which implies $w^{-1}w' = 1$. □

Define the function d_λ on Φ^+ by

$$d_\lambda(\alpha) = \frac{2\langle \rho + \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = B(\rho + \lambda, \alpha^\vee). \tag{9}$$

Lemma 2 *We have $d_\lambda(\alpha) \in \mathbb{Z}^+$ for all $\alpha \in \Phi^+$, and $d_\lambda(\alpha_i) = l_i + 1$ if α_i is a simple positive root.*

Proof This follows from (7), expressing ρ as the sum of the fundamental dominant weights. □

2.3 Hilbert symbols

Let $n > 1$ be an integer and let F be a number field containing the n-th roots of unity. Let S be a finite set of places of F such that S contains all archimedean places, all places ramified over $\mathbb{Q}$, and that is sufficiently large that the ring of S-integers $\mathfrak{o}_S$ is a principal ideal domain. Embed $\mathfrak{o}_S$ in $F_S = \prod_{v \in S} F_v$ diagonally.

The product of local Hilbert symbols gives rise to a pairing $(\ ,\ )_S : F_S^\times \times F_S^\times \longrightarrow \mu_n$ by $(a, b)_S = \prod_{v \in S} (a, b)_v$. A subgroup Ω of $F_S^\times$ is called *isotropic* if $(\varepsilon, \delta)_S = 1$ for $\varepsilon, \delta \in \Omega$. Let Ω be the subgroup $\mathfrak{o}_S^\times F_S^{\times, n}$, which is maximal isotropic. If t is a positive integer, let $\mathcal{M}_t(\Omega)$ be the vector space of functions $\Psi : F_{\text{fin}}^\times \longrightarrow \mathbb{C}$ that satisfy

$$\Psi(\varepsilon c) = (c, \varepsilon)_S^{-t} \Psi(c), \tag{10}$$

when $\varepsilon \in \Omega$. We denote $\mathcal{M}_1(\Omega)$ by $\mathcal{M}(\Omega)$. Note that if ε is sufficiently close to the identity in $F_{\text{fin}}^\times$ it is an n-th power at every place in S_{fin}, so such a function is locally constant. It is easy to see that the dimension of $\mathcal{M}(\Omega)$ is $[F_S^\times : \Omega] < \infty$.

2.4 Gauss sums

If $a \in \mathfrak{o}_S$ and $\mathfrak{b}$ is an ideal of $\mathfrak{o}_S$ let $\left(\frac{a}{\mathfrak{b}}\right)$ be the nth order power residue symbol as defined in [4]. (This depends on S, but we suppress this dependence from the notation.) If $a, c \in \mathfrak{o}_S$ and $c \neq 0$, and if t is a positive integer, define the Gauss sum $g_t(a, c)$ as follows. We choose a nontrivial additive character ψ of F_S such that $\psi(x\mathfrak{o}_S) = 1$ if and only if $x \in \mathfrak{o}_S$. (See Brubaker and Bump [2], Lemma 1.) Then the Gauss sum is given by

$$g_t(a, c) = \sum_{d \bmod c} \left(\frac{d}{c\mathfrak{o}_S}\right)^t \psi\left(\frac{ad}{c}\right). \tag{11}$$

We will also denote $g_1(a, c) = g(a, c)$.

2.5 Kubota Dirichlet series

If $\Psi \in \mathcal{M}_t(\Omega)$, the space of functions defined in (10), let

$$\mathcal{D}_t(s, \Psi, a) = \sum_{0 \neq c \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g_t(a, c)\, \Psi(c)\, \mathbb{N}(c)^{-2s} \quad .$$

We will also denote $\mathcal{D}_1(s, \Psi, a) = \mathcal{D}(s, \Psi, a)$. Here $\mathbb{N}(c)$ is the order of $\mathfrak{o}_S / c\mathfrak{o}_S$. The term $g_t(a, c)\, \Psi(c)\, \mathbb{N}(c)^{-2s}$ is independent of the choice of representative c modulo S-units. It follows easily from standard estimates for Gauss sums that the series is convergent if $\Re(s) > \frac{3}{4}$.

Let

$$\boldsymbol{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-1} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{12}$$

In view of the multiplication formula for the Gamma function, we may also write

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Let

$$\mathcal{D}_t^*(s,\Psi,a) = \boldsymbol{G}_m(s)^{[F:\mathbb{Q}]/2}\, \zeta_F(2ms-m+1)\, \mathcal{D}_t(s,\Psi,a), \tag{13}$$

where $m = n/\gcd(n,t)$, $\frac{1}{2}[F:\mathbb{Q}]$ is the number of archimedean places of the totally complex field F, and ζ_F is the Dedekind zeta function of F.

If $v \in S_{\text{fin}}$ let q_v denote the cardinality of the residue class field $\mathfrak{o}_v/\mathfrak{p}_v$, where $\mathfrak{o}_v$ is the local ring in F_v and $\mathfrak{p}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}.

If $\Psi \in \mathcal{M}(\Omega)$ and $\eta \in F_S^\times$ denote

$$\tilde{\Psi}_\eta(c) = (\eta, c)_S\ \Psi(c^{-1}\eta^{-1}). \tag{14}$$

It is easy to check that $\tilde{\Psi}_\eta \in \mathcal{M}(\Omega)$ and that it depends only on the class of η in $F_S^\times/F_S^{\times,n}$.

Then we have the following result, which follows easily from the work of Brubaker and Bump [2].

Theorem 1 *Let $\Psi \in \mathcal{M}_t(\Omega)$, and let $a \in \mathfrak{o}_S$. Let $m = n/\gcd(n,t)$. Then $\mathcal{D}_t^*(s,\Psi,a)$ has meromorphic continuation to all s, analytic except possibly at $s = \frac{1}{2} \pm \frac{1}{2m}$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta^t(s)$ that depend only on the image of η in $F_S^\times/F_S^{\times,n}$ such that*

$$\mathcal{D}_t^*(s,\Psi,a) = \mathbb{N}(a)^{1-2s} \sum_{\eta \in F_S^\times/F_S^{\times,n}} P_{a\eta}^t(s) \mathcal{D}_t^*(1-s,\tilde{\Psi}_\eta,a). \tag{15}$$

This result, based on ideas of Kubota [12], relies on the theory of Eisenstein series. The case $t=1$ is to be found in [2]; the general case follows as discussed in the proof of Proposition 5.2 of [4]. Importantly, the factor $\mathbb{N}(a)^{1-2s}$ does not depend on t.

2.6 Normalizing factors

As a final preliminary, we record the zeta and gamma factors that will be needed to normalize the Weyl group multiple Dirichlet series. These will be used to prove global functional equations.

Let Φ be a reduced root system of rank r, with inner product $\langle\ ,\ \rangle$ chosen such that $||\alpha|| = \sqrt{\langle\alpha,\alpha\rangle}$ and $2\langle\alpha,\beta\rangle$ are integral for all $\alpha,\beta \in \Phi$. Let

$$n(\alpha) = \frac{n}{\gcd(n, \|\alpha\|^2)}. \tag{16}$$

If Φ is simply-laced, then we may take all roots to have length 1 and then $n(\alpha) = n$ for every $\alpha \in \Phi$. If Φ is not simply-laced but irreducible, and if $\langle\ ,\ \rangle$ is normalized so that the short roots have length 1, then

$$n(\alpha) = \begin{cases} n & \text{if } \alpha \text{ is a short root,} \\ n & \text{if } \alpha \text{ is a long root and } \Phi \neq G_2, \text{ and } n \text{ is odd} \\ \frac{n}{2} & \text{if } \alpha \text{ is a long root and } \Phi \neq G_2, \text{ and } n \text{ is even} \\ n & \text{if } \alpha \text{ is a long root and } \Phi = G_2, \text{ and } 3 \nmid n \\ \frac{n}{3} & \text{if } \alpha \text{ is a long root and } \Phi = G_2, \text{ and } 3|n. \end{cases}$$

If α is a positive root, write $\alpha = \sum k_i\alpha_i$ as before. Let

$$\zeta_\alpha(\boldsymbol{s}) = \zeta_F\left(1 + 2n(\alpha)\sum_{i=1}^r k_i(s_i - \frac{1}{2})\right) = \zeta_F\left(1 + 2n(\alpha)\, B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right)\right). \tag{17}$$

Also let

$$G_\alpha(\boldsymbol{s}) = \boldsymbol{G}_{n(\alpha)}\left(\frac{1}{2} + \sum_{i=1}^{r} k_i\left(s_i - \frac{1}{2}\right)\right) = \boldsymbol{G}_{n(\alpha)}\left(\frac{1}{2} + B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right)\right),$$

where $\boldsymbol{G}_n(s)$ is defined as in (12). Define the *normalized* multiple Dirichlet series by

$$Z_\Psi^*(\boldsymbol{s}) = \left[\prod_{\alpha\in\Phi^+} G_\alpha(\boldsymbol{s})\,\zeta_\alpha(\boldsymbol{s})\right] Z_\Psi(\boldsymbol{s}). \tag{18}$$

3 Stability Assumption

All of our subsequent computations rely on a critical assumption that n, the order of the power residue symbols appearing in all our definitions, is sufficiently large. This dependence appears only once in the section on global functional equations, but is crucial in simplifying the proof that the multiple Dirichlet series can be understood in terms of Kubota Dirichlet series. This dependence is also crucial in making the bridge between Weyl group multiple Dirichlet series and those series defined by Gelfand-Tsetlin patterns.

Let $\sigma_i \in W$ be a fixed simple reflection about $\alpha_i \in \Phi$. Let $m_1, \cdots, m_r$ be fixed. For p a prime, let $l_i = \mathrm{ord}_p(m_i)$. (For convenience, we suppress the dependence of l_i on p in the notation.) Let

$$\lambda_p = \sum_{i=1}^{r} l_i \varepsilon_i. \tag{19}$$

Stability Assumption. *The positive integer n satisfies the following property. Let $\alpha = \sum_{i=1}^r t_i\alpha_i$ be the largest positive root in the partial ordering. Then for every prime p,*

$$n \geq \gcd(n, ||\alpha||^2)\cdot d_{\lambda_p}(\alpha) = \gcd(n, ||\alpha||^2)\cdot \sum_{i=1}^{r} t_i(\,l_i + 1)\,. \tag{20}$$

Note that the right-hand side of (20) is clearly bounded for fixed choice of $m_1, \cdots, m_r$. We fix an n satisfying this assumption for the rest of the paper.

For example, if $\Phi = A_r$ and the inner product is chosen so that that $\|\alpha\| = 1$ for each root α, the condition (20) becomes $n \geqslant \sum_{i=1}^r l_i$.

4 Definition of the twisted multiple Dirichlet series

Let $\mathcal{M}(\Omega^r)$ be as in [4], and let $\Psi \in \mathcal{M}(\Omega^r)$. We will define

$$Z_\Psi(s_1, \cdots, s_r; m_1, \cdots, m_r) = \sum_{\mathfrak{c}_1,\cdots,\mathfrak{c}_r} H\Psi(\mathfrak{c}_1, \cdots, \mathfrak{c}_r; m_1, \cdots, m_r)\ \mathbb{N}\mathfrak{c}_1^{-2s_1}\cdots\mathbb{N}\mathfrak{c}_r^{-2s_r}, \tag{21}$$

where the coefficients H will be described next; as in [4], the product

$$H\Psi(C_1, \cdots, C_r; m_1, \cdots, m_r) = H(C_1, \cdots, C_r; m_1, \cdots, m_r)\,\Psi(C_1, \cdots, C_r)$$

will be unchanged if C_i is multiplied by a unit, so (21) can be regarded as either a sum over $C_i \in \mathfrak{o}_S/\mathfrak{o}_S^\times$ or of the ideals $\mathfrak{c}_i$ that the C_i generate.

It remains to describe the twisted coefficients H. If

$$\gcd(C_1 \cdots C_r, C'_1 \cdots C'_r) = 1, \tag{22}$$

then

$$\frac{H(C_1C'_1, \cdots, C_rC'_r; m_1, \cdots, m_r)}{H(C_1, \cdots, C_r; m_1, \cdots, m_r)\, H(C'_1, \cdots, C'_r; m_1, \cdots, m_r)} = \prod_{i=1}^{r} \left(\frac{C_i}{C'_i}\right)^{\|\alpha_i\|^2} \left(\frac{C'_i}{C_i}\right)^{\|\alpha_i\|^2} \prod_{i<j} \left(\frac{C_i}{C'_j}\right)^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{C'_i}{C_j}\right)^{2\langle\alpha_i,\alpha_j\rangle}. \tag{23}$$

Moreover if $\gcd(m'_1 \cdots m'_r, C_1 \cdots C_r) = 1$ we will have the multiplicativity

$$H(C_1, \cdots, C_r; m_1m'_1, \cdots, m_rm'_r) = \left(\frac{m'_1}{C_1}\right)^{-||\alpha_1||^2} \cdots \left(\frac{m'_r}{C_r}\right)^{-||\alpha_r||^2} H(C_1, \cdots, C_r; m_1, \cdots, m_r). \tag{24}$$

Equations (23) and (24) reduce the specification of the coefficients $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$ to those of the form $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ where p is a prime. To give these, let Φ_w be the set of all positive roots α such that $w(\alpha)$ is a negative root. The cardinality of Φ_w is equal to the length $l(w)$ of w in the Weyl group. Then we define

$$H(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{d_{\lambda_p}(\alpha)-1}, p^{d_{\lambda_p}(\alpha)}), \tag{25}$$

where $d_{\lambda_p}(\alpha)$ is given by (9).

5 Local computations

In this section, we analyze our multiple Dirichlet series coefficients at powers of a single fixed prime p, and show that they contain Gauss sums. These will be used to form Kubota Dirichlet series in the next section.

For the remainder of this section, let $l_1, \cdots, l_r$ be fixed non-negative integers, and let

$$\lambda = \sum_{i=1}^{r} l_i \varepsilon_i$$

as in the previous section.

We recall that on prime powers, the choices of k_i for which $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ is non-zero are in one-to-one correspondence with elements $w \in W$, the Weyl group. We say that $(k_1, \cdots, k_r) \in \mathbb{Z}^r$ is *associated to $w \in W$ with respect to λ* if (8) is satisfied; in this case, we write

$$(k_1, \cdots, k_r) = \mathrm{assoc}_\lambda(w).$$

The following results are generalizations of Propositions 4.1, 4.2 of [4].

Proposition 2 *Let $w \in W$ be such that $l(\sigma_i w) = l(w)+1$. Suppose that $\mathrm{assoc}_\lambda(w) = (k_1, \cdots, k_r) \in \mathbb{Z}^r$ and $\mathrm{assoc}_\lambda(\sigma_i w) = (h_1, \cdots, h_r)$. Let $d_\lambda = d_\lambda(w^{-1}\alpha_i)$ in the notation (9). Then*

$$h_j = \begin{cases} k_i + d_\lambda & \text{if } j = i; \\ k_i & \text{if } j \neq i, \end{cases} \tag{26}$$

and

$$H(p^{h_1}, \ldots, p^{h_r}) = g_{\|\alpha_i\|^2}(p^{d_\lambda - 1}, p^{d_\lambda})\, H(p^{k_1}, \ldots, p^{k_r}). \tag{27}$$

Proof This is proved similarly to Prop. 4.1 of [4], but replacing ρ by $\rho + \lambda$ and d by d_λ. □

Proposition 3 *Let $d_\lambda = d_\lambda(w^{-1}(\alpha_i))$ and let $l_1, \ldots, l_r$ be fixed as above. For any $w \in W$, the monomial in the r complex variables $\boldsymbol{s} = (s_1, \ldots, s_r)$*

$$\mathbb{N}p^{(s_i - \frac{1}{2})(d_\lambda - l_i - 1)} \prod_{\alpha \in \Phi_w} \mathbb{N}p^{-2B(\rho+\lambda - w(\rho+\lambda), \mathbf{s})}$$

is invariant under the action of σ_i given in (6).

Proof The statement is equivalent to showing that

$$\frac{1}{2}(d_\lambda - l_i - 1)\alpha_i + \rho + \lambda - w(\rho + \lambda) \tag{28}$$

is orthogonal to α_i. Hence, it suffices to show (28) is fixed by σ_i, i.e.,

$$\sigma_i(\rho + \lambda) - \sigma_i w(\rho + \lambda) = (d_\lambda - l_i - 1)\alpha_i + (\rho + \lambda) - w(\rho + \lambda). \tag{29}$$

Since $\rho + \lambda - \sigma_i(\rho + \lambda) = (1 + l_i)\alpha_i$ we can write (29) as $w(\rho + \lambda) - \sigma_i w(\rho + \lambda) = d_\lambda \alpha_i$, and indeed applying the definition (9)

$$d_\lambda \alpha_i = \frac{2\left\langle w^{-1}\alpha_i, \rho + \lambda\right\rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i = \frac{2\left\langle \alpha_i, w(\rho + \lambda)\right\rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i = w(\rho + \lambda) - \sigma_i w(\rho + \lambda).$$

□

6 Preparing the global Dirichlet series

We preserve the notations above. In particular, $m_1, \cdots, m_r$ are fixed integers with corresponding λ_p defined for each prime p as in (19). In our local computations, we showed a connection between the prime-power coefficients $H(p^{k_1}, \cdots, p^{k_r})$ associated to pairs of Weyl group elements w and $\sigma_i w$ for a fixed simple reflection σ_i and Gauss sums. The next step is to translate this into a global notion. Once the correct definitions are given, it turns out to be relatively straightforward to generalize the proofs in [4], so we will omit many proof details which follow by very similar methods to [4].

In [4], the notion of admissibility for r-tuples of integers $(C_1, \cdots, C_r)$ in $(\mathfrak{o}_S)^r$ was defined. We generalize this in the following definition.

Definition 1 *We say that* $(C_1, \cdots, C_r)$ *in* $(\mathfrak{o}_S)^r$ *is* ***admissible with respect to*** λ *if, for each prime* p*, there exists a Weyl group element* $w_p \in W$ *such that*

$$(\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_r)) = \mathrm{assoc}_{\lambda_p}(w_p).$$

For such $(C_1, \cdots, C_r)$*, we say that* C_i *is* i***-reduced*** *if, for every* p*, we have* $l(\sigma_i w_p) = l(w_p) + 1$.

We note that if $C_1, \dots, C_r$ are nonzero elements of $\mathfrak{o}_S$, then $(C_1, \dots, C_r)$ is admissible with respect to λ if and only if $H(C_1, \dots, C_r; m_1, \dots, m_r) \neq 0$. This is immediate from the definition of H.

We have the following results.

Proposition 4 *Let* $C_1, \cdots, C_{i-1}, C_{i+1}, \cdots, C_r$ *be nonzero elements of* $\mathfrak{o}_S$*. If there exists a* C_i *such that* $(C_1, \cdots, C_r)$ *is admissible with respect to* λ*, then there exists a* C_i' *(modulo the action of* $\mathfrak{o}_S^\times$*) that is* i*-reduced. This* C_i' *divides* C_i *and is uniquely determined up to multiplication by a unit. Moreover, for each prime* p*, if* w_p' *is determined by the equality*

$$(\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_i'), \cdots, \mathrm{ord}_p(C_r)) = (k_1, \cdots, k_r) = \mathrm{assoc}_{\lambda_p}(w_p'),$$

then either $\mathrm{ord}_p(C_i) = k_i$ *or* $\mathrm{ord}_p(C_i) = k_i + d_\lambda$*, where* $d_\lambda = d_\lambda((w_p')^{-1}\alpha_i)$.

Proof The proof is similar to Proposition 5.2 of [4], replacing admissible by admissible with respect to λ and d by d_λ. □

The multiple Dirichlet series is built out of H-coefficients of the form $H(C_1, \dots, C_r; m_1, \dots, m_r)$ satisfying the multiplicativity relation (24). However, for convenience we will suppress the m_i's from the notation. By Proposition 4, $Z_\Psi(s_1, \cdots, s_r) =$

$$= \sum_{\substack{0 \neq C_j \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S \\ 1 \leqslant j \leqslant r \\ C_1, \cdots, C_r \text{ admissible w.r.t. } \lambda \\ C_i \; i\text{-reduced}}} \mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r} H(C_1, \cdots, C_r) \sum_{0 \neq D \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S} (D, C_i)_S^{\|\alpha_i\|^2}$$
$$\times \frac{H(C_1, C_2, \cdots, DC_i, \cdots, C_r)}{H(C_1, C_2, \cdots, C_i, \cdots, C_r)} \prod_{j>i} (D, C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \Psi_i^{C_1, \cdots, C_r}(D)\, \mathbb{N}D^{-2s_i}, \tag{30}$$

where we define

$$\Psi_i^{C_1, \dots, C_r}(D) = \Psi(C_1, \dots, C_i D, \dots, C_r)(D, C_i)_S^{-\|\alpha_i\|^2} \prod_{j>i} (D, C_j)_S^{-2\langle \alpha_i, \alpha_j \rangle} \tag{31}$$

to emphasize the dependence on D for fixed parameters $C_1, \dots, C_r$ in the inner sum. We recall

Lemma 3 ([4], Lemma 5.3) *Let* $C_1, \dots, C_r$ *be fixed nonzero elements of* $\mathfrak{o}_S$*. Then with the notation (31), the function* $\Psi_i^{C_1, \dots, C_r} \in \mathcal{M}_{\|\alpha_i\|^2}(\Omega)$.

One can now show that the inner sum in (30) is a Kubota Dirichlet series. The key is to identify the quotient of H's and Hilbert symbols in (30) as a Gauss sum. If the C_i and D are powers of a single prime p, this is (27), which is generalized in (33) below.

Lemma 4 *Fix an integer $i \in \{1, \ldots, r\}$ and integers $(m_1, \ldots, m_r)$. If $(C_1, \ldots, C_r) \in \mathfrak{o}_S^r$ is admissible with respect to λ with C_i i-reduced, then*

$$B_i = \prod_{j=1}^{r} C_j^{-2\langle\alpha_j,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle} \tag{32}$$

is an $\mathfrak{o}_S$ integer and for every $D \in \mathfrak{o}_S$ we have

$$\frac{H(C_1, \ldots, DC_i, \ldots, C_r)}{H(C_1, \ldots, C_r)}(D, C_i)_S^{\|\alpha_i\|^2} \prod_{j>i}(D, C_j)_S^{2\langle\alpha_i,\alpha_j\rangle} = g_{\|\alpha_i\|^2}(m_i B_i, D). \tag{33}$$

Moreover for each prime p of $\mathfrak{o}_S$ we have

$$\mathrm{ord}_p(B_i) = d_{\lambda_p}(w_p^{-1}\alpha_i) - l_i - 1, \tag{34}$$

where λ_p corresponds to the m_j as in (19), and w_p is determined by the condition

$$\mathrm{assoc}_{\lambda_p}(w_p) = (\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_r)).$$

The proof of this is similar to Lemma 5.3 of [4], with modifications similar to those above, and is omitted.

Using Lemmas 3 and 4, we may rewrite the Dirichlet series $Z_\Psi(s_1, \ldots, s_r)$ in terms of a Kubota Dirichlet series in the variable s_i.

Proposition 5 *With notations as above, we have*

$$Z_\Psi(s_1, \ldots, s_r) =$$

$$\sum_{\substack{0 \neq C_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (C_1, \cdots, C_r) \text{ admissible w.r.t. } \lambda \\ C_i \text{ } i\text{-reduced}}} \mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r} H(C_1, \ldots, C_r)\, \mathcal{D}_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1, \ldots, C_r}, m_i B_i),$$

where, for fixed $C_1, \cdots, C_r$, the coefficient B_i is defined in (32).

Proof We have already rewritten the Dirichlet series $Z_\Psi(s_1, \ldots, s_r)$ in equation (30) in terms of sums over C_j, $j = 1, \ldots, r$ with C_i i-reduced. The proposition then follows immediately from the previous two lemmas and the definition of $\mathcal{D}_t(s, \Psi, C)$ for S-integer C and $\Psi \in \mathcal{M}_t(\Omega)$, where $t = \|\alpha_i\|^2$. □

7 Global functional equations

Using Proposition 5 as our starting point, we are finally ready to prove functional equations corresponding to the transformations σ_i defined in (6), for each $i = 1, \cdots, r$. First we recall some notation from [4].

Let $\mathcal{A}$ be the ring of (Dirichlet) polynomials in $q_v^{\pm 2s_1}, \ldots, q_v^{\pm 2s_r}$ where v runs through the finite set of places S_{fin}, and let $\mathfrak{M} = \mathcal{A} \otimes \mathcal{M}(\Omega^r)$. We may regard elements of $\mathfrak{M}$ as functions $\Psi : \mathbb{C}^r \times (F_S^\times)^r \longrightarrow \mathbb{C}$ such that for any fixed $(s_1, \ldots, s_r) \in \mathbb{C}^r$ the function

$$(C_1, C_2, \ldots, C_r) \longmapsto \Psi(s_1, \ldots, s_r; C_1, \ldots, C_r)$$

defines an element of $\mathcal{M}(\Omega^r)$, while for any $(C_1, \ldots, C_r) \in (F_S^\times)^r$, the function

$$(s_1, \ldots, s_r) \longmapsto \Psi(s_1, \ldots, s_r; C_1, \ldots, C_r)$$

is an element of $\mathcal{A}$. We will sometimes use the notation

$$\Psi_{\boldsymbol{s}}(C_1, \ldots, C_r) = \Psi(s_1, \ldots, s_r; C_1, \ldots, C_r), \qquad \boldsymbol{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r. \tag{35}$$

We identify $\mathcal{M}(\Omega^r)$ with its image $1 \otimes \mathcal{M}(\Omega^r)$ in $\mathfrak{M}$; this just consists of the $\Psi_{\boldsymbol{s}}$ that are independent of $\boldsymbol{s} \in \mathbb{C}^r$.

The operators σ_i on $\mathbb{C}^r$ are defined in (6). Define corresponding operators σ_i on $\mathfrak{M}$ by

$$\begin{aligned}(\sigma_i \Psi_{\boldsymbol{s}})(C_1, \cdots, C_r) = (\sigma_i \Psi)(s_1, \cdots, s_r; C_1, \cdots, C_r) = \\ \sum_{\eta \in F_S^\times / F_S^{\times, n}} (\eta, C_i)_S^{\|\alpha_i\|^2} \prod_{j > i} (\eta, C_j^{2\langle \alpha_i, \alpha_j \rangle})_S P_{\eta m_i B_i}(s_i) \\ \Psi(\sigma_i(s_1, \cdots, s_r); C_1, C_2, \cdots, \eta^{-1} C_i, \cdots, C_r)\end{aligned} \tag{36}$$

where, as in (32),

$$B_i = \prod_j C_j^{-2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle} = C_i^{-2} \prod_{j \neq i} C_j^{-2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle}.$$

We have arranged this definition to give a clean formulation of the functional equations. Note that the Dirichlet polynomials P are associated to nth power classes which depend on the fixed parameter m_i in $(m_1, \cdots, m_r)$, though we suppress this from the notation for the action σ_i on $\mathfrak{M}$.

Proposition 6 *If $\Psi \in \mathfrak{M}$, then $\sigma_i \Psi$ is in $\mathfrak{M}$.*

Proof See [4], Prop. 8, for a proof (replace all instances of C_0 there by $m_i B_i$ to obtain the present result). $\square$

Each functional equation corresponding to $\sigma_i \in W$ is inherited from a functional equation for the Kubota Dirichlet series appearing in Proposition 5. These functional equations are formalized in the following result.

Lemma 5 *Given an element $\Psi_{\boldsymbol{s}}(C_1, \ldots, C_r) \in \mathfrak{M}$, we have*

$$\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1, \ldots, C_r}, m_i B_i) = \mathbb{N}(m_i B_i)^{1 - 2s_i} \mathcal{D}^*_{\|\alpha_i\|^2}(1 - s_i, (\sigma_i \Psi)_i^{C_1, \ldots, C_r}, m_i B_i),$$

where $\mathcal{D}^*_{\|\alpha_i\|^2}$ is as in (13).

Proof This follows from (15). To check the way in which the Ψ function changes under the functional equation, this follows from the definition in (36) as in Lemma 5.7 of [4], with the substitution $C_0 = m_i B_i$ in every instance it appears. $\square$

Let $\mathfrak{W}$ denote the group of automorphisms of $\mathfrak{M}$ generated by σ_i. This will turn out to be the group of functional equations for the multiple Dirichlet series. The natural homomorphism $\mathfrak{W} \longrightarrow W$ gives an action of $\mathfrak{W}$ on $\mathbb{C}^r$ induced by the action of W, and if $w \in \mathfrak{W}$ we will denote by $w\boldsymbol{s}$ the effect of w on $\boldsymbol{s} \in \mathbb{C}^r$ in this induced action. Further, recall the definition of $n(\alpha)$ for $\alpha \in \Phi$ given in (16) by

$$n(\alpha) = \frac{n}{\gcd(n, \|\alpha\|^2)}.$$

Theorem 2 *The function $Z^*_\Psi(\mathbf{s}; m_1, \dots, m_r)$ has meromorphic continuation to the complex space $\mathbb{C}^r$. Moreover, for each $w \in \mathfrak{W}$ we may identify w with its image in the Weyl group and writing $w = \sigma_{j_1} \cdots \sigma_{j_k}$ as a product of simple reflections, $Z^*_\Psi(\mathbf{s}; m_1, \dots, m_r)$ satisfies the functional equation*

$$Z^*_{w\Psi}(w\boldsymbol{s}; m_1, \dots, m_r) = \prod_{i=1}^{k} m_{j_i}^{1-2(\sigma_{j_1}\cdots\sigma_{j_{i-1}})(s_{j_i})} Z^*_\Psi(\boldsymbol{s}; m_1, \dots, m_r) \tag{37}$$

where the action of w on $\mathfrak{M}$ is similarly given by the composition of simple reflections σ_i. It is analytic except along the hyperplanes $B(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = \frac{1}{2n(\alpha)}$, where α runs through Φ, $\frac{1}{2}\rho^\vee = \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$, and B is defined by (2); along these hyperplanes it can have simple poles.

Observe that the equation $B(-\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = \frac{1}{2n(\alpha)}$ is equivalent to $B(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = -\frac{1}{2n(\alpha)}$, so the polar hyperplanes occur in parallel pairs.

Proof The proof is based on Bochner's tube-domain Theorem. We sketch this argument below, and refer the reader to [3] where the case $\Phi = A_2$ is worked out in full detail, and to [4] where a similar argument to the one below is applied for $(m_1, \cdots, m_r) = (1, \cdots, 1)$.

From the standard estimates for the Gauss sums and Proposition 5, it follows that the original Dirichlet series defining Z_Ψ is absolutely convergent in a translate of the fundamental Weyl chamber, denoted

$$\Lambda_0 = \left\{ \boldsymbol{s} = (s_1, \cdots, s_r) \quad | \quad \Re(s_j) > \frac{3}{4}, \quad j = 1, \cdots, r \right\}.$$

Using standard growth estimates for the Kubota Dirichlet series, one sees that the expression in Proposition 5 is analytic in the convex hull of $\Lambda_0 \cup \sigma_i \Lambda_0$, which we will denote by Λ_i. On this region, we claim that for the simple reflection σ_i one has

$$Z^*_{\sigma_i \Psi}(\sigma_i \boldsymbol{s}) = \mathbb{N}(m_i)^{1-2s_i} Z^*_\Psi(\boldsymbol{s}).$$

Recalling the effect of the transformation σ_i on $\mathbb{C}^r$ from (6), we note that the Kubota Dirichlet series appearing in Proposition 5 is essentially invariant under this transformation by Lemma 5, with the implicit Ψ function mapped to the appropriately defined $\sigma_i \Psi$. That is, the Kubota Dirichlet series functional equation produces a factor of $\mathbb{N}(m_i B_i)^{1-2s_i}$. Thus, it remains to show that for each of the r-tuples $(C_1, \cdots, C_r)$ that are admissible with respect to λ with C_i i-reduced, the corresponding terms on the right-hand side of Proposition 5, multiplied by $\mathbb{N}(m_i B_i)^{1/2-s_i}$ are invariant under

$\boldsymbol{s} \mapsto \sigma_i(\boldsymbol{s})$ given explicitly in (6). This follows from Proposition 3, since with our definitions, when $C_1, \cdots, C_r$ is admissible with respect to λ and C_i is i-reduced,

$$\mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r} = \prod_p \prod_{\alpha \in \Phi_w} \mathbb{N}p^{-2B(\rho+\lambda-w(\rho+\lambda),\mathbf{s})}$$

and, by (34)

$$\mathbb{N}B_i = \mathbb{N}m_i^{-1} \prod_p \mathbb{N}p^{d_{\lambda_p}(w_p^{-1}\alpha_i)-1}.$$

Finally, regarding the normalizing factor, one factor $G_{\alpha_i}(\boldsymbol{s})\zeta_{\alpha_i}(\boldsymbol{s})$ from (18) is needed to normalize $\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0)$ in Proposition 5; the remaining factors are permuted amongst themselves since σ_i permutes $\Phi^+ - \{\alpha_i\}$.

Arguing as in [3] we obtain analytic continuation to any simply-connected region Λ' that is a union of W-translates of the Λ_i obtained by composing functional equations. We may choose Λ' so that its convex hull is all of $\mathbb{C}^r$. The meromorphic continuation to all of $\mathbb{C}^r$ follows from Bochner's tube-domain Theorem (Bochner [1] or Hörmander [9], Theorem 2.5.10). As in [3], one actually applies Bochner's theorem to the function

$$Z_\Psi(\boldsymbol{s}) \prod_{\alpha \in \Phi} \left(B\left(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee\right) - \frac{1}{2n(\alpha)} \right),$$

since inclusion of the factors to cancel the poles of Z_Ψ gives a function that is everywhere analytic. $\square$

We have not proved that the natural map $\mathfrak{W} \longrightarrow W$ is an isomorphism. It is highly likely that this is true, but not too important for the functional equations as we will now see. We recall from the introduction that we defined $\mathfrak{M}'$ to be the quotient of $\mathfrak{M}$ by the kernel of the map $\Psi \longmapsto Z_\Psi$.

Corollary 1 *If $w \in \mathfrak{W}$ is in the kernel of the map $\mathfrak{W} \longrightarrow W$ then $Z_\Psi = Z_{w\Psi}$. Thus there is an action of W on $\mathfrak{M}'$ that is compatible with the action of $\mathfrak{W}$ on $\mathfrak{M}$.*

Proof Since such a w acts trivially on $\mathbb{C}^r$, (37) implies that Z_Ψ and $Z_{w\Psi}$ are multiple Dirichlet series that agree in the region of absolute convergence. Hence they are equal. $\square$

Thus Z_Ψ can be regarded as depending on $\Psi \in \mathfrak{M}'$, on which W acts, and so W can truly be regarded as the group of functional equations of the Weyl group multiple Dirichlet series. This completes the generalization of results from [4].

8 The Gelfand-Tsetlin pattern conjecture

In this section, we turn to the case $\Phi = A_r$ that was investigated in [5]. In this section $\Phi = A_r$ and $\|\alpha\| = 1$ for all roots α. We will then show that the multiple Dirichlet series constructed above (now known to have a group of functional equations according to Theorem 2) are in fact the same as the multiple Dirichlet series given in [5] via a combinatorial prescription in terms of strict Gelfand-Tsetlin patterns.

Let us recall the description of these Weyl group multiple Dirichlet series in [5]. Recall that a Gelfand-Tsetlin pattern is a triangular array of integers

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{array} \right\} \tag{38}$$

where the rows interleave; that is, $a_{i-1,j-1} \geqslant a_{i,j} \geqslant a_{i-1,j}$. The pattern is *strict* if each row is strictly decreasing. The strict Gelfand-Tsetlin pattern $\mathfrak{T}$ in (38) is *left-leaning* at (i,j) if $a_{i,j} = a_{i-1,j-1}$, *right-leaning* at (i,j) if $a_{i,j} = a_{i-1,j}$, and *special* at (i,j) if $a_{i-1,j-1} > a_{i,j} > a_{i-1,j}$.

Given a strict Gelfand-Tsetlin pattern, for $j \geq i$ let

$$s_{ij} = \sum_{k=j}^{r} a_{ik} - \sum_{k=j}^{r} a_{i-1,k}, \tag{39}$$

and define

$$\gamma(i,j) = \begin{cases} \mathbb{N}p^{s_{ij}} & \text{if } \mathfrak{T} \text{ is right-leaning at } (i,j), \\ g(p^{s_{ij}-1}, p^{s_{ij}}) & \text{if } \mathfrak{T} \text{ is left-leaning at } (i,j), \\ \mathbb{N}p^{s_{ij}}(1-\mathbb{N}p^{-1}) & \text{if } (i,j) \text{ is special and } n|s_{ij}; \\ 0 & \text{if } (i,j) \text{ is special and } n \nmid s_{ij}. \end{cases}$$

Also, define

$$G(\mathfrak{T}) = \prod_{j \geqslant i \geqslant 1} \gamma(i,j). \tag{40}$$

Given non-negative integers k_i, l_i, $1 \leq i \leq r$, and a prime p, we define the p-th contribution to the coefficient of a multiple Dirichlet series by

$$H_{GT}(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\mathfrak{T}} G(\mathfrak{T}) \tag{41}$$

where the sum is over all strict Gelfand-Tsetlin patterns $\mathfrak{T}$ with top row

$$l_1 + \ldots + l_r + r, l_2 + \ldots + l_r + r - 1, \cdots, l_r + 1, 0$$

such that for each i, $1 \leq i \leq r$,

$$\sum_{j=i}^{r} (a_{ij} - a_{0,j}) = k_i. \tag{42}$$

Note that $(k_1, \cdots, k_r) = k(\mathfrak{T})$ in the notation of [5]. The general coefficient of the multiple Dirichlet series, $H_{GT}(C_1, \cdots, C_r; m_1, \cdots, m_r)$, is then defined by means of twisted multiplicativity as in (23), (24). In [5] we conjecture that these multiple Dirichlet series have meromorphic continuation and satisfy functional equations. We prove this below for n satisfying the Stability Assumption.

We begin by relating the Stability Assumption to the Gelfand-Tsetlin patterns. We recall from [5] that a strict Gelfand-Tsetlin pattern is *stable* if every entry equals one of the two directly above it (unless, of course, it is in the top row). If the top row is fixed, there are $(r+1)!$ strict stable patterns.

Proposition 7 *Suppose that the Stability Assumption (20) holds. If $\mathfrak{T}$ appearing in the sum (41) is not stable, then $G(\mathfrak{T}) = 0$.*

Proof Suppose that $\mathfrak{T}$ is special at (i, j) and that (20) holds. Recall that s_{ij} is given by (39). Since $a_{i,k} \geq a_{i-1,k}$ for all $k \geq i$ and $a_{i,j} > a_{i-1,j}$ it follows that $s_{ij} > 0$. Similarly, since $a_{i,k} \leq a_{i-1,k-1}$ for all $k \geq i$ and $a_{i,j} < a_{i-1,j-1}$, it follows that

$$s_{ij} = \sum_{k=j}^{r} a_{ik} - \sum_{k=j}^{r} a_{i-1,k} < \sum_{k=j}^{r} a_{i-1,k-1} - \sum_{k=j}^{r} a_{i-1,k} = a_{i-1,j-1} - a_{i-1,r}.$$

Since each entry of $\mathfrak{T}$ is at most $l_1 + \cdots + l_r + r$, it follows that $s_{i,j} < n$. But then $0 < s_{ij} < n$, and this implies that n does not divide s_{ij}. Hence $\gamma(i, j) = 0$, and $G(\mathfrak{T}) = 0$, as claimed. $\square$

We identify the A_r root system with the set of vectors $\mathrm{e}_i - \mathrm{e}_j$ with $i \neq j$ where

$$\mathrm{e}_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{r+1}, \qquad 1 \text{ in the } i\text{-th position.}$$

The simple positive roots are $\alpha_i = \mathrm{e}_i - \mathrm{e}_{i+1}$. The root system lies in the hyperplane V of $\mathbb{R}^{r+1}$ orthogonal to the vector $\sum \mathrm{e}_i$. Particularly $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and the fundamental dominant weights ε_i are given by

$$\rho = \left(\frac{r}{2}, \frac{r}{2} - 1, \cdots, -\frac{r}{2}\right), \qquad \varepsilon_i = (1, \cdots, 1, 0, \cdots, 0) - \frac{i}{r}(1, \cdots, 1),$$

where in the definition of ε_i there are i 1's in the first vector. The action of $W = S_{r+1}$ on vectors in $\mathbb{R}^{r+1}$ is by

$$w(t_1, t_2, t_3, \cdots, t_{r+1}) = (t_{w^{-1}(1)}, t_{w^{-1}(2)}, t_{w^{-1}(3)}, t_{w^{-1}(4)}, \cdots, t_{w^{-1}(r+1)}).$$

Suppose that $\mathfrak{T}$ is a strict Gelfand-Tsetlin pattern with $a_{0r} = 0$. We may find nonnegative integers l_i so that the top row of the pattern is

$$l(r), \cdots, l(0) \tag{43}$$

with

$$l(k) = k + l_{r-k+1} + \cdots + l_r, \qquad 1 \leqslant k \leqslant r, \qquad l(0) = 0.$$

Thus $a_{0j} = l(r - j)$. Let $\lambda = \sum l_i \varepsilon_i$. We call λ the *dominant weight associated with* $\mathfrak{T}$. We will associate a Weyl group element $w \in W$ with each $\mathfrak{T}$ that is stable in the next result.

Proposition 8 *Let $\mathfrak{T}$ be a stable strict Gelfand-Tsetlin pattern with $a_{0r} = 0$, and with associated dominant weight vector λ. Define nonnegative integers $k_1, \cdots, k_r$ by (42) and also let $k_{r+1} = k_0 = 0$. Then there exists a unique element $w \in W = S_{r+1}$ such that*

$$\rho + \lambda - w(\rho + \lambda) = (k_1, k_2 - k_1, k_3 - k_2, \cdots, -k_r) = \sum k_i \alpha_i. \tag{44}$$

In fact, for $0 \leqslant i \leqslant r$

$$k_i - k_{i+1} + l(r - i) = l(r + 1 - w^{-1}(i + 1)) \tag{45}$$

is the unique element in the i-th row that is not in the $(i + 1)$-th row. We have

$$\begin{aligned} \rho + \lambda - w(\rho + \lambda) &= \\ (l(r) - l(r + 1 - w^{-1}(1)), \cdots, l(0) - l(r + 1 - w^{-1}(r + 1))). \end{aligned} \tag{46}$$

Proof Let R be the top row (43) of $\mathfrak{T}$. With $\lambda = \sum l_i \varepsilon_i$ the vector $\rho + \lambda$ differs from the top row (43) by a multiple of $(1, \cdots, 1)$, which is canceled away when we compute $\rho + \lambda - w(\rho + \lambda)$. Thus it is equivalent to show that $R - w(R) = \sum k_i \alpha_i$ for a unique permutation $w \in S_{r+1}$, or in other words, that $R - \sum k_i \alpha_i$ is a permutation of R. By (42)

$$k_i - k_{i+1} + l(r-i) = k_i - k_{i+1} + a_{0,i} = \left(\sum_{j=i}^{r} a_{i,j} \right) - \left(\sum_{j=i+1}^{r} a_{i+1,j} \right).$$

Remembering the pattern is stable, the terms in the second sum may all be found in the first sum, so

$$k_i - k_{i+1} + l(r-i) = a_{ij},$$

where a_{ij} is the unique element of the i-th row that is not in the $(i+1)$-th row. Now (46) and (45) are also clear. $\square$

We now develop some facts necessary to compare the Gelfand-Tsetlin multiple Dirichlet series to the Weyl group multiple Dirichlet series in the twisted stable case.

Lemma 6 *Let $\mathfrak{T}$ be a stable strict Gelfand-Tsetlin pattern with $a_{0r} = 0$. For $j \geq 1$, we have*

$$\begin{aligned} \{a_{jj}, a_{j,j+1}, \cdots, a_{j,r}\} &= \\ \{k_j - k_{j+1} + l(r-j), k_{j+1} - k_{j+2} + l(r-j-1), \cdots, k_{r-1} - k_r + l(1), k_r + l(0)\} &= \\ \{l(r+1-w^{-1}(j+1)), l(r+1-w^{-1}(j+2)), \cdots, l(r+1-w^{-1}(r+1))\}. \end{aligned}$$

Proof The statement follows by induction from the fact that $k_i - k_{i+1} + l(r-i)$ is the unique element of the i-th row that is not in the $(i-1)$-th row. $\square$

For $w \in S_{r+1}$, an *i-inversion* is a j such that $i < j \leq r+1$ but $w(i) > w(j)$.

Proposition 9 *Let $\mathfrak{T}$ be a stable strict Gelfand-Tsetlin pattern with $a_{0r} = 0$, and let $w \in S_{r+1}$ be associated to $\mathfrak{T}$ as in Proposition 8. Then the number of i-inversions of w^{-1} equals the number of left-leaning entries in the i-th row of $\mathfrak{T}$.*

For example, let w^{-1} be the permutation (143), and take $\lambda = 0$. We find that $\rho - w(\rho) = (3, 0, -2, -1) = 3\alpha_1 + 3\alpha_2 + \alpha_3$ and so the corresponding Gelfand-Tsetlin pattern is the unique pattern with $(k_1, k_2, k_3) = (3, 3, 1)$. This pattern is

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} 3 & & 2 & & 1 & & 0 \\ & \hat{3} & & \hat{2} & & \hat{1} & \\ & & \hat{3} & & 1 & & \\ & & & 1 & & & \end{array} \right\},$$

where we have marked the location of the left-leaning entries. The number of i-inversions of w^{-1} is:

i	i-inversions of w^{-1}	number
1	$(1,2), (1,3), (1,4)$	3
2	$(2,3)$	1
3	none	0

As Proposition 9 states, the number of i-inversions of w^{-1} determines the number of left-leaning entries in the i-th row; since the i-inversions are obviously forced to the left in a stable pattern, this number is also the location of the last left-leaning entry.

Proof The i-th row, together with the rows immediately above and below, are:

$$\begin{array}{ccccccccccccc} a_{i-1,i-1} & & a_{i-1,i} & & a_{i-1,i+1} & & \cdots & & a_{i-1,r-1} & & a_{i-1,r} \\ & a_{i,i} & & a_{i,i+1} & & \cdots & & a_{i,r-1} & & a_{i,r} & \\ & & a_{i+1,i+1} & & a_{i+1,i+2} & & \cdots & & a_{i+1,r} & & \end{array}$$

If we assume that there are exactly m left-leaning entries in the i-th row, then

$$a_{i,i} = a_{i-1,i-1}, \cdots, a_{i,i+m-1} = a_{i-1,i-2+m} \tag{47}$$

while

$$a_{i,i+m} = a_{i-1,i+m}, \cdots, a_{i,r} = a_{i-1,r}. \tag{48}$$

The number of i-inversions of w^{-1} is the number of elements of the set

$$\{w^{-1}(i+1), \cdots, w^{-1}(r+1)\}$$

that are less than $w^{-1}(i)$. Since the function l is monotone, this equals the number of elements of the set

$$\left\{l(r+1-w^{-1}(i+1)), \cdots, l(r+1-w^{-1}(r+1))\right\} \tag{49}$$

that are greater than $l(r+1-w^{-1}(i))$. By Lemma 6, the numbers in the set (49) are just the elements of the i-th row of $\mathfrak{T}$, and $l(r+1-w^{-1}(i))$ is the unique element of the $(i-1)$-th row that doesn't occur in the i-th row. Thus the elements of the i-th row that are greater than $l(r+1-w^{-1}(i))$ are precisely the left-leaning entries in the row. $\square$

Theorem 3 *Suppose that $\Phi = A_r$ and that $\langle\ ,\ \rangle$ is chosen so that $\|\alpha\| = 1$ for all $\alpha \in \Phi$. Suppose also that the Stability Assumption (20) holds.*
(i) Let $\mathfrak{T}$ be a stable strict Gelfand-Tsetlin pattern, and let $G(\mathfrak{T})$ be the product of Gauss sums defined in (40). Let w be the Weyl group element associated to $\mathfrak{T}$ in Proposition 8. Then

$$G(\mathfrak{T}) = \prod_{\alpha\in\Phi_w} g(p^{d_\lambda(\alpha)-1}, p^{d_\lambda(\alpha)}),$$

matching the definition as in (25) where $d_\lambda(\alpha)$ is given by (9).
(ii)

$$H(C_1, \cdots, C_r; m_1 \cdots, m_r) = H_{GT}(C_1, \cdots, C_r; m_1 \cdots, m_r).$$

That is, the Weyl group multiple Dirichlet series is the same as the series defined by the Gelfand-Tsetlin description in the twisted stable case.

Proof Since both coefficients are obtained from their prime-power parts by means of twisted multiplicativity, part (i) implies part (ii).

We turn to the proof of part (i). Since $\mathfrak{T}$ is stable, we have $s_{ij} = 0$ if $\mathfrak{T}$ is right-leaning at (i,j). Thus

$$G(\mathfrak{T}) = \prod_{(i,j)\text{left-leaning}} g(p^{s_{ij}-1}, p^{s_{ij}}),$$

where the product is over the left-leaning entries of the Gelfand-Tsetlin pattern corresponding to w whose top row is (43).

It suffices to check that the set of s_{ij} at left-leaning entries in the Gelfand-Tsetlin pattern corresponding to w coincides with the set of $d_\lambda(\alpha)$ as α runs over Φ_w. In fact we shall show a slightly sharper statement, namely that the left-leaning entries in row i correspond exactly to a certain set of roots in Φ_w.

To give this more precisely, we require some notation. Recall that we have identified the roots of A_r with the vectors $\mathrm{e}_i - \mathrm{e}_j$, $1 \leq i \neq j \leq r+1$. The action of a permutation $w \in S_{r+1}$ on the corresponding vectors then becomes:

$$w(\mathrm{e}_i - \mathrm{e}_j) = \mathrm{e}_{w(i)} - \mathrm{e}_{w(j)}.$$

Fix w. Observe that (i,j) is an i-inversion for w^{-1} (that is, $i < j$ but $w^{-1}(j) < w^{-1}(i)$) if and only if the root

$$\alpha_{i,j,w} := \mathrm{e}_{w^{-1}(j)} - \mathrm{e}_{w^{-1}(i)}$$

is in Φ_w. Indeed, $\alpha_{i,j,w}$ is positive if and only if $w^{-1}(j) < w^{-1}(i)$, and $w(\alpha_{i,j,w}) = \mathrm{e}_j - \mathrm{e}_i$, which is negative if and only if $j > i$. We will compute the contribution from the set of $\alpha_{i,j,w}$ for each fixed i.

First, we compute $d_\lambda(\alpha_{i,j,w})$. We have

$$\rho = \frac{1}{2}\sum_{m=1}^{r+1}(r+2-2m)\mathrm{e}_m.$$

Also, since $\alpha_i = \mathrm{e}_i - \mathrm{e}_{i+1}$, we have

$$\alpha_{i,j,w} = \sum_{k=w^{-1}(j)}^{w^{-1}(i)-1} \alpha_k.$$

Recall that $\lambda = \sum_{i=1}^{r} l_i \varepsilon_i$, where $\{\varepsilon_i\}$ are the fundamental dominant weights. Since $\langle \alpha_{i,j,w}, \alpha_{i,j,w}\rangle = 2$, we find that

$$d_\lambda(\alpha_{i,j,w}) = 2\frac{\langle \rho + \lambda, \alpha_{i,j,w}\rangle}{\langle \alpha_{i,j,w}, \alpha_{i,j,w}\rangle} = w^{-1}(i) - w^{-1}(j) + \sum_{k=w^{-1}(j)}^{w^{-1}(i)-1} l_k.$$

Now we consider the set of $d_\lambda(\alpha_{i,j,w})$ as j varies over the numbers such that (i,j) is a i-inversion for w^{-1}. We see that $w^{-1}(j)$ runs through the set

$$\{1, \cdots, w^{-1}(i) - 1\} - \{w^{-1}(1), \cdots, w^{-1}(i-1)\},$$

where as usual if X and Y are sets then $X - Y = \{x \in X | x \notin Y\}$. Let

$$D_i = \{d_\lambda(\alpha_{i,j,w}) \mid (i,j) \text{ is an } i\text{-inversion for } w^{-1}\}.$$

Then we obtain the following value for the set D_i:

$$\begin{aligned} D_i = \quad & \{1 + l_{w^{-1}(i)-1}, 2 + l_{w^{-1}(i)-2} + l_{w^{-1}(i)-1}, \cdots, w^{-1}(i) - 1 + l_1 + \cdots + l_{w^{-1}(i)-1}\} \\ & -\{w^{-1}(i) - w^{-1}(j) + l_{w^{-1}(j)} + \cdots l_{w^{-1}(i)-1} \mid j < i \text{ and } w^{-1}(j) < w^{-1}(i)\}. \end{aligned}$$

Now we turn to the Gauss sums obtained from the Gelfand-Tsetlin pattern. Suppose that there are b_i i-inversions for w^{-1}. By Proposition 9 the first b_i entries of the i-th row are the left-leaning entries, and the nontrivial Gauss sums in the i-th row come from the quantities s_{ij}, $i \leqslant j \leqslant i+b_i-1$. Recall that every entry in the stable strict Gelfand-Tsetlin pattern is either left-leaning or right-leaning. We thus have $a_{ij} = a_{i-1,j-1}$ for $i \leqslant j \leqslant i+b_i-1$ and $a_{ij} = a_{i-1,j}$ for $j \geqslant i+b_i$. The sum for s_{ij} telescopes:

$$\begin{aligned} s_{ij} &= (a_{ij} - a_{i-1,j}) + (a_{i,j+1} - a_{i-1,j+1}) + \cdots + (a_{ir} - a_{i-1,r}) \\ &= (a_{i-1,j-1} - a_{i-1,j}) + (a_{i-1,j} - a_{i-1,j+1}) + \cdots \\ &\qquad + (a_{i-1,b_i+i-2} - a_{i-1,b_i+i-1}) + 0 + \cdots + 0 \\ &= a_{i-1,j-1} - a_{i-1,b_i+i-1}. \end{aligned}$$

By Lemma 6, we have

$$a_{i-1,i+b_i-1} = r + 1 - w^{-1}(i) + l_{w^{-1}(i)} + \cdots + l_r.$$

To compute the s_{ij} as j varies, we must subtract this quantity from $a_{i-1,i-1+k}$ for each k, $0 \leqslant k \leqslant b_i - 1$. So we must compute the quantities $a_{i-1,i-1+k}$. Recall that the 0-th row of the Gelfand-Tsetlin pattern is $\{l(r), l(r-1), \cdots, l(0)\}$. By Lemma 6 again, the entries of the $(i-1)$-th row of the Gelfand-Tsetlin pattern are given by

$$\{l(r), \cdots, l(0)\} - \{l(r+1-w^{-1}(m)) \mid 1 \leqslant m \leqslant i-1\}. \tag{50}$$

We need to specify the b_i entries with largest argument in the set (50); these are the elements from which we will subtract the term $l(r+1-w^{-1}(i))$. We have

$$b_i = |\{j > i \mid w^{-1}(j) < w^{-1}(i)\}| = w^{-1}(i) - 1 - |\{j < i \mid w^{-1}(j) < w^{-1}(i)\}|.$$

Let $h_1, \cdots, h_{b_i}$ be the integers in the interval $[1, w^{-1}(i)-1]$ that are not of the form $w^{-1}(j)$ with some j, $j < i$. These are the only integers h in the interval $[1, w^{-1}(i)-1]$ such that $l(r+1-h)$ is not removed from the $(i-1)$-th row of the Gelfand-Tsetlin pattern, by (50). Hence the only terms of the form $l(r+1-k)$ with $1 \leqslant k < w^{-1}(i)$ that are in the Gelfand-Tsetlin pattern are exactly the numbers of the form $l(r+1-h_m)$, $1 \leqslant m \leqslant b_i$. Since these are visibly the b_i entries with largest argument, we have determined the entries $a_{i-1,i-1+k}$, $0 \leqslant k \leqslant b_i - 1$. We have

$$l(r+1-h_m) - l(r+1-w^{-1}(i)) = w^{-1}(i) - h_m + l_{h_m} + \cdots + l_{w^{-1}(i)-1}.$$

As m varies from 1 to b_i, these quantities give exactly the set D_i that we obtained above from the formula for the coefficients (25).

This completes the proof of Theorem 3. □

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