Weyl Group Multiple Dirichlet Series II: The Stable Case

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Abstract. To each reduced root system Φ of rank r, and each sufficiently large integer n, we define a family of multiple Dirichlet series in r complex variables, whose group of functional equations is isomorphic to the Weyl group of Φ. The coefficients in these Dirichlet series exhibit a multiplicativity that reduces the specification of the coefficients to those that are powers of a single prime p. For each p, the number of nonzero such coefficients is equal to the order of the Weyl group, and each nonzero coefficient is a product of n-th order Gauss sums. The root system plays a basic role in the combinatorics underlying the proof of the functional equations.

1. Introduction

Let V be a real vector space of dimension r, and let $\Phi \subset V$ be a reduced root system spanning V. Let n be an integer > 1, and let F be an algebraic number field that contains the group μ_n of n-th roots of unity; we assume also that -1 is an n-th power in F, which implies that it is totally complex. With these data we described in [5] a family of Dirichlet series in several complex variables, to be called *Weyl group multiple Dirichlet series*. Their coefficients involve n-th order Gauss sums. In this paper we will give a complete proof of the analytic continuation and functional equations of these multiple Dirichlet series subject to the *stability assumption*, which states that n is sufficently large, depending on Φ. The stability assumption is stated precisely in Section 3.

We recall that the dual space $V^\vee$ of V contains a root system $\Phi^\vee$, whose elements are called *coroots*. (See, for example, Springer [20], Section 1.) There is a bijection $\alpha \longmapsto \alpha^\vee$ from $\Phi \longrightarrow \Phi^\vee$ that sends long roots to short roots and conversely. We write the dual pairing $V \times V^\vee \longrightarrow \mathbb{R}$ as $(x, y) \longmapsto B(x, y)$. We have $B(\alpha, \alpha^\vee) = 2$. The simple reflection $\sigma_\alpha : V \longrightarrow V$ corresponding to α is given by $\sigma_\alpha(x) = x - B(x, \alpha^\vee)\,\alpha$, and similarly $\sigma_{\alpha^\vee} : V^\vee \longrightarrow V^\vee$ is defined by $\sigma_{\alpha^\vee}(x) = x - B(\alpha, x)\,\alpha^\vee$. We have $\sigma_\alpha(\Phi) = \Phi$ and $\sigma_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$.

Let S be a finite set of places of F containing the archimedean ones, all those which are ramified over $\mathbb{Q}$ (in particular those dividing n) and enough others so that the ring $\mathfrak{o}_S$ of S-integers is a principal ideal domain. We embed $\mathfrak{o}_S$ into $F_S = \prod_{v \in S} F_v$ along the diagonal. It is discrete and cocompact.

We will exhibit a finite-dimensional vector space $\mathcal{M}(\Omega^r)$ of functions on F_S^r, to be defined below in (8), and for each function $\Psi \in \mathcal{M}(\Omega^r)$, we will exhibit a function Z_Ψ on $V_{\mathbb{C}}^\vee = \mathbb{C} \otimes V^\vee$. Each $Z_\Psi(\boldsymbol{s})$ will be a multiple Dirichlet series, defined as a sum over r-tuples of ideals of $\mathfrak{o}_S$, convergent when the real part $\Re(B(\alpha, \boldsymbol{s})) > \frac{3}{4}$ for every simple positive root α. These are the *Weyl group multiple Dirichlet series* of the title. After multiplying by a normalizing factor, consisting of some gamma and zeta functions, one obtains the normalized function $Z_\Psi^*(\boldsymbol{s})$, whose analytic continuation and functional equations we will prove.

The linear map $\Psi \longmapsto Z_\Psi$ may naturally be extended to $\mathfrak{M} = \mathcal{A} \otimes \mathcal{M}(\Omega^r)$, where $\mathcal{A}$ is the ring of (Dirichlet) polynomials in the $q_v^{\pm 2B(\alpha,\boldsymbol{s})}$, where v runs through the finite set of nonarchimedean places in S, α runs through the set of simple positive roots and q_v denotes the cardinality of the residue field. Let $\mathfrak{M}'$ be the quotient of $\mathfrak{M}$ by the kernel of the map $\Psi \longmapsto Z_\Psi$. Let W be the Weyl group of Φ, which acts on V and $V^\vee$. We will define an action of W on $\mathfrak{M}'$.

Let $\rho^\vee$ be half the sum of the positive coroots. There is naturally a linear action of W on $V^\vee$, in which W appears as the group generated by the $\sigma_{a^\vee}$. However we will modify this action by moving the center to $\frac{1}{2}\rho^\vee$. Thus $w \in W$ induces a transformation $V_{\mathbb{C}}^\vee \longrightarrow V_{\mathbb{C}}^\vee$, also to be denoted w, defined by requiring that

$$B\left(w\alpha, w(\boldsymbol{s}) - \frac{1}{2}\rho^\vee\right) = B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right), \tag{1}$$

and $\frac{1}{2}\rho^\vee$ is fixed by this action.

Our main result is the following.

Theorem. *The function $Z_\Psi^*(\boldsymbol{s})$, originally convergent when $\Re(B(\alpha, \boldsymbol{s})) > \frac{3}{4}$ for all simple positive roots α, has meromorphic continuation to all $V_{\mathbb{C}}^\vee$ and satisfies functional equations*

$$Z_{w\Psi}^*(w\boldsymbol{s}) = Z_\Psi^*(\boldsymbol{s})$$

when $w \in W$, $\Psi \in \mathfrak{M}'$ and $\boldsymbol{s} \in V_{\mathbb{C}}^\vee$.

A more precise version of this theorem may be found in Theorem 5.8. Our immediate goal is to give the definition of Z_Ψ.

We will introduce coordinates on $V_{\mathbb{C}}^\vee$ as follows. Let $\Delta = \{\alpha_1, \cdots, \alpha_r\}$ be the set of simple positive roots. Define an isomorphism $V_{\mathbb{C}}^\vee \longrightarrow \mathbb{C}^r$ by mapping $\boldsymbol{s} \in V_{\mathbb{C}}^\vee$ to

$$(s_1, \cdots, s_r), \qquad s_i = B(\alpha_i, \boldsymbol{s}).$$

We may identify $V_{\mathbb{C}}^\vee$ with $\mathbb{C}^r$ and write $\boldsymbol{s} = (s_1, \cdots, s_r)$. The Weyl group multiple Dirichlet series has the form

$$Z_\Psi(\boldsymbol{s}) = Z_\Psi(s_1, \cdots, s_r) = \sum_{\mathfrak{c}_1, \cdots, \mathfrak{c}_r} H\Psi(\mathfrak{c}_1, \cdots, \mathfrak{c}_r)\ \mathbb{N}\mathfrak{c}_1^{-2s_1} \cdots \mathbb{N}\mathfrak{c}_r^{-2s_r},$$

where the sum is over nonzero ideals $\mathfrak{c}_1, \cdots, \mathfrak{c}_r$ of $\mathfrak{o}_S$. The coefficients in Z thus have two parts, to be denoted $H(C_1, \cdots, C_r)$ and $\Psi(C_1, \cdots, C_r)$, defined for nonzero $C_i \in \mathfrak{o}_S$. The product $H\Psi$ is unchanged if C_i is multiplied by a unit, and so is a function of r-tuples of ideals in the principal ideal domain $\mathfrak{o}_S$.

The function H is more arithmetically interesting and important than Ψ. To specify it, we require first that it obeys a "twisted" multiplicativity. We will denote by $\left(\frac{c}{d}\right)$

the n-th power residue symbol, defined when c and d are coprime elements of $\mathfrak{o}_S$ and $\gcd(n,d)=1$. It depends only on c modulo d, and satisfies the reciprocity law

$$\left(\frac{c}{d}\right) = (d,c)_S \left(\frac{d}{c}\right),$$

where $(d,c)_S \in \mu_n$ denotes the S-Hilbert symbol, defined for $c,d \in F_S^\times$. (The properties of the Hilbert and power residue symbols will be reviewed later in Section 2.) If

$$\gcd(C_1 \cdots C_r, C_1' \cdots C_r') = 1, \tag{2}$$

then

$$\frac{H(C_1C_1', \cdots, C_rC_r')}{H(C_1,\cdots,C_r)\, H(C_1',\cdots,C_r')} = \prod_{i=1}^{r} \left(\frac{C_i}{C_i'}\right)^{\|\alpha_i\|^2} \left(\frac{C_i'}{C_i}\right)^{\|\alpha_i\|^2} \prod_{i<j} \left(\frac{C_i}{C_j'}\right)^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{C_i'}{C_j}\right)^{2\langle\alpha_i,\alpha_j\rangle}. \tag{3}$$

Here we have chosen a Weyl group-invariant inner product $\langle\alpha,\beta\rangle$ on V that is normalized so that if $\alpha,\beta \in \Phi$ then $\|\alpha\|^2 = \langle\alpha,\alpha\rangle$ and $2\langle\alpha,\beta\rangle$ are integers. In particular, if the inner product is chosen so that all short roots have length 1, then this is true. For example, if Φ is of type A_2 and the roots are normalized to have length 1 then the twisted multiplicativity condition states that

$$H(C_1C_1', C_2C_2') = H(C_1,C_2)H(C_1',C_2') \left(\frac{C_1}{C_1'}\right) \left(\frac{C_1'}{C_1}\right) \left(\frac{C_2}{C_2'}\right) \left(\frac{C_2'}{C_2}\right) \left(\frac{C_1}{C_2'}\right)^{-1} \left(\frac{C_1'}{C_2}\right)^{-1}.$$

The "twistedness" of the multiplicativity (3) means that $Z_\Psi(\boldsymbol{s})$ is not an Euler product. On the other hand, this multiplicativity reduces specification of the coefficients H to the case where the C_i are all powers of the same prime. We describe these next.

We choose a nontrivial additive character ψ of F_S such that $\psi(x\mathfrak{o}_S)=1$ if and only if $x \in \mathfrak{o}_S$. (See Brubaker and Bump [3], Lemma 1.) If t is any positive integer and $a,c \in \mathfrak{o}_S$ we define the Gauss sum

$$g_t(a,c) = \sum_{d \bmod c} \left(\frac{d}{c}\right)^t \psi\left(\frac{ad}{c}\right). \tag{4}$$

The properties of Gauss sums are recalled later in Section 2.

For each prime p of $\mathfrak{o}_S$, the coefficient $H(p^{k_1},\cdots,p^{k_r})=0$ except for a finite number of particular values of $(k_1,\cdots,k_r)$; these values are in one-to-one correspondence with the elements of the Weyl group W, which acts on V. To describe these nonzero values, let $\rho \in V$ be half the sum of the positive roots. Given $w \in W$, the value of $(k_1,\cdots,k_r)$ corresponding to w is determined by the condition that

$$\rho - w\rho = \sum_{\alpha_i \in \Delta} k_i \alpha_i.$$

Define a function d on the set Φ^+ of positive roots, taking values in the positive integers, by

$$d(\alpha) = \frac{2\langle\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} = B(\rho,\alpha^\vee). \tag{5}$$

Let Φ_w be the set of all positive roots α such that $w(\alpha)$ is a negative root. The cardinality of Φ_w is equal to the length $l(w)$ of w in the Weyl group. Then

$$H(p^{k_1}, \dots, p^{k_r}) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{d(\alpha)-1}, p^{d(\alpha)}). \tag{6}$$

If $w = 1$ then $\Phi_w = \varnothing$, so $H(1, \cdots, 1) = 1$. By Proposition 2.6 (iv) we can also write

$$H(p^{k_1}, \dots, p^{k_r}) = \prod_{\alpha \in \Phi_w} \mathbb{N}p^{d(\alpha)-1} \cdot g_{d(\alpha)\|\alpha\|^2}(1, p).$$

Let Ω be a subgroup of $F_S^\times$. We say that Ω is *isotropic* if $(\varepsilon, \delta)_S = 1$ for $\varepsilon, \delta \in \Omega$. In particular

$$\Omega_0 = \mathfrak{o}_S^\times F_S^{\times, n}, \tag{7}$$

where $F_S^{\times, n}$ is the subgroup of n-th powers in $F_S^\times$, is isotropic (Lemma 2 of [3]). Note that this requires our assumption that -1 is an n-th power. We will fix an isotropic subgroup Ω containing Ω_0. Let $\mathcal{M}(\Omega^r)$ be the space of functions on $\Psi : (F_S^\times)^r \longrightarrow \mathbb{C}$ that satisfy

$$\Psi(\varepsilon_1 C_1, \cdots, \varepsilon_r C_r) = \prod_{i=1}^{r} (\varepsilon_i, C_i)_S^{\|\alpha_i\|^2} \left\{ \prod_{i<j} (\varepsilon_i, C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \right\} \Psi(C_1, \cdots, C_r) \tag{8}$$

when $\varepsilon_1, \cdots, \varepsilon_r \in \Omega$ and $C_i \in F_{\text{fin}}^\times$.

Lemma 1.1. *The dimension of* $\mathcal{M}(\Omega^r)$ *is* $|(\Omega \backslash F_{\text{fin}}^\times)^r| < \infty$. *Thus* $\dim \mathcal{M}(\Omega^r) > 0$.

Proof. Using the fact that Ω is isotropic, it is easy to see that the value Ψ may be specified arbitrarily on a set of representatives of $\Omega^r \backslash (F_{\text{fin}}^\times)^r$. □

Lemma 1.2. *If* $\varepsilon_1, \cdots, \varepsilon_r \in \mathfrak{o}_S$ *then*

$$H(\varepsilon_1 C_1, \cdots, \varepsilon_r C_r) \Psi(\varepsilon_1 C_1, \cdots, \varepsilon_r C_r) = H(C_1, \cdots, C_r) \Psi(C_1, \cdots, C_r).$$

Proof. In (3) we may take $C_i' = \varepsilon_i$. The statement follows from (8) and Lemma 2.5 below. □

Remark 1.3. The definitions (8) and (3) depend on the choice of ordering of the simple roots α_i for $i = 1, \dots, r$. In all that follows, we will assume that such a choice is fixed. If one took a different ordering of the roots, then the residue symbols and Hilbert symbols appearing in all subsequent definitions would be slightly altered, but the space of multiple Dirichlet series that we define would be unchanged, since there would be compensating changes to both H and the space of Ψ in (9).

We now define

$$Z_\Psi(\boldsymbol{s}) = Z_\Psi(s_1, \cdots, s_r) = \sum_{0 \neq C_1 \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S} \cdots \sum_{0 \neq C_r \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S} H(C_1, \cdots, C_r)\, \Psi(C_1, \cdots, C_r)\, \mathbb{N}C_1^{-2s_1} \mathbb{N}C_2^{-2s_2} \cdots \mathbb{N}C_r^{-2s_r}. \tag{9}$$

By Lemma 1.2 the summand is unchanged if C_i is changed by a unit, so Z_Ψ is well-defined and may be regarded as a sum over r-tuples of ideals of $\mathfrak{o}_S$. The series is absolutely convergent if $\Re(s_i) > \frac{3}{4}$. We will describe the normalizing factor in Section 3.3.

The method by which we prove our main theorem is based on a theorem of several complex variables due to Bochner [1]. This is similar to the method by which Jacquet proved the functional equations of Whittaker functions in [14]; the same method underlies the reduction of the functional equations of Eisenstein series to the rank one case, and goes back to Selberg [19].

This paper is one of several recent ones on this subject. The paper of Brubaker, Bump, Chinta, Friedberg and Hoffstein [5] gives specific examples and an expanded discussion; it can be regarded as an introduction to this paper. The paper of Brubaker and Bump [3] treats the case $\Phi = A_1$ (in Cartan's classification) and contains a specific foundational result that we will require, while [5] discusses A_2 and C_2. The paper [4] shows how a double Dirichlet series previously defined by Friedberg, Hoffstein and Lieman [11] can be regarded as a residue of a Weyl group multiple Dirichlet series of type A_r. The paper of Brubaker, Bump, Friedberg and Hoffstein [6] returns to A_2, introducing instability through twisting; it also marks progress towards the removal of the stability assumption, giving a conjectural description of the Weyl group multiple Dirichlet series when $\Phi = A_r$ that is valid for all n. This description expresses the coefficients of the multiple Dirichlet series as a sum of products of Gauss sums indexed by strict Gelfand-Tsetlin patterns (or alternating sign matrices), and has some common characteristics with the formula of Casselman and Shalika [9] for spherical Whittaker functions. Using a different approach, the earlier paper of Chinta [10] applies the case $n = 2$ and $\Phi = A_5$ to a problem in analytic number theory; this example is outside the stable range. What is specific to the current paper is the complete proof of the analytic continuation and functional equations of the Weyl group multiple Dirichlet series for general Φ in the stable range.

We believe that these multiple Dirichlet series can be understood in the context of automorphic forms. More precisely, let G be a simply connected algebraic group over F whose root system is the *dual* of the root system Φ; in other words, Φ is the root system of the L-group ${}^LG$.

Conjecture 1.4. The Weyl group multiple Dirichlet series $Z_\Psi(s_1, \cdots, s_r)$ is a Whittaker coefficient of an Eisenstein series on an n-fold metaplectic cover $\tilde{G}$ of G.

Evidence for this conjecture is described in [6]. In summary,

- The normalizing factors and groups of functional equations for Z_Ψ are the same as for the Eisenstein series on $\tilde{G}$. The normalizing factors can be computed by the formula of Gindikin and Karpelevich; this computation is similar to the normalization of the intertwining integrals on the metaplectic group in Kazhdan and Patterson [15], Proposition I.2.4. The normalizing factors are in bijection with the positive roots, and their form depends on $n(\alpha)$ as defined in Section 3, precisely as in (31).
- It is clear that the Whittaker coefficients of metaplectic Eisenstein series are multiple Dirichlet series whose coefficients involve Gauss sums.
- We compute the Whittaker coefficients of Eisenstein series on the metaplectic covers of GL_3 in [6], and we find that they agree with the Weyl group multiple Dirichlet series of type A_2 as defined in this paper. We refer to [6] for further discussion of the conjecture.

It should be noted that while Conjecture 1.4 would imply a version of Theorem 5.8, the precise statement of Theorem 5.8 would be difficult to obtain by that method. Instead, we rely on the precise results in Brubaker and Bump [3] that are obtained by proving local functional equations for Dirichlet series arising from test data with small support in the big Bruhat cell of SL_2. Adapting this argument to an arbitrary G would be harder than the argument we give here. Thus we believe that the best way to study the analytic properties of the Weyl group multiple Dirichlet series is that of this paper, though the connection with Eisenstein series is important for philosophical reasons.

The Weyl group multiple Dirichlet series unify and extend many examples that have previously been treated on a case-by-case basis. This is particularly true when one considers residues of these multiple Dirichlet series. Some of these instances are surveyed in the final Section of [5], and another instructive one in [4].

To provide the reader with a glimpse of the possibilities that appear when one considers the residues of Weyl group multiple Dirichlet series, we discuss one particular example in detail – a connection with the still-mysterious Whittaker-Fourier coefficients of the generalized theta series of Kubota [16] and Kazhdan and Patterson [15]. These automorphic forms, which are residues of Eisenstein series on the n-fold metaplectic cover of GL_r, are intimately connected with the reciprocity laws. Their Fourier-Whittaker coefficients are incompletely known except when $r = n$ or $n - 1$, in which case they are products of Gauss sums. Yet there is strong evidence that unproved relations exist between these coefficients, such as the one conjectured by Patterson [18], and those conjectured by Bump and Hoffstein [8].

The Fourier coefficients $\tau(c)$ of the generalized theta series $\theta^{(4)}$ on the 4-fold cover of GL(2) were studied by Suzuki [21], who was able to evaluate only some of these coefficients, for reasons which eventually were understood as relating to the nonuniqueness of Whittaker models. Still Patterson [18] was able to make a conjecture about the nature of these coefficients. Despite partial proofs by Suzuki in [22] and [23], the Patterson conjecture remains unproved. We will explain that the conjecture amounts to a relationship between specializations of two different Weyl group multiple Dirichlet series.

Since different root systems Φ and different values of n will appear in the following discussion, we will denote $Z_\Psi(\boldsymbol{s})$ by $Z(\boldsymbol{s};\Phi^{(n)})$; we suppress the Ψ since it plays no important role in this heuristic discussion. The function $Z(s_1, s_2, s_3; A_3^{(n)})$ is in the stable range if $n \geqslant 3$. It is a function of three complex variables, s_1, s_2 and s_3, and taking the residue in s_3 yields a function of two complex variables. One can compute this residue using Proposition 5.5; that proposition expresses Z in terms of Kubota Dirichlet series $\mathcal{D}$ introduced below in Section 2.3. The coefficients τ appear as a consequence of taking the residue of the Kubota Dirichlet series. One may then take a second residue in s_1 obtaining a Dirichlet series $\Upsilon^{(n)}(s)$ in one variable $s = s_2$. The Dirichlet series $\Upsilon^{(n)}(s)$ is not multiplicative, but for the purpose of discussion and comparison with Bump and Hoffstein [8] we give its p-part, where p is a prime of F. Using results of this paper this can be computed to be

$$1 + g(1,p)\mathbb{N}p^{-\frac{1}{n}-2s_2}\tau(p)^2 + 2g(1,p)g(p,p^2)\mathbb{N}p^{-1-\frac{2}{n}-4s_2}\tau(p^2) +$$
$$g(1,p)^2g(p^2,p^3)\mathbb{N}p^{-2-\frac{3}{n}-6s_2}\tau(p)^2 + g(1,p)g(p,p^2)^2g(p^2,p^3)\mathbb{N}p^{-4-\frac{4}{n}-8s_2}.$$

The Patterson conjecture states that when $n = 4$, we have $\tau(p)^2 = 2\,\overline{g(1,p)}\,\mathbb{N}p^{-1/2}$.

The Dirichlet series $\Upsilon^{(4)}(s)$ was constructed by Bump and Hoffstein [8] as a Whittaker coefficient of a metaplectic Eisenstein series on the 4-fold metaplectic cover of GL_4, induced from two copies of $\theta^{(4)}$. We note that this equivalence of $\Upsilon(s)$ with their Dirichlet series can be checked directly, but it would follow from Conjecture 1.4, and so it is evidence for Conjecture 1.4.

Bump and Hoffstein [8] described a relationship that can be expressed in the equation

$$\Upsilon^{(4)}(s) = \zeta_F(8s-2)\,\zeta_F(8s-3)\,Z\left(s+\tfrac{1}{8}, s+\tfrac{1}{8}; A_2^{(2)}\right).$$

Much evidence supports this guess, which is equivalent to the Patterson conjecture.

Let us make a remark on the poles of these Dirichlet series. It follows from Theorem 5.8 that $\Upsilon^{(4)}$ has the following poles:

s	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
multiplicity	1	2	2	2	1

Dividing by $\zeta_F(8s-2)\zeta_F(8s-3)$, this gives a Dirichlet series with the following poles:

s	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$
multiplicity	1	1	1	1

(10)

Also by Theorem 5.8 the function

$$Z(s_1, s_2; A_2^{(2)})$$

has poles along the lines $s_1, s_2, s_1+s_2-\frac{1}{2} = \frac{1}{4}, \frac{3}{4}$. This would suggest that $Z(s+\frac{1}{8}, s+\frac{1}{8}; A_2^{(2)})$ should have the poles in (10), but that the poles at $\frac{1}{8}$ and $\frac{5}{8}$ should be double. That this doubling of the poles is illusory can be seen from Conjecture 1.4 as follows. The Taylor expansion of $Z(s_1, s_2; A_2^{(2)})$ near $s_1 = s_2 = \frac{3}{4}$ has the form $F(s_1, s_2)(s_1-\frac{3}{4})^{-1}(s_2-\frac{3}{4})^{-1}$, and by Conjecture 1.4 the function $F(s_1, s_2)$ should be a Whittaker coefficient of the theta function on the double cover of GL_3. Yet Kazhdan and Patterson [15] have shown that this theta function has no Whittaker model, so $F(\frac{3}{4}, \frac{3}{4}) = 0$. This causes the disappearance of one of the poles of

$$Z\left(s+\tfrac{1}{8}, s+\tfrac{1}{8}; A_2^{(2)}\right) = F\left(s+\tfrac{1}{8}, s+\tfrac{1}{8}\right)\left(s-\tfrac{5}{8}\right)^{-2},$$

consistent with the poles of $\zeta_F(8s-2)^{-1}\zeta_F(8s-3)^{-1}\Upsilon^{(4)}(s)$ described above.

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2. Preliminaries

In this section we collect some foundational material, and describe our basic setup.

Let $n > 1$ be an integer and let F, S, $\mathfrak{o}_S$, ψ and F_S be as in the introduction. We recall that the S-integer ring $\mathfrak{o}_S$ is the set of elements $\alpha \in F$ such that $|\alpha_v|_v \leqslant 1$ for all places v of F not in S. We will partition $S = S_\infty \cup S_{\text{fin}}$, where S_∞ is the set of archimedean places, and S_{fin} are the remaining nonarchimedean ones. We will denote

$$F_S = \prod_{v \in S} F_v \quad \text{with} \quad F_\infty = \prod_{v \in S_\infty} F_v, \qquad F_{\text{fin}} = \prod_{v \in S_{\text{fin}}} F_v.$$

2.1. Hilbert symbol

Remark 2.1. In what follows, we will maintain an ambiguity whether the group μ_n is a subgroup of $F^\times$ or of $\mathbb{C}^\times$. To be correct, μ_n should be regarded as a subgroup of $F^\times$, and there will be fixed an embedding $j : \mu_n \longrightarrow \mathbb{C}^\times$; we suppress j from the notation.

If v is a place of F, the v-Hilbert symbol $(\ ,\ )_v$ is a map $F_v^\times \times F_v^\times \longrightarrow \mu_n$. We will take the symbol to be the *inverse* of the symbol defined by Neukirch [17].

Proposition 2.2. *(i) The Hilbert symbol is a skew-symmetric bilinear pairing on $F_v^\times$. That is,*

$$\begin{aligned}(aa', b)_v &= (a, b)_v(a', b)_v,\\ (a, bb')_v &= (a, b)_v(a, b')_v,\\ (a, b)_v &= (b, a)_v^{-1}.\end{aligned}$$

(ii) We have $(a, b)_v = 1$ if and only if a is a norm from $F_v(b^{1/n})$.
(iii) We have $(a, -a)_v = 1$, and if $a \neq 1$ we have $(a, 1-a)_v = 1$.
(iv) If v is complex, then $(a, b)_v = 1$ for all $a, b \in F_v^\times$.
(v) If v is nonarchimedean and n is prime to the residue characteristic, then $(a, b)_v = 1$ when a, b are both elements of the group $\mathfrak{o}_v^\times$ of units of the ring $\mathfrak{o}_v$ of integers in F_v.

Proof. See Neukirch [17], Chapter V Section 3, particularly Proposition 3.2 on p. 334. □

Theorem 2.3 (Hilbert). *If $a, b \in F^\times$, then*

$$\prod_v (a, b)_v = 1. \tag{11}$$

This is the *Hilbert reciprocity law.*

Proof. See Neukirch [17] Chapter VI, Theorem 8.1 on p. 414. □

Let $F_S = \prod_{v\in S} F_v$. Define a pairing $(\ \ ,\ \ )_S : F_S^\times \times F_S^\times \longrightarrow \mu_n$ by

$$(a, b)_S = \prod_{v\in S} (a, b)_v.$$

If $a, b \in \mathfrak{o}_S$ then we may also write

$$(a, b)_S = \prod_{v\notin S} (a, b)_v^{-1}$$

by the Hilbert reciprocity law (11). We will refer to this pairing $(\ ,\ )_S$ as the *S-Hilbert Symbol*.

We recall from the introduction that a subgroup Ω of $F_S^\times$ is called *isotropic* if $(\varepsilon, \delta)_S = 1$ for $\varepsilon, \delta \in \Omega$. In this paper we fix an isotropic subgroup $\Omega \supset \Omega_0$, where the isotropic subgroup Ω_0 was defined in (7). There are advantages to taking Ω to be maximal isotropic, but we will not need this assumption. If t is a positive integer, let $\mathcal{M}_t(\Omega)$ be the vector space of functions $\Psi : F_{\mathrm{fin}}^\times \longrightarrow \mathbb{C}$ that satisfy

$$\Psi(\varepsilon c) = (c, \varepsilon)_S^{-t}\Psi(c), \tag{12}$$

when $\varepsilon \in \Omega$. We denote $\mathcal{M}_1(\Omega)$ by $\mathcal{M}(\Omega)$. This is consistent with (8). Note that if ε is sufficiently close to the identity in $F_{\mathrm{fin}}^\times$ it is an n-th power at every place in S_{fin}, so such a function is locally constant. It is easy to see that the dimension of $\mathcal{M}(\Omega)$ is $[F_S^\times : \Omega] < \infty$.

2.2. Power residue symbol

If $a \in \mathfrak{o}_S$ and $\mathfrak{b}$ is an ideal of $\mathfrak{o}_S$ then the *power residue symbol* $\left(\frac{a}{\mathfrak{b}}\right)$ is defined as follows. If a is not coprime to $\mathfrak{b}$ then it is defined to be zero. If $\mathfrak{b} = \mathfrak{p}$ is prime and a is coprime to $\mathfrak{p}$, then $\left(\frac{a}{\mathfrak{p}}\right)$ is defined to be the unique element of μ_n such that

$$\left(\frac{a}{\mathfrak{p}}\right) \equiv a^{(\mathbb{N}\mathfrak{p}-1)/n} \mod \mathfrak{p}.$$

Note that $(\mathbb{N}\mathfrak{p} - 1)/n$ is an integer since we are assuming that $\mu_n \subset F$. Finally, the definition of $\left(\frac{a}{\mathfrak{b}}\right)$ is extended to all $\mathfrak{b}$ by multiplicativity, that is, by the condition

$$\left(\frac{a}{\mathfrak{bc}}\right) = \left(\frac{a}{\mathfrak{b}}\right)\left(\frac{a}{\mathfrak{c}}\right).$$

If $a, b \in \mathfrak{o}_S$ we will also denote by $\left(\frac{a}{b}\right) = \left(\frac{a}{b\mathfrak{o}_S}\right)$ where $b\mathfrak{o}_S$ is the principal ideal generated by b.

Proposition 2.4. (Gauss, Eisenstein, Kummer, Hilbert) *If a and b are coprime then*

$$\left(\frac{a}{b}\right) = (b, a)_S \left(\frac{b}{a}\right). \tag{13}$$

Proof. See Neukirch [17], Theorem 8.3 of Chapter 6. □

Lemma 2.5. *If $\varepsilon \in \mathfrak{o}_S^\times$ and $0 \neq c \in \mathfrak{o}_S$ then*

$$\left(\frac{\varepsilon}{c}\right) = (c, \varepsilon)_S. \tag{14}$$

Proof. This follows from (13) since ε is a unit, so $\left(\frac{c}{\varepsilon}\right) = 1$. □

If $a, c \in \mathfrak{o}_S$ and $c \neq 0$, and if t is a positive integer, define the Gauss sum $g_t(a, c)$ by (4). We will also denote $g_1(a, c) = g(a, c)$.

Proposition 2.6. *The Gauss sum has the following properties.*
(i) We have

$$g(m, cc') = \left(\frac{c}{c'}\right)\left(\frac{c'}{c}\right) g(m, c)\, g(m, c'), \qquad \text{if } c, c' \text{ are coprime.}$$

(ii) We have

$$g(am, c) = \left(\frac{a}{c}\right)^{-1} g(m, c) \qquad \text{if } a, c \text{ are coprime.}$$

(iii) If $\varepsilon \in \mathfrak{o}_S^\times$ we have

$$g(a, \varepsilon c) = (c, \varepsilon)_S\, g(a, c).$$

(iv) If $c \in \mathfrak{o}_S$ let $\phi(c)$ denote the cardinality of $(\mathfrak{o}_S/c\mathfrak{o}_S)^\times$. If p is prime, then

$$g_t(p^k, p^l) = \begin{cases} \mathbb{N}(p)^k\, g_{tl}(1, p) & \text{if } l = k+1; \\ \phi(p^l) & \text{if } n|tl \text{ and } k \geqslant l; \\ 0 & \text{otherwise.} \end{cases}$$

(v) If a and p are coprime, then

$$|g(a, p)| = \sqrt{\mathbb{N}p}.$$

Proof. These properties of the Gauss sums are well known. See, for example, Ireland and Rosen [13]. Note that (iii) follows from (i) and (14).

2.3. Kubota Dirichlet series

If $\Psi \in \mathcal{M}(\Omega)$, the space of functions defined in (12), let

$$\mathcal{D}_t(s,\Psi,a) = \sum_{0\neq c\in\mathfrak{o}_S/\mathfrak{o}_S^\times} g_t(a,c)\,\Psi(c)\,\mathbb{N}(c)^{-2s} \quad .$$

We will also denote $\mathcal{D}_1(s,\Psi,a) = \mathcal{D}(s,\Psi,a)$. Here $\mathbb{N}(c) = |c|$ is the order of $\mathfrak{o}_S/c\mathfrak{o}_S$. The term $g(a,c)\,\Psi(c)\,\mathbb{N}(c)^{-2s}$ is independent of the choice of representative c modulo S-units by (12) and Proposition 2.6. It follows easily from Proposition 2.6 that the series is convergent if $\Re(s) > \frac{3}{4}$.

Let

$$\boldsymbol{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-1} \Gamma\left(2s-1+\frac{j}{n}\right). \tag{15}$$

In view of the multiplication formula for the Gamma function, we may also write

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)}\frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Let

$$\mathcal{D}^*(s,\Psi,a) = \boldsymbol{G}_n(s)^{\frac{1}{2}[F:\mathbb{Q}]}\zeta_F(2ns-n+1)\,\mathcal{D}(s,\Psi,a)$$

where we note that $\frac{1}{2}[F:\mathbb{Q}]$ is the number of archimedean places of the totally complex field F and ζ_F is the Dedekind zeta function of F.

If $v \in S_{\text{fin}}$ let q_v denote the cardinality of the residue class field $\mathfrak{o}_v/\mathfrak{p}_v$, where $\mathfrak{o}_v$ is the local ring in F_v and $\mathfrak{p}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}.

If $\Psi \in \mathcal{M}(\Omega)$ and $\eta \in F_S^\times$ denote

$$\tilde{\Psi}_\eta(c) = (\eta,c)_S\,\Psi(c^{-1}\eta^{-1}). \tag{16}$$

It is easy to check that $\tilde{\Psi}_\eta \in \mathcal{M}(\Omega)$ and that it depends only on the class of η in $F_S^\times/F_S^{\times,n}$.

The following result will be fundamental in our proofs.

Theorem 2.7. (Brubaker and Bump [3]) *Let $\Psi \in \mathcal{M}(\Omega)$, and let $a \in \mathfrak{o}_S$. Then $\mathcal{D}^*(s,\Psi,a)$ has meromorphic continuation to all s, analytic except possibly at $s = \frac{1}{2} \pm \frac{1}{2n}$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta(s)$ that depend only on the image of η in $F_S^\times/F_S^{\times,n}$ such that*

$$\mathcal{D}^*(s,\Psi,a) = \mathbb{N}(a)^{1-2s} \sum_{\eta\in F_S^\times/F_S^{\times,n}} P_{a\eta}(s)\mathcal{D}^*(1-s,\tilde{\Psi}_\eta,a). \tag{17}$$

This result, based on ideas of Kubota [16], relies on the theory of Eisenstein series.

2.4. Root systems

Background information on root systems and Weyl groups may be found in Bump [7], Chapter 21, or Bourbaki [2], as well as many other sources. A *root system of rank* r is a finite subset Φ of nonzero vectors in the Euclidean space V that span V as a vector space satisfying the following properties. If $\alpha \in \Phi$, let $\sigma_\alpha : V \longrightarrow V$ denote the reflection in the hyperplane through the origin perpendicular to the vector α. The definition of a root system requires that $\sigma_\alpha(\Phi) = \Phi$, and if $\alpha, \beta \in \Phi$, then $\beta - \sigma_\alpha(\beta)$ is an integer multiple of α. We will denote by $\langle\ ,\ \rangle$ the Euclidean inner product on V, and by $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$ the Euclidean norm. We make the following
Normalization Assumption: *We require that* $\|\alpha\|^2 \in \mathbb{Z}$ *and that* $2\langle \alpha, \beta \rangle \in \mathbb{Z}$ *for all* $\alpha, \beta \in \Phi$.

Let us observe that this can always be arranged. There is no loss of generality in assuming that Φ is irreducible, that is, that there is no way to write $\Phi = \Phi_1 \cup \Phi_2$ where Φ_1 and Φ_2 lie in orthogonal subspaces of V. Assuming this, the normalization assumption will be satisfied if we adjust the inner product so that the short roots have length 1.

We have

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \tag{18}$$

so

$$\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

when α and β are roots. Since $-\alpha = \sigma_\alpha(\alpha)$, these axioms imply that $-\alpha \in \Phi$. The root system is called *reduced* if α and 2α are not both in Φ, and it is called *irreducible* if it is not the union of two smaller root systems that span orthogonal subspaces of V.

Let $\Phi \subset V$ be a reduced root system of rank r. We partition Φ as usual into positive roots Φ^+ and negative roots Φ^-. Let $\Delta = \{\alpha_1, \cdots, \alpha_r\} \subset \Phi^+$ be the subset of *simple positive roots*. These are the positive roots that cannot be decomposed into sums of other positive roots. It is well-known that α_i are a linear basis of V, and that if $\alpha \in F$, then $\alpha = \sum k_i \alpha_i$ where $k_i \in \mathbb{Z}$; moreover, k_i are all positive if $\alpha \in \Phi^+$, and all negative if $\alpha \in \Phi^-$. The σ_{α_i} are called *simple reflections*. We will sometimes denote $\sigma_i = \sigma_{\alpha_i}$.

Let W denote the Weyl group of Φ, which is the group of transformations of V generated by the reflection σ_α in the hyperplane through the origin perpendicular to the root α. The inner product $\langle\ ,\ \rangle$ is W-invariant. It is well-known that W is generated by the σ_α for α in the set Δ of simple roots. To each element $w \in W$ let $\Phi_w \subset \Phi^+$ be

$$\Phi_w = \Phi^+ \cap w^{-1}\Phi^- = \{\alpha \mid \alpha \in \Phi^+, w(\alpha) \in \Phi^-\}.$$

We will denote

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

We also recall that the *length function* $l : W \longrightarrow \mathbb{Z}$ is defined by letting $l(w)$ be the smallest number of simple reflections into which $w \in W$ may be decomposed.

Lemma 2.8. *Let* $w \in W$.
(i) The cardinality of Φ_w *is the length* $l(w)$ *of* w.

(ii) We have

$$\rho - w(\rho) = \sum_{\alpha \in \Phi_w} (-w\alpha) = \sum_{\substack{\alpha \in \Phi^+ \\ w^{-1}\alpha \in \Phi^-}} \alpha \tag{19}$$

while

$$\rho - w^{-1}(\rho) = \sum_{\alpha \in \Phi_w} \alpha. \tag{20}$$

(iii) Express $\rho - w(\rho)$ as a linear combination of the simple roots:

$$\rho - w(\rho) = \sum_{i=1}^{r} k_i \alpha_i. \tag{21}$$

Then the k_i are nonnegative integers.
(iv) If $w, w' \in W$ such that $\rho - w(\rho) = \rho - w'(\rho)$ then $w = w'$.
(v) For every simple root α_i we have

$$\frac{2\langle \rho, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1. \tag{22}$$

Proof. Part (i) follows from Proposition 21.2 of [7].

For (ii), note that $\rho - w(\rho)$ equals

$$\begin{aligned}
&\tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha - \tfrac{1}{2} \sum_{\alpha \in \Phi^+} w(\alpha) \\
&= \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha - \tfrac{1}{2} \sum_{w^{-1}\alpha \in \Phi^+} \alpha = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha + \tfrac{1}{2} \sum_{w^{-1}\alpha \in \Phi^-} \alpha \\
&= \tfrac{1}{2} \sum_{\substack{\alpha \in \Phi^+ \\ w^{-1}\alpha \in \Phi^+}} \alpha + \tfrac{1}{2} \sum_{\substack{\alpha \in \Phi^+ \\ w^{-1}\alpha \in \Phi^-}} \alpha + \tfrac{1}{2} \sum_{\substack{w^{-1}\alpha \in \Phi^- \\ \alpha \in \Phi^+}} \alpha + \tfrac{1}{2} \sum_{\substack{w^{-1}\alpha \in \Phi^- \\ \alpha \in \Phi^-}} \alpha.
\end{aligned}$$

The first term and the last term cancel, but the second and third term are equal, so this equals the right-hand side of (19). On the other hand,

$$\sum_{\alpha \in \Phi_w} (-w\alpha) = \sum_{\substack{\alpha \in \Phi^+ \\ w(\alpha) \in \Phi^-}} (-w\alpha) = \sum_{\substack{\alpha \in \Phi^- \\ w(\alpha) \in \Phi^+}} w(\alpha) = \sum_{\substack{w^{-1}(\alpha) \in \Phi^- \\ \alpha \in \Phi^+}} \alpha,$$

where we have replaced α first by $-\alpha$, then $w^{-1}(\alpha)$. Now (19) is proved. Replacing w by w^{-1} gives (20).

Since (19) shows that $\rho - w(\rho)$ is a sum of positive roots, it is a sum of simple positive roots, and (iii) is also clear.

For (iv), we have $w(\rho) = w'(\rho)$, so $w^{-1}w'$ fixes ρ. It follows that $\Phi_{w^{-1}w'}$ is the empty set, so $l(w^{-1}w') = 0$, that is, $w^{-1}w' = 1$.

For (v), when α_i is a simple root, σ_i changes α_i to $-\alpha_i$ and permutes the remaining positive roots. Thus $\sigma_i(\rho) = \rho - \alpha_i$. By (18) we obtain (22).

As in the introduction, define the function d on Φ^+ by (5).

Lemma 2.9. *We have $d(\alpha) \in \mathbb{Z}^+$ for all $\alpha \in \Phi^+$, and $d(\alpha) = 1$ if α is a simple positive root.*

Proof. We recall that $\lambda \in V$ is a *weight* if $2\langle\lambda, \alpha\rangle / \langle\alpha, \alpha\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$, and the weight is *dominant* if $2\langle\lambda, \alpha\rangle / \langle\alpha, \alpha\rangle \geqslant 0$ for all $\alpha \in \Phi^+$. It is well-known that ρ is a dominant weight; in fact it is the sum of the fundamental dominant weights ([7], Proposition 21.16). If ε_i are the fundamental dominant weights then $2\langle\varepsilon_i, \alpha_j\rangle / \langle\alpha_j, \alpha_j\rangle = \delta_{ij}$ (Kronecker delta) which implies that $d(\alpha_i) = 1$ for the simple roots α_i. □

Lemma 2.10. *If Φ is reduced then given any simple root α_i, and $w \in W$ a fixed element of the Weyl group, then*

$$\sum_{\alpha\in\Phi_w} \frac{2\langle w\alpha, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} = d - 1 \tag{23}$$

where $d = d(w^{-1}\alpha_i)$ is defined in (5).

Proof. By (20),

$$\sum_{\alpha\in\Phi_w} \frac{2\langle w\alpha, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} = \frac{2\left\langle w(\rho - w^{-1}\rho), \alpha_i\right\rangle}{\langle\alpha_i, \alpha_i\rangle} = \frac{2\langle w\rho - \rho, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}.$$

On the other hand, using the W-invariance of $\langle\ ,\ \rangle$ and (22) the right-hand side of (23) is

$$\frac{2\left\langle\rho, w^{-1}(\alpha_i)\right\rangle}{\langle\alpha_i, \alpha_i\rangle} - 1 = \frac{2\langle w\rho, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} - \frac{2\langle\rho, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle},$$

and the statement follows. □

3. Stability, normalizing factors and Weyl group action

3.1. Stability

Let Φ be a reduced root system of rank r. We assume that $\langle\ ,\ \rangle$ satisfies the Normalization Assumption of Section 2.4. We will also make a certain *stability assumption* to the effect that n is sufficiently large, depending on Φ. Without this assumption, we expect that Z_Ψ can be defined and satisfies a similar group of functional equations, but that the definition of Z_Ψ will be more complicated. (See [6].)
Stability Assumption. *We will assume that $n \geqslant d(\alpha) \cdot \gcd(n, \|\alpha\|^2)$ for every positive root α.*

3.2. Weyl group action

We now describe the precise action of W on $(s_1, \cdots, s_r)$ that will be used to state the functional equations of Z_Ψ. We start with a pairing $B : V \times \mathbb{C}^r \longrightarrow \mathbb{C}$ defined as follows. If $\alpha \in V$ and $\boldsymbol{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r$, write $\alpha = \sum k_i\alpha_i$, and let

$$B(\alpha, \boldsymbol{s}) = \sum k_i s_i. \tag{24}$$

We do not insist on identifying $\mathbb{C}^r$ with $V_\mathbb{C}^\vee$ as in the introduction. Indeed, our proofs will be formulated without explicit mention of the coroots, and the reader may disregard the dual root system entirely. However if the identification $\mathbb{C}^r = V_{\mathbb{C}}^\vee$ is made, this B is precisely the complexification of the dual pairing $V \times V^\vee \longrightarrow \mathbb{R}$ that was denoted B in the introduction. The coroot $\alpha^\vee$ is then the linear functional $x \longmapsto 2\langle x, \alpha\rangle / \langle \alpha, \alpha\rangle$ on V.

At this point, we may simply define $\rho^\vee = (1, \cdots, 1)$. If we identify $V_{\mathbb{C}}^\vee$ with $\mathbb{C}^r$ as in the introduction, this is half the sum of the positive coroots, as can be seen from (22) below, applied to the dual root system $\Phi^\vee$ in $V^\vee$. The reader does not need this fact, however, and can take the formula

$$\frac{1}{2}\rho^\vee = \left(\frac{1}{2}, \cdots, \frac{1}{2}\right) \tag{25}$$

to be a definition. We define the action of W on $\mathbb{C}^r$ as in the introduction, by (1). This action will appear also in the following guise. If p is a prime, let

$$M_p(\alpha; \boldsymbol{s}) = \prod_{i=1}^{r} \mathbb{N}p^{-2k_i s_i} = \mathbb{N}p^{-2B(\alpha, \boldsymbol{s})}.$$

Then (1) gives

$$M_p\left(w\alpha, w(\boldsymbol{s}) - \frac{1}{2}\rho^\vee\right) = M_p\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right). \tag{26}$$

In order to describe this action more explicitly in $\boldsymbol{s}$, it is enough to examine the simple reflections σ_i in the root α_i, which generate the group W. The effect of σ_i on the root system is given by the map

$$\sigma_i : \alpha \longmapsto \alpha - \frac{2\langle \alpha, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i,$$

in terms of the standard Euclidean inner product $\langle\ ,\ \rangle$ on V.

Proposition 3.1. *The action of σ_i on $\boldsymbol{s} = (s_1, \ldots, s_r)$ according to (1) is given by:*

$$s_j \longmapsto s_j - \frac{2\langle \alpha_j, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\left(s_i - \tfrac{1}{2}\right), \qquad j = 1, \cdots, r. \tag{27}$$

In particular, $s_i \longmapsto 1 - s_i$.

Note that $-\frac{2\langle \alpha_j, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} \geqslant 0$ if $j \neq i$.

Proof. Take $\alpha = \alpha_j$. The right-hand side of (1) equals $s_j - \frac{1}{2}$. Denoting $\sigma_i \boldsymbol{s} = (s_1', \cdots, s_r')$, the left-hand side of (1) equals

$$B\left(\alpha_j - \frac{2\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i, \sigma_i \boldsymbol{s} - \rho^\vee\right) = (s_j' - \tfrac{1}{2}) - \frac{2\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}(s_i' - \tfrac{1}{2}).$$

So

$$s_j - \tfrac{1}{2} = (s_j' - \tfrac{1}{2}) - \frac{2\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}\left(s_i' - \tfrac{1}{2}\right). \tag{28}$$

This reduces to (27). $\square$

3.3. Normalizing factors

The multiple Dirichlet series must be normalized with gamma and zeta factors. Let

$$n(\alpha) = \frac{n}{\gcd(n, \|\alpha\|^2)}. \tag{29}$$

For example, if all roots have length 1 (so Φ is simply-laced), then we have $n(\alpha) = n$ for every $\alpha \in \Phi$. On the other hand, if Φ is not simply-laced but irreducible, and if $\langle\ ,\ \rangle$ is normalized so that the short roots have length 1, we have

$$n(\alpha) = \begin{cases} n \text{ if } \alpha \text{ is a short root,} \\ n \text{ if } \alpha \text{ is a long root and } \Phi \neq G_2, \text{ and } n \text{ is odd} \\ \frac{n}{2} \text{ if } \alpha \text{ is a long root and } \Phi \neq G_2, \text{ and } n \text{ is even} \\ n \text{ if } \alpha \text{ is a long root and } \Phi = G_2, \text{ and } 3 \nmid n \\ \frac{n}{3} \text{ if } \alpha \text{ is a long root and } \Phi = G_2, \text{ and } 3|n. \end{cases}$$

If α is a positive root, write $\alpha = \sum k_i \alpha_i$ as before. Let ζ be the Dedekind zeta-function of F and

$$\zeta_\alpha(\boldsymbol{s}) = \zeta\left(1 + 2n(\alpha) \sum_{i=1}^{r} k_i (s_i - \frac{1}{2})\right) = \zeta\left(1 + 2n(\alpha)\, B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right)\right). \tag{30}$$

Also let

$$G_\alpha(\boldsymbol{s}) = \boldsymbol{G}_{n(\alpha)}\left(\frac{1}{2} + \sum_{i=1}^{r} k_i \left(s_i - \frac{1}{2}\right)\right) = \boldsymbol{G}_{n(\alpha)}\left(\frac{1}{2} + B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right)\right),$$

where $\boldsymbol{G}_n(s)$ is defined as in (15). We define the *normalized* multiple Dirichlet series by

$$Z^*_\Psi(\boldsymbol{s}) = \left[\prod_{\alpha \in \Phi^+} G_\alpha(\boldsymbol{s})\, \zeta_\alpha(\boldsymbol{s})\right] Z_\Psi(\boldsymbol{s}). \tag{31}$$

4. Local computations

As before, we let Φ denote a reduced root system with r simple roots. The basic idea in our combinatorial proof is to rearrange the Dirichlet series $Z_\Psi(\boldsymbol{s})$ so that the inner sum is over C_i for some fixed index i. This allows us to prove that the resulting sum can be realized as a finite linear combination of Dirichlet polynomials in terms of primes dividing C_j $(j \neq i)$, multiplied by Kubota Dirichlet series of the form $D(s, \Psi, \alpha)$ introduced earlier. We will see the precise way in which α is related to the parameters C_j in the next section. Here we take a first step in this direction by proving a relationship between the functions $H(C_1, \cdots, C_r)$ on prime powers.

We recall that on prime powers, $H(p^{k_1}, \cdots, p^{k_r})$ has finitely many non-zero terms as the k_i range over the non-negative integers. They are in one-to-one correspondence with elements $w \in W$, the Weyl group. We say that $(k_1, \cdots, k_r) \in \mathbb{Z}^r$ is *associated* to $w \in W$ if (21) is satisfied; in this case, we write

$$(k_1, \cdots, k_r) = \text{assoc}(w).$$

Proposition 4.1. *Let $w \in W$ be such that $l(\sigma_i w) = l(w) + 1$. Suppose that* $\operatorname{assoc}(w) = (k_1, \cdots, k_r) \in \mathbb{Z}^r$ *and* $\operatorname{assoc}(\sigma_i w) = (l_1, \cdots, l_r)$. *Let $d = d(w^{-1}\alpha_i)$ in the notation (5). Then*

$$l_j = \begin{cases} k_i + d & \text{if } j = i; \\ k_i & \text{if } j \neq i, \end{cases} \tag{32}$$

and

$$H(p^{l_1}, \ldots, p^{l_r}) = g_{\|\alpha_i\|^2}(p^{d-1}, p^d)\, H(p^{k_1}, \ldots, p^{k_r}). \tag{33}$$

Moreover

$$d - 1 = \sum_{j=1}^{r} \frac{-2\,\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} k_j\ . \tag{34}$$

Proof. Propositions 21.2 and 21.10 of [7] show that the assumption that $l(\sigma_i w) = l(w)+1$ implies that $w^{-1}(\alpha_i) \in \Phi^+$ is the unique root appearing in the difference of sets $\Phi_{\sigma_i w} - \Phi_w$. Now (33) follows from (6) since $\|w^{-1}\alpha_i\| = \|\alpha_i\|$. Moreover,

$$\begin{aligned} \sum_{j=1}^{r} l_j \alpha_j &= \rho - \sigma_i w\rho = (\rho - w\rho) + w\rho - \sigma_i w\rho \\ [\text{by } (18)] &= (\rho - w\rho) + \frac{2\,\langle w\rho, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \\ [\text{by } W\text{-invariance of } \langle\ ,\ \rangle] &= (\rho - w\rho) + \frac{2\,\langle \rho, w^{-1}\alpha_i \rangle}{\langle w^{-1}\alpha_i, w^{-1}\alpha_i \rangle}\alpha_i \\ [\text{by definition of } d = d(w^{-1}\alpha_i)] &= \sum_{j=1}^{r} k_j \alpha_j + d\alpha_i, \end{aligned}$$

whence (32). Finally, by (19) we have

$$\sum_{\alpha \in \Phi_w} (-w\alpha) = \rho - w\rho = \sum_{j=1}^{r} k_j \alpha_j.$$

Since by Lemma 2.10

$$d - 1 = \sum_{\alpha \in \Phi_w} \frac{2\,\langle w\alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

we obtain (34).

Proposition 4.2. *Let $d = d(w^{-1}(\alpha_i))$. For any $w \in W$, the monomial*

$$\mathbb{N}p^{(s_i - \frac{1}{2})(d-1)} \prod_{\alpha \in \Phi_w} M_p(-w\alpha; \boldsymbol{s})$$

is invariant under the transformation (27).

Proof. Reinterpreting this in terms of the action of W given by (1), we must show equivalently that

$$\tfrac{1}{2}(d-1)\alpha_i + \sum_{\substack{\alpha>0 \\ w\alpha<0}} (-w\alpha) = \tfrac{1}{2}(d-1)\alpha_i + \rho - w\rho \tag{35}$$

is orthogonal to α_i. We will show that the right-hand side of (35) is fixed by σ_i, i.e.,

$$\sigma_i\rho - \sigma_i w\rho = (d-1)\alpha_i + \rho - w\rho.$$

Since $\rho - \sigma_i\rho = \alpha_i$ we can write this more simply as $w\rho - \sigma_i w\rho = d\alpha_i$, and indeed

$$d\alpha_i = \frac{2\left\langle w^{-1}\alpha_i, \rho\right\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i = \frac{2\left\langle \alpha_i, w\rho\right\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i = w\rho - \sigma_i w\rho.$$

This concludes the proof of the Proposition. □

5. Global functional equations

Let $\sigma_i \in W$ be a fixed simple reflection about $\alpha_i \in \Phi$.

Definition 5.1. *We say that $(C_1, \cdots, C_r)$ in $(\mathfrak{o}_S)^r$ is **admissible** if, for each prime p, there exists a Weyl group element $w_p \in W$ such that*

$$(\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_r)) = \mathrm{assoc}(w_p).$$

*For such $(C_1, \cdots, C_r)$, we say that C_i is i-**reduced** if, for every p, we have $l(\sigma_i w_p) = l(w_p) + 1$.*

We note that if $C_1, \ldots, C_r$ are nonzero elements of $\mathfrak{o}_S$, then $(C_1, \ldots, C_r)$ is admissible if and only if $H(C_1, \cdots, C_r) \neq 0$. This is immediate from the definition of H.

Proposition 5.2. *Let $C_1, \cdots, C_{i-1}, C_{i+1}, \cdots, C_r$ be nonzero elements of $\mathfrak{o}_S$. If there exists a C_i such that $(C_1, \cdots, C_r)$ is admissible, then there exists a C_i' (modulo the action of $\mathfrak{o}_S^\times$) that is i-reduced. This C_i' divides C_i and is uniquely determined up to multiplication by a unit. Moreover, if*

$$(\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_i'), \cdots, \mathrm{ord}_p(C_r)) = (k_1, \cdots, k_r) = \mathrm{assoc}(w_p')$$

then either $\mathrm{ord}_p(C_i) = k_i$ or $\mathrm{ord}_p(C_i) = k_i + d$, where $d = d((w_p')^{-1}\alpha_i)$.

Proof. Let w_p be such that $(\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_r)) = \mathrm{assoc}(w_p)$. Let

$$w_p' = \begin{cases} w_p & \text{if } l(\sigma_i w_p) = l(w_p) + 1; \\ \sigma_i w_p & \text{if } l(\sigma_i w_p) = l(w_p) - 1. \end{cases}$$

In either case $l(\sigma_i w_p') = l(w_p') + 1$ and so there exists admissible $(C_1', \cdots, C_r')$ such that $(\mathrm{ord}_p(C_1'), \cdots, \mathrm{ord}_p(C_r')) = \mathrm{assoc}(w_p')$, and C_i' is i-reduced with respect to $C_1', \cdots, C_{i-1}'$, $C_{i+1}', \cdots, C_r'$. By Proposition 4.1, we have

$$\mathrm{assoc}(w_p) = \begin{cases} \mathrm{assoc}(w_p') & \text{if } l(\sigma_i w_p) = l(w_p) + 1; \\ \mathrm{assoc}(w_p') + (0, \cdots, d, \cdots, 0) & \text{if } l(\sigma_i w_p) = l(w_p) - 1, \end{cases}$$

where $d = d((w_p')^{-1}\alpha_i)$. Thus C_j' differs from C_j by a unit if $j \neq i$, so we may as well take $C_j' = C_j$. Moreover, $\mathrm{ord}_p(C_i) \geqslant \mathrm{ord}_p(C_i')$, so C_i' divides C_i. □

By Proposition 5.2,

$$Z_\Psi(s_1,\cdots,s_r)=$$

$$\sum_{\substack{0\neq C_j\in\mathfrak{o}_S^\times\backslash\mathfrak{o}_S\\ 1\leqslant j\leqslant r\\ C_1,\cdots,C_r\text{ admissible}\\ C_i\ i\text{-reduced}}} \mathbb{N}C_1^{-2s_1}\cdots\mathbb{N}C_r^{-2s_r}H(C_1,\cdots,C_r)\sum_{0\neq D\in\mathfrak{o}_S^\times\backslash\mathfrak{o}_S}(D,C_i)_S^{\|\alpha_i\|^2}$$

$$\times\frac{H(C_1,C_2,\cdots,DC_i,\cdots,C_r)}{H(C_1,C_2,\cdots,C_i,\cdots,C_r)}\prod_{j>i}(D,C_j)_S^{2\langle\alpha_i,\alpha_j\rangle}\Psi_i^{C_1,\cdots,C_r}(D)\,\mathbb{N}D^{-2s_i}, \tag{36}$$

where we define

$$\Psi_i^{C_1,\dots,C_r}(D)=\Psi(C_1,\dots,C_iD,\dots,C_r)(D,C_i)_S^{-\|\alpha_i\|^2}\prod_{j>i}(D,C_j)_S^{-2\langle\alpha_i,\alpha_j\rangle} \tag{37}$$

to emphasize the dependence on D for fixed parameters $C_1,\dots,C_r$ in the inner sum. The choice of Hilbert symbols is explained by Lemma 5.3 below.

We will show that the inner sum is a Kubota Dirichlet series. The key is to identify the quotient of H's in (36) as a Gauss sum. If the C_i and D are powers of a single prime p, this is (33), which we must generalize in (42) below. First we need to explain some nuances related to the fact that Kubota Dirichlet series involving Gauss sums of differing degrees can occur.

Let $t=||\alpha_i\|^2$; both t and i will be fixed in the following discussion. Here $t=1,2$ or 3 if Φ is irreducible and the short roots have length 1. If $\Psi\in\mathcal{M}_t(\Omega)$ then we may consider Dirichlet series of the form

$$\mathcal{D}_t^*(s,\Psi,a)=\boldsymbol{G}_m(s)^{\frac12[F:\mathbb{Q}]}\,\zeta_F(2ms-m+1)\,\mathcal{D}_t(s,\Psi,a), \tag{38}$$

where $m=n/\gcd(n,t)$ and

$$\mathcal{D}_t(s,\Psi,a)=\sum_{0\neq c\in\mathfrak{o}_S/\mathfrak{o}_S^\times}g_t(a,c)\,\Psi(c)\,\mathbb{N}(c)^{-2s}.$$

These are Kubota Dirichlet series for Gauss sums relative to the symbol $\left(\text{-}\right)^t$. This is a power residue symbol of degree m; if $\gcd(n,t)=1$, this means that we still have functional equations for $\mathcal{D}_t$ implied by Theorem 2.7, but that the embedding $j:\mu_n\longrightarrow\mathbb{C}^\times$ in Remark 2.1 will be raised to the t-th power. In every case Theorem 2.7 is applicable with n replaced by m. Because $P_{\alpha\eta}$ in Theorem 2.7 will arise for different values of t (with n replaced by m) we will add t to the notation to write (17) in the form

$$\mathcal{D}_t^*(s,\Psi,a)=\mathbb{N}(a)^{1-2s}\sum_{\eta\in F_S^\times/F_S^{\times,n}}P_{a\eta}^t(s)\,\mathcal{D}_t^*(1-s,\tilde{\Psi}_\eta^t,a). \tag{39}$$

As in (16), if $\Psi\in\mathcal{M}_t(\Omega)$ and $\eta\in F_S^\times$ denote

$$\tilde{\Psi}_\eta^t(c)=(\eta,c)_S^t\,\Psi(c^{-1}\eta^{-1}). \tag{40}$$

Note that Theorem 2.7 would actually have us summing over $\eta \in F_S^\times/F_S^{\times,m}$ but there is no harm in summing over the larger set $\eta \in F_S^\times/F_S^{\times,n}$, and dividing the value of $P^t_{a\eta}(s)$ by a factor of $\gcd(n,t)$. What will prove important is that the factor $\mathbb{N}(a)^{1-2s}$ does not depend on t.

Lemma 5.3. *Let $C_1, \ldots, C_r$ be fixed nonzero elements of $\mathfrak{o}_S$. Then with the notation (37), the function $\Psi_i^{C_1,\ldots,C_r} \in \mathcal{M}_t(\Omega)$.*

Proof. We check the transformation property for $\Psi_i^{C_1,\ldots,C_r}(D)$ when D is multiplied by a unit ε, using the definition (8):

$$\begin{aligned}
\Psi_i^{C_1,\ldots,C_r}(\varepsilon D) &= \Psi(C_1,\ldots,C_i\varepsilon D,\ldots,C_r)(\varepsilon D, C_i)_S^{-\|\alpha_i\|^2}\prod_{j>i}(\varepsilon D, C_j)_S^{-2\langle\alpha_i,\alpha_j\rangle}\\
&= (\varepsilon, DC_i)_S^{\|\alpha_i\|^2}\left\{\prod_{j>i}(\varepsilon, C_j)_S^{2\langle\alpha_i,\alpha_j\rangle}\right\}\Psi(C_1,\ldots,C_iD,\ldots,C_r)\\
&\quad\times(\varepsilon D, C_i)_S^{-\|\alpha_i\|^2}\prod_{j>i}(\varepsilon D, C_j)_S^{-2\langle\alpha_i,\alpha_j\rangle}\\
&= (\varepsilon, D)_S^{\|\alpha_i\|^2}\Psi_i^{C_1,\ldots,C_r}(D).
\end{aligned}$$

□

Lemma 5.4. *Fix an integer $i \in \{1,\ldots,r\}$. Given admissible $(C_1,\ldots,C_r) \in \mathfrak{o}_S^r$ with C_i i-reduced, then*

$$C_0 = \prod_j C_j^{-2\langle\alpha_j,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle} \tag{41}$$

is an $\mathfrak{o}_S$ integer and for every $D \in \mathfrak{o}_S$ we have

$$\frac{H(C_1,\ldots,DC_i,\ldots,C_r)}{H(C_1,\ldots,C_r)}(D,C_i)_S^{\|\alpha_i\|^2}\prod_{j>i}(D,C_j)_S^{2\langle\alpha_i,\alpha_j\rangle} = g_{\|\alpha_i\|^2}(C_0,D). \tag{42}$$

Moreover for each prime p of $\mathfrak{o}_S$ we have

$$\mathrm{ord}_p(C_0) = d(w_p^{-1}\alpha_i) - 1, \tag{43}$$

where w_p is determined by the condition

$$\mathrm{assoc}(w_p) = (\mathrm{ord}_p(C_1),\cdots,\mathrm{ord}_p(C_r)).$$

Proof. To prove (42), we first check that the multiplicativity of the left-hand side of (42) is the same as that of the Gauss sum. Let $h(D)$ denote the left-hand side. Let $\gcd(D,E) = 1$ and for $j = 1,\ldots,r$, write $C_j = C_j'C_j''$ so that $\gcd(DC_1'\cdots C_r', EC_1''\cdots C_r'') = 1$. To check the multiplicativity we consider $h(DE)h(D)^{-1}h(E)^{-1}$, which equals

$$\frac{H(C_1'C_1'',\ldots,DEC_i'C_i'',\ldots,C_r'C_r'')H(C_1'C_1'',\ldots,C_r'C_r'')}{H(C_1'C_1'',\ldots,DC_i'C_i'',\ldots,C_r'C_r'')\,H(C_1'C_1'',\ldots,EC_i'C_i'',\ldots,C_r'C_r'')} \tag{44}$$

since the Hilbert symbols appearing in (42) are multiplicative in the first component, and hence cancel. Then according to the multiplicativity for H given in (3), the quantity (44) equals

$$\frac{F(D,E)\,F(1,1)}{F(D,1)\,F(1,E)}$$

where

$$F(D,E)=\left(\frac{DC_i'}{EC_i''}\right)^{\|\alpha_i\|^2}\left(\frac{EC_i''}{DC_i'}\right)^{\|\alpha_i\|^2}\prod_{\substack{j\,=\,1\\ j\,\neq\,i}}^{r}\left(\frac{C_j'}{C_j''}\right)^{\|\alpha_j\|^2}\left(\frac{C_j''}{C_j'}\right)^{\|\alpha_j\|^2}$$
$$\times\prod_{j<i}\left(\frac{C_j'}{EC_i''}\right)^{2\langle\alpha_i,\alpha_j\rangle}\left(\frac{C_j''}{DC_i'}\right)^{2\langle\alpha_i,\alpha_j\rangle}\prod_{j>i}\left(\frac{DC_i'}{C_j''}\right)^{2\langle\alpha_i,\alpha_j\rangle}\left(\frac{EC_i''}{C_j'}\right)^{2\langle\alpha_i,\alpha_j\rangle}$$
$$\times\prod_{\substack{j\,<\,k\\ j,\,k\,\neq\,i}}\left(\frac{C_j'}{C_k''}\right)^{2\langle\alpha_j,\alpha_k\rangle}\left(\frac{C_j''}{C_k'}\right)^{2\langle\alpha_j,\alpha_k\rangle}.$$

After cancellation, this equals $\left(\frac{D}{E}\right)^{\|\alpha_i\|^2}\left(\frac{E}{D}\right)^{\|\alpha_i\|^2}$.

We have verified that the left-hand side of (42) has the same multiplicativity as Proposition 2.6 (i) predicts for the Gauss sum on the right-hand side. It remains to check that C_0 is an integer and that the term $h(D)$ agrees with $g_{\|\alpha_i\|^2}(C_0,D)$ at prime powers.

To this end, let $D=p^r$ and write $C_j=C_j'p^{k_j}$ with $p\nmid C_j'$ for $j=1,\dots,r$. Then by definition $h(p^r)$ is

$$\frac{H(C_1'p^{k_1},\dots,C_i'p^{k_i+r},\dots,C_r'p^{k_r})}{H(C_1'p^{k_1},\dots,C_i'p^{k_i},\dots,C_r'p^{k_r})}\quad(p^r,C_i'p^{k_i})_S^{\|\alpha_i\|^2}\prod_{j>i}(p^r,C_j'\,p^{k_j})_S^{2\langle\alpha_i,\alpha_j\rangle}.$$

Since $(p,p)=1$ we can immediately simplify the Hilbert symbols. By (3) we can simplify this as above. We need only analyze the residue symbols involving the index i as all other residue symbols immediately cancel. Hence,

$$h(p^r)=(p^r,C_i')_S^{||\alpha_i||^2}\prod_{j>i}(p^r,C_j')_S^{2\langle\alpha_i,\alpha_j\rangle}\left(\frac{C_i'}{p^r}\right)^{\|\alpha_i\|^2}\left(\frac{p^r}{C_i'}\right)^{\|\alpha_i\|^2}$$
$$\times\prod_{j>i}\left(\frac{p^r}{C_j'}\right)^{2\langle\alpha_i,\alpha_j\rangle}\prod_{j<i}\left(\frac{C_j'}{p^r}\right)^{2\langle\alpha_i,\alpha_j\rangle}\frac{H(p^{k_1},\dots,p^{k_i+r},\dots,p^{k_r})}{H(p^{k_1},\dots,p^{k_i},\dots,p^{k_r})}$$
$$=\left(\frac{(C_i')^2}{p^r}\right)^{\|\alpha_i\|^2}\prod_{j\neq i}\left(\frac{C_j'}{p^r}\right)^{2\langle\alpha_i,\alpha_j\rangle}\quad\frac{H(p^{k_1},\dots,p^{k_i+r},\dots,p^{k_r})}{H(p^{k_1},\dots,p^{k_i},\dots,p^{k_r})}$$

where we have used the reciprocity law in the final step.

Applying Proposition 5.2, we find that $H(p^{k_1},\dots,p^{k_i+r},\dots,p^{k_r})$ is non-zero for precisely two choices of r, namely $r=0$ and $r=d(w_p^{-1}\alpha_i)$. By Proposition 4.1, (43) is true, so $C_0 \in \mathfrak{o}_S$ and the two possible values of r are 0 and $\mathrm{ord}_p(C_0)+1$. On the other hand by the stability hypothesis $n/\gcd(\|\alpha_i\|^2, n) \geqslant \mathrm{ord}_p(C_0)+1$, so by Proposition 2.6, there are only two values of r such that $g_{\|\alpha_i\|^2}(C_0, p^r) \neq 0$; again, they are $r = 0$ and $\mathrm{ord}_p(C_0)+1$. Thus for these two values, we have to check that

$$H(p^{k_1}, \ldots, p^{k_i+r}, \ldots, p^{k_r}) = g_{\|\alpha_i\|^2}(C_0, p^r).$$

If $r = 0$, both sides are 1, so assume that $r = \mathrm{ord}_p(C_0)+1$. By (33) we have

$$\begin{aligned} h(p^r) &= \left(\frac{(C_i')^2}{p^r}\right)^{\|\alpha_i\|^2} \prod_{j\neq i} \left(\frac{C_j'}{p^r}\right)^{2\langle\alpha_i,\alpha_j\rangle} g_{\|\alpha_i\|^2}(p^{r-1}, p^r) \\ &= g_{\|\alpha_i\|^2}\left((C_i')^{-2}\prod_{j\neq i}(C_j')^{-2\langle\alpha_j,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle} p^{r-1}, p^r\right)_S = g_{\|\alpha_i\|^2}(C_0, p^r). \end{aligned}$$

□

Using Lemmas 5.3 and 5.4, we may rewrite the Dirichlet series $Z_\Psi(s_1, \ldots, s_r)$ in terms of a Kubota Dirichlet series in the variable s_i.

Proposition 5.5. *We have*

$$Z_\Psi(s_1, \ldots, s_r) = \sum_{\substack{0 \neq C_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (C_1, \cdots, C_r) \text{ admissible} \\ C_i \text{ } i\text{-reduced}}} \mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r} H(C_1, \ldots, C_r)\, \mathcal{D}_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0),$$

where, for fixed $C_1, \cdots, C_r$, the coefficient C_0 is defined in (41).

Proof. We have already rewritten the Dirichlet series $Z_\Psi(s_1, \ldots, s_r)$ in equation (36) in terms of sums over C_j, $j = 1, \ldots, r$ with C_i i-reduced. The proposition then follows immediately from the previous two lemmas and the definition of $\mathcal{D}_t(s, \Psi, C)$ for S-integer C and $\Psi \in \mathcal{M}_t(\Omega)$, where $t = \|\alpha_i\|^2$.

We are preparing to prove a functional equation corresponding to the transformation σ_i defined in (27).

Let $\mathcal{A}$ be the ring of (Dirichlet) polynomials in $q_v^{\pm 2s_1}, \ldots, q_v^{\pm 2s_r}$ where v runs through the finite set of places S_{fin}. Further define $\mathfrak{M} = \mathcal{A} \otimes \mathcal{M}(\Omega^r)$. We may regard elements of $\mathfrak{M}$ as functions $\Psi : \mathbb{C}^r \times (F_S^\times)^r \longrightarrow \mathbb{C}$ such that for any fixed $(s_1, \ldots, s_r) \in \mathbb{C}^r$ the function

$$(C_1, C_2, \ldots, C_r) \longmapsto \Psi(s_1, \ldots, s_r; C_1, \ldots, C_r)$$

defines an element of $\mathcal{M}(\Omega^r)$, while for any $(C_1, \ldots, C_r) \in (F_S^\times)^r$, the function

$$(s_1, \ldots, s_r) \longmapsto \Psi(s_1, \ldots, s_r; C_1, \ldots, C_r)$$

is an element of $\mathcal{A}$. We will sometimes use the notation

$$\Psi_s(C_1, \ldots, C_r) = \Psi(s_1, \ldots, s_r; C_1, \ldots, C_r), \qquad \boldsymbol{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r. \tag{45}$$

We identify $\mathcal{M}(\Omega^r)$ with its image $1 \otimes \mathcal{M}(\Omega^r)$ in $\mathfrak{M}$; this just consists of the $\Psi_{\boldsymbol{s}}$ that are independent of $\boldsymbol{s} \in \mathbb{C}^r$.

Recall that we defined operators σ_i on $\mathbb{C}^r$ in (27). We define corresponding operators σ_i on $\mathfrak{M}$ by

$$\begin{aligned}(\sigma_i \Psi_{\boldsymbol{s}})(C_1, \cdots, C_r) = (\sigma_i \Psi)(s_1, \cdots, s_r; C_1, \cdots, C_r) = \\ \sum_{\eta \in F_S^\times / F_S^{\times, n}} (\eta, C_i)_S^{\|\alpha_i\|^2} \prod_{j>i} (\eta, C_j^{2\langle \alpha_i, \alpha_j \rangle})_S \, P_{\eta C_0}(s_i) \\ \Psi(\sigma_i(s_1, \cdots, s_r); C_1, C_2, \cdots, \eta^{-1} C_i, \cdots, C_r)\end{aligned} \tag{46}$$

where, as in (41),

$$C_0 = \prod_j C_j^{-2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle} = C_i^{-2} \prod_{j \neq i} C_j^{-2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle}.$$

Proposition 5.6. *If $\Psi \in \mathfrak{M}$, then $\sigma_i \Psi$ is in $\mathfrak{M}$.*

Proof. Let $\varepsilon_1, \ldots, \varepsilon_r \in \Omega$. We have

$$\begin{aligned}(\sigma_i \Psi)(s_1, \cdots, s_r; \varepsilon_1 C_1, \cdots, \varepsilon_r C_r) = \\ \sum_{\eta \in F_S^\times / F_S^{\times, n}} (\eta, \varepsilon_i C_i)_S^{\|\alpha_i\|^2} \prod_{j>i} (\eta, (\varepsilon_j C_j)^{2\langle \alpha_i, \alpha_j \rangle})_S \, P_{\eta \varepsilon_0 C_0}(s_i) \\ \Psi(\sigma_i(s_1, \cdots, s_r); \varepsilon_1 C_1, \cdots, \eta^{-1} \varepsilon_i C_i, \cdots, \varepsilon_r C_r)\end{aligned}$$

where, in parallel with the definition of C_0, we let

$$\varepsilon_0 = \varepsilon_i^{-2} \prod_{j \neq i} \varepsilon_j^{-2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle}.$$

Making the variable change $\eta \longmapsto \varepsilon_0^{-1} \eta$, we may rewrite the above as

$$\begin{aligned}\sum_{\eta \in F_S^\times / F_S^{\times, n}} (\varepsilon_0^{-1} \eta, \varepsilon_i C_i)_S^{\|\alpha_i\|^2} \prod_{j>i} (\varepsilon_0^{-1} \eta, \varepsilon_j C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} P_{\eta C_0}(s_i) \\ \Psi(\sigma_i(s_1, \cdots, s_r); \varepsilon_1 C_1, \cdots, \eta^{-1} \varepsilon_0 \varepsilon_i C_i, \cdots, \varepsilon_r C_r).\end{aligned}$$

By the transformation property (8) of the function Ψ in $\mathcal{M}(\Omega^r)$ this equals

$$\begin{aligned}\sum_{\eta \in F_S^\times / F_S^{\times, n}} (\varepsilon_0^{-1} \eta, \varepsilon_i C_i)_S^{\|\alpha_i\|^2} \prod_{j>i} (\varepsilon_0^{-1} \eta, \varepsilon_j C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} P_{\eta C_0}(s_i) \\ (\varepsilon_0 \varepsilon_i, \eta^{-1} C_i)_S^{\|\alpha_i\|^2} \prod_{j \neq i} (\varepsilon_j, C_j)_S^{\|\alpha_j\|^2} \\ \prod_{j>i} (\varepsilon_0 \varepsilon_i, C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \prod_{j<i} (\varepsilon_j, \eta^{-1} C_i)_S^{2\langle \alpha_i, \alpha_j \rangle} \prod_{\substack{j < k \\ j, k \neq i}} (\varepsilon_j, C_k)_S^{2\langle \alpha_j, \alpha_k \rangle} \\ \Psi(\sigma_i(s_1, \cdots, s_r); C_1, \cdots, \eta^{-1} C_i, \cdots, C_r).\end{aligned}$$

Looking just at factors of the form $(\eta, \varepsilon_j)_S$ we find the following:

$$(\eta, \varepsilon_i)_S^{\|\alpha_i\|^2} (\varepsilon_0\varepsilon_i, \eta^{-1})_S^{\|\alpha_i\|^2} \prod_{j>i} (\eta, \varepsilon_j)_S^{2\langle\alpha_i,\alpha_j\rangle} \prod_{j<i} (\varepsilon_j, \eta^{-1})_S^{2\langle\alpha_i,\alpha_j\rangle}.$$

These cancel due to the definition of ε_0. We cancel these, and remember also that due to the isotropy of Ω, we have $(\varepsilon_j, \varepsilon_k)_S = 1$. Rearranging the terms and using the definition of $\sigma_i\Psi$ gives

$$\begin{aligned}(\varepsilon_0^{-1}, C_i)_S^{\|\alpha_i\|^2} \prod_{j>i} (\varepsilon_0^{-1}, C_j)_S^{2\langle\alpha_i,\alpha_j\rangle} (\varepsilon_0\varepsilon_i, C_i)_S^{\|\alpha_i\|^2} \prod_{j\neq i} (\varepsilon_j, C_j)_S^{\|\alpha_j\|^2} \\ \prod_{j>i} (\varepsilon_0\varepsilon_i, C_j)_S^{2\langle\alpha_i,\alpha_j\rangle} \prod_{j<i} (\varepsilon_j, C_i)_S^{2\langle\alpha_i,\alpha_j\rangle} \prod_{\substack{j<k \\ j,k\neq i}} (\varepsilon_j, C_k)_S^{2\langle\alpha_j,\alpha_k\rangle} \\ \sigma_i\Psi_{\boldsymbol{s}}(C_1, \cdots, C_r).\end{aligned}$$

Expanding the Hilbert symbols, everything involving ε_0 cancels and we are left with

$$\begin{aligned}(\varepsilon_i, C_i)_S^{\|\alpha_i\|^2} \prod_{j\neq i} (\varepsilon_j, C_j)_S^{\|\alpha_j\|^2} \prod_{j>i} (\varepsilon_i, C_j)_S^{2\langle\alpha_i,\alpha_j\rangle} \\ \prod_{j<i} (\varepsilon_j, C_i)_S^{2\langle\alpha_i,\alpha_j\rangle} \prod_{\substack{j<k \\ j,k\neq i}} (\varepsilon_j, C_k)_S^{2\langle\alpha_j,\alpha_k\rangle} \sigma_i\Psi_{\boldsymbol{s}}(C_1, \cdots, C_r) = \\ \prod_{j} (\varepsilon_j, C_j)_S^{\|\alpha_j\|^2} \prod_{j<k} (\varepsilon_j, C_k)_S^{2\langle\alpha_j,\alpha_k\rangle} \sigma_i\Psi_{\boldsymbol{s}}(C_1, \cdots, C_r).\end{aligned}$$

This proves that $\sigma_i\Psi$ satisfies (8). □

Let $\mathfrak{W}$ denote the group of automorphisms of $\mathfrak{M}$ generated by σ_i. This will turn out to be the group of functional equations for the multiple Dirichlet series. Each functional equation corresponding to $\sigma_i \in W$ is inherited from a functional equation for the Kubota Dirichlet series appearing in Proposition 5.5. There is a natural homomorphism $\mathfrak{W} \longrightarrow W$, which is expected to be an isomorphism, though we do not prove this. (But see Corollary 5.9 and the discussion at the end.) There is an action of $\mathfrak{W}$ on $\mathbb{C}^r$ induced by the action of W, and if $w \in \mathfrak{W}$ we will denote by $w\boldsymbol{s}$ the effect of w on $\boldsymbol{s} \in \mathbb{C}^r$ in this induced action.

These functional equations are formalized in the following result.

Lemma 5.7. *Given an element $\Psi_{\boldsymbol{s}}(C_1, \ldots, C_r) \in \mathfrak{M}$, we have*

$$\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0) = \mathbb{N}C_0^{1-2s_i} \mathcal{D}^*_{\|\alpha_i\|^2}(1-s_i, (\sigma_i\Psi)_i^{C_1,\ldots,C_r}, C_0),$$

where $\mathcal{D}^*_{\|\alpha_i\|^2}$ is as in (38).

Proof. By Theorem 2.7, and using (39)

$$\begin{aligned}\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0) = \\ \sum_{\eta\in F_S^\times/F_S^{\times,n}} \mathbb{N}C_0^{1-2s_i} P_{\eta C_0}^{\|\alpha_i\|^2}(s_i) \mathcal{D}^*_{\|\alpha_i\|^2}(1-s_i, \tilde{\Psi}_{i,\eta}^{C_1,\ldots,C_r}, C_0),\end{aligned}$$

where, in this context, by (40) and (37)

$$\begin{aligned}\tilde{\Psi}_{i,\eta}^{C_1,\dots,C_r}(D) &= (\eta, D)_S^{\|\alpha_i\|^2} \Psi_i^{C_1,\dots,C_r}(\eta^{-1}D^{-1}) \\ &= (\eta, D)_S^{\|\alpha_i\|^2} (\eta D, C_i)_S^{\|\alpha_i\|^2} \prod_{j>i} (\eta D, C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \\ &\quad \Psi(C_1, \dots, C_{i-1}, D^{-1}\eta^{-1}C_i, C_{i+1}, \dots, C_r).\end{aligned}$$

If we take the s_i to have sufficiently large negative real part, we may expand the right-hand side of the above functional equation as a Dirichlet series. That is,

$$\sum_{\eta \in F_S^\times / F_S^{\times,n}} \mathbb{N}C_0^{1-2s_i} P_{\eta C_0}^{\|\alpha_i\|^2}(s_i)\, \mathcal{D}^*_{\|\alpha_i\|^2}(1-s, \tilde{\Psi}_{i,\eta}^{C_1,\cdots,C_r}, C_0) =$$

$$\sum_{\eta \in F_S^\times / F_S^{\times,n}} \mathbb{N}C_0^{1-2s_i} P_{\eta C_0}^{\|\alpha_i\|^2}(s_i) \sum_{0 \neq D \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g_{\|\alpha_i\|^2}(C_0, D)$$

$$\mathbb{N}D^{2s_i - 2} (\eta, D)_S^{\|\alpha_i\|^2} (D\eta, C_i)_S^{\|\alpha_i\|^2}$$

$$\times \prod_{j>i} (D\eta, C_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \Psi(C_1, \cdots, C_{i-1}, D^{-1}\eta^{-1}C_i, C_{i+1}, \cdots, C_r).$$

We now apply the change of variables $\eta \longrightarrow D^{-2}\eta$ (noting that $(D, D)_S = 1$) giving

$$\sum_{\eta \in F_S^\times / F_S^{\times,n}} \mathbb{N}C_0^{1-2s_i} P_{\eta D^{-2} C_0}^{\|\alpha_i\|^2}(s_i) \sum_{0 \neq D \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g_{\|\alpha_i\|^2}(C_0, D)\, \mathbb{N}D^{2s_i - 2}$$

$$\times \left(\eta, (DC_i)^{\|\alpha_i\|^2} \prod_{j>i} C_j^{2\langle \alpha_i, \alpha_j \rangle} \right)_S (D, C_i)_S^{-\|\alpha_i\|^2} \prod_{j>i} (D, C_j)_S^{-2\langle \alpha_i, \alpha_j \rangle}$$

$$\times \Psi(C_1, \cdots, C_{i-1}, \eta^{-1}DC_i, C_{i+1}, \cdots, C_r).$$

Rewriting this in terms of the operator σ_i on $\mathfrak{M}$ defined in (46), we have

$$\sum_{0 \neq D \in \mathfrak{o}_S / \mathfrak{o}_S^\times} \mathbb{N}C_0^{1-2s_i} g_{\|\alpha_i\|^2}(C_0, D)\, \mathbb{N}D^{2s_i - 2} (D, C_i)_S^{-\|\alpha_i\|^2}$$

$$\prod_{j>i} (D, C_j)_S^{-2\langle \alpha_i, \alpha_j \rangle} (\sigma_i \Psi)(C_1, \dots, DC_i, \dots, C_r).$$

According to (37), this is simply

$$\sum_{0 \neq D \in \mathfrak{o}_S / \mathfrak{o}_S^\times} \mathbb{N}C_0^{1-2s_i} g_{\|\alpha_i\|^2}(C_0, D) \mathbb{N}D^{2s_i - 2} (\sigma_i \Psi)_i^{C_1,\dots,C_r}(D) =$$

$$\mathbb{N}C_0^{1-2s_i} \mathcal{D}^*_{\|\alpha_i\|^2}(1 - s_i, (\sigma_i \Psi)_i^{C_1,\dots,C_r}, C_0).$$

Theorem 5.8. *The function $Z_\Psi^*(\mathbf{s})$ has meromorphic continuation to the complex space $\mathbb{C}^r$ and satisfies the functional equation*

$$Z_{w\Psi}^*(w\boldsymbol{s}) = Z_\Psi^*(\boldsymbol{s}) \tag{47}$$

for each $w \in \mathfrak{W}$, where the action of w on $\mathfrak{M}$ is given by the composition of simple reflections σ_i. It is analytic except along the hyperplanes $B(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = \frac{1}{2n(\alpha)}$, where α runs through Φ, $n(\alpha)$ is defined by (29), $\frac{1}{2}\rho^\vee = \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$ and B is defined by (24); along these hyperplanes it can have simple poles.

Observe that the equation $B(-\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = \frac{1}{2n(\alpha)}$ is equivalent to $B(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = -\frac{1}{2n(\alpha)}$, so the polar hyperplanes occur in parallel pairs.

Proof. The reader may find it useful to look at the special case where $\Phi = A_2$ in [5], which is worked out in full detail.

It may be checked that the original Dirichlet series defining Z_Ψ is absolutely convergent in the region Λ_0 defined by

$$\Re(s_j) > \tfrac{3}{4}.$$

The expression in Proposition 5.5 is analytic in the region Λ_i which is the convex hull of $\Lambda_0 \cup \sigma_i \Lambda_0$. On this region, we claim that for each simple reflection σ_i we have

$$Z^*_{\sigma_i \Psi}(\sigma_i \boldsymbol{s}) = Z^*_\Psi(\boldsymbol{s}).$$

In order to deduce this from Proposition 5.5 and Lemma 5.7 one must consider the effect of the transformation (27). It remains to be shown that under this change of variables the quantity $\mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r}$ gets multiplied by $\mathbb{N}C_0^{1-2s_i}$; in other words, that $\mathbb{N}C_0^{s_i - \frac{1}{2}} \cdot \mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r}$ is invariant under (27) for the fixed index i. This follows from Proposition 4.2, since with our definitions, when $C_1, \cdots, C_r$ is admissible and C_i is i-reduced (19) implies that

$$\mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r} = \prod_p \prod_{\alpha \in \Phi_w} M_p(-w\alpha; \boldsymbol{s})$$

and, by (43)

$$\mathbb{N}C_0 = \prod_p \mathbb{N}p^{d(w_p^{-1}\alpha_i) - 1}.$$

Concerning the normalizing factor, one factor $G_{\alpha_i}(\boldsymbol{s})\zeta_{\alpha_i}(\boldsymbol{s})$ from (31) is needed to normalize $\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0)$ in Proposition 5.5; the remaining factors are permuted amongst themselves since σ_i permutes $\Phi^+ - \{\alpha_i\}$.

Arguing as in [5] we obtain analytic continuation to any simply-connected region Λ' that is a union of W-translates of the Λ_i obtained by composing functional equations. We may choose Λ' so that its convex hull is all of $\mathbb{C}^r$. The meromorphic continuation to all of $\mathbb{C}^r$ follows from Bochner's tube-domain Theorem (Bochner [1] or Hörmander [12], Theorem 2.5.10). As in [5], one actually applies Bochner's theorem to the function

$$Z_\Psi(\boldsymbol{s}) \prod_{\alpha \in \Phi} \left(B\left(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee\right) - \frac{1}{2n(\alpha)} \right),$$

since inclusion of the factors to cancel the poles of Z_Ψ gives a function that is everywhere analytic.

We have not proved that the natural map $\mathfrak{W} \longrightarrow W$ is an isomorphism, so we cannot assert that W itself acts on $\mathfrak{M}$. To prove that $\mathfrak{W} \cong W$, one would need to check that σ_i and $\sigma_j \in \mathfrak{W}$ satisfy the same braid relation $(\sigma_i \sigma_j)^{r_{ij}} = 1$ as their images in W. It is highly likely that this is true, but not too important for the functional equations as we will now see. We recall from the introduction that we defined $\mathfrak{M}'$ to be the quotient of $\mathfrak{M}$ by the kernel of the map $\Psi \longmapsto Z_\Psi$.

Corollary 5.9. *If $w \in \mathfrak{W}$ is in the kernel of the map $\mathfrak{W} \longrightarrow W$ then $Z_\Psi = Z_{w\Psi}$. Thus there is an action of W on $\mathfrak{M}'$ that is compatible with the action of $\mathfrak{W}$ on $\mathfrak{M}$.*

Proof. Since such a w acts trivially on $\mathbb{C}^r$, (47) implies that Z_Ψ and $Z_{w\Psi}$ are multiple Dirichlet series that agree in the region of absolute convergence. Hence they are equal. □

Thus Z_Ψ can be regarded as depending on $\Psi \in \mathfrak{M}'$, on which W acts, and so W can truly be regarded as the group of functional equations of the Weyl group multiple Dirichlet series.

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