Twisted Conjugacy

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If G is a finite group and $\tau : G \longrightarrow G$ is an automorphism, then *twisted conjugacy* with respect to τ is the action of G on itself by $g : x \longmapsto gxg^{-\tau}$. The orbits of this action are called τ-twisted conjugacy classes.

Theorem 1 *Let G be a finite group, and let $\tau : G \longrightarrow G$ an automorphism of degree m. The following three numbers are equal:*
(i) The number of τ-twisted conjugacy classes;
(ii) The number of τ-invariant conjugacy classes;
(iii) The number of τ-invariant irreducible representations of G.

Proof There are two permutation actions of $\langle\tau\rangle$ which we may consider: the action on the conjugacy classes of G, and the action on the irreducible representations of G. Both permutation actions are realized on the center of $\mathbb{C}[G]$, the first by taking the basis of the center comprised of the conjugacy class sums, the second by taking the basis of central idempotents parametrized by the irreducible representations. Thus these actions are equivalent, and the number of τ invariant representations equals the number of τ-invariant conjugacy classes.

We must therefore show that the number of τ-invariant conjugacy classes equals the number of twisted conjugacy classes. We will count the number N of solutions (x, g) to the equation $g^{-1}xg =^\tau x$ in two different ways.

First, N equals

$$\sum_{x \sim^\tau x} \#\{g \in G | g^{-1}xg =^\tau x\} = \sum_{x \sim^\tau x} |C_G(x)|$$

because given x which is conjugate to ${}^\tau x$, we may count the number of g conjugating x to ${}^\tau x$ by fixing one such g, then noting that any other such g

must differ from that one by an element of the centralizer $|C_G(x)|$. Because the number of elements of the conjugacy class of x is $|G|/|C_G(x)|$, we see that N is $|G|$ times the number of τ-invariant conjugacy classes.

On the other hand, N equals

$$\sum_{g\in G} \#\{x \in G | xg^\tau x^{-1} = g\}.$$

The index in G of $\#\{x \in G | xg^\tau x^{-1} = g\}$ is the order of the twisted conjugacy class of g, so this is $|G|$ times the number of twisted conjugacy classes. $\square$

We are interested in twisted conjugacy for the group $G(\mathbb{F}_{q^m}) = \mathrm{GL}(n, \mathbb{F}_{q^m})$, where τ is the Frobenius $x \longmapsto x^{(q)}$. The story of twisted conjugacy is intimately connected with the *Norm map* $N : G(\mathbb{F}_{q^m}) \longrightarrow G(\mathbb{F}_{q^m})$, defined by

$$N(g) = g \cdot {}^\tau g \cdot {}^{\tau^2} g \cdot \ldots \cdot {}^{\tau^{m-1}} g.$$

If $n = 1$, so that $G = \mathrm{GL}(1)$ is abelian, this map takes values in $\mathrm{GL}(1, \mathbb{F}_q)$, so we might hope that the norm map takes $\mathrm{GL}(n, \mathbb{F}_{q^m})$ into $\mathrm{GL}(n, \mathbb{F}_q)$. This is not true. What *is* true is that the map N induces a map from *conjugacy classes* of $\mathrm{GL}(n, \mathbb{F}_{q^m})$ to conjugacy classes of $\mathrm{GL}(n, \mathbb{F}_q)$.

Theorem 2 (Shintani) *Let τ be the Frobenius endomorphism of $\mathrm{GL}(n, \mathbb{F}_{q^m})$.*
(i) If g_1 and g_2 are τ-twisted conjugates then $N(g_1)$ and $N(g_2)$ are conjugates;
(ii) If $g \in G(\mathbb{F}_{q^m})$ then $N(g)$ is conjugate to an element of $G(\mathbb{F}_q)$;
(iii) The intersection of $N(x)$ with $\mathrm{GL}(n, \mathbb{F}_q)$ as x runs through the twisted conjugacy classes of $\mathrm{GL}(n, \mathbb{F}_{q^m})$ is a single conjugacy class of $\mathrm{GL}(n, \mathbb{F}_q)$.

Proof We have $N(xg_1x^{-\tau}) = xN(g_1)x^{-1}$, so (i) is clear.

For (ii), applying τ to $N(g)$ gives $g^{-1}N(g)g$, so the conjugacy class of $N(g)$ is Galois invariant; we have shown in *The Lang Map* (using both the Rational Canonical Form and Lang's theorem) that this implies that the $\mathrm{GL}(n, \mathbb{F}_{q^m})$-conjugacy class of $N(g)$ contains an element of $\mathrm{GL}(n, \mathbb{F}_q)$.

For (iii), we also proved in *The Lang Map* (using both the Rational Canonical Form and Lang's theorem) that two elements of $\mathrm{GL}(n, \mathbb{F}_q)$ are conjugate in $\mathrm{GL}(n, \mathbb{F}_{q^m})$ if and only if they are already conjugate in $\mathrm{GL}(n, \mathbb{F}_q)$, and from this and (i) and (ii), the assertion is clear. $\square$

Theorem 3 (Shintani) *The number of τ-invariant irreducible representations of $\mathrm{GL}(n, \mathbb{F}_{q^m})$ is equal to the number of irreducible representations of $\mathrm{GL}(n, \mathbb{F}_q)$.*

Proof This follows from Theorems 1 and 2, and the fact that the number of irreducible representations of a group is equal to the number of conjugacy classes. $\square$

Note that the proofs above depend only on the Lang map and are therefore also valid for parabolic subgroups.