Shintani's Theorem

March 2, 2004

We proved earlier, in *Twisted Conjugacy* that the numbers of Galois invariant irreducible representations of $\mathrm{GL}(n, \mathbb{F}_{q^m})$ and irreducible representations of $\mathrm{GL}(n, \mathbb{F}_q)$ are equal. Shintani describes a bijection between these two sets of representations by character identities, which are the basis of the Shintani-Langlands-Arthur-Clozel theory of base change for automorphic forms.

Let (π, W) be an irreducible representation of $\mathrm{GL}(n, \mathbb{F}_q)$ and let (Π, V) be a Galois invariant irreducible representation of $\mathrm{GL}(n, \mathbb{F}_{q^m})$. Since Π is isomorphic to its twist by the Frobenius automorphism τ of $\mathrm{GL}(n, \mathbb{F}_{q^m})$, there exists an intertwining operator $j : \Pi \longrightarrow \Pi^\tau$. In concrete terms, j is an endomorphism of V such that

$$j \circ \pi(g) = \pi({}^\tau g) \circ j. \tag{1}$$

We note that $j^m \circ \pi(g) = \pi(g) \circ j^m$, so by Schur's Lemma, j^m is a scalar. Multiplying by a suitable constant, we may assume that $j^m = 1$. We will denote by χ_π the character of π. We will say that Π is a *Shintani-lift* of π if

$$\chi_\pi(N(g)) = \varepsilon \ \mathrm{tr}(\pi(g) \circ j), \qquad \varepsilon = \pm 1.$$

Theorem 1 (Shintani) *There is a unique bijection between the the set of irreducible representations of* $\mathrm{GL}(n, \mathbb{F}_q)$ *and the set of Galois invariant irreducible representations of* $\mathrm{GL}(n, \mathbb{F}_{q^m})$ *taking a representation* π *of* $\mathrm{GL}(n, \mathbb{F}_q)$ *to a Shintani-lift.*

We will prove this, eventually. We begin by noting a reformulation. Let $\widehat{\mathrm{GL}}(n, \mathbb{F}_{q^m})$ be the semidirect product of $\mathrm{GL}(n, \mathbb{F}_{q^m})$ by a cyclic group $\langle t \rangle$ acting on $\mathrm{GL}(n, \mathbb{F}_{q^m})$ by conjugation, so that $t\,g\,t^{-1} = {}^\tau g$. By (1), we can then extend Π to $\widetilde{\mathrm{GL}}(n, \mathbb{F}_{q^m})$ so that $\Pi(t) = j$. Let χ_Π be the character of this extension. Then the identity is that

$$\chi_\pi(N(g)) = \varepsilon\, \chi_\Pi(tg). \tag{2}$$

In this installment, we will prove a sequence of Lemmas, due to Shintani.

1 Inner Product Formula

Let $G = \mathrm{GL}(n)$, or a parabolic subgroup of $\mathrm{GL}(n)$.

Theorem 2 *There exists an isometry $f \longmapsto f'$ between the Hilbert space of functions on $G(\mathbb{F}_{q^m})$ which are constant on twisted conjugacy classes and the space of class functions on $G(\mathbb{F}_q)$, such that $f(h) = f'(g)$ if $g \in G(\mathbb{F}_q)$ is conjugate to $N(h)$, with $g \in h(\mathbb{F}_{q^m})$.*

Proof We recall from Theorem 2 (iii) in *Twisted Conjugacy* that if $y \in P(\mathbb{F}_{q^m})$ then $N(y)$ may not be an element of $P(\mathbb{F}_q)$ but it is always conjugate to an element of $G(\mathbb{F}_q)$, and that the set of $N(x)$ which are in $G(\mathbb{F}_q)$, as x runs through the twisted conjugacy class of y is a single conjugacy class in $G(\mathbb{F}_q)$. Thus the map $f \longmapsto f'$ exists as a bijection between the twisted conjugacy classes in $G(\mathbb{F}_{q^m})$ and the conjugacy classes in $G(\mathbb{F}_q)$. Further work is needed to show that it is an isometry.

We will refine this bijection. Suppose that $g \in G(\mathbb{F}_q)$. We will prove that there exists $h \in G(\mathbb{F}_{q^m})$ such that $N(h) = g$, and such that the twisted centralizer of h in $G(\mathbb{F}_{q^m})$ equals the centralizer of g, that is,

$$u \in G(\mathbb{F}_{q^m}) \text{ and } u \cdot h \cdot {}^{\tau}u^{-1} = h \iff u \in G(\mathbb{F}_q) \text{ and } ugu^{-1} = g. \tag{3}$$

Let K be the algebraic closure of $\mathbb{F}_q$. Let $A(K)$ be the K-algebra generated by g in $\mathrm{Mat}_n(K)$. By Theorem 4 in *The Lang Map* we can find invertible $x \in A(K)$ such that $x^{-1} \cdot {}^{\tau^m}x = g$. Now consider $h = x^{-1} \cdot {}^{\tau}x$. We claim that $h \in G(\mathbb{F}_{q^m})$. To see this, note that $g = {}^{\tau}g$, so $x^{-1} \cdot {}^{\tau^m}x = {}^{\tau}x^{-1} \cdot {}^{\tau^{m+1}}x$. Since $A(K)$ is commutative, this means that $x^{-1} \cdot {}^{\tau}x = {}^{\tau^m}x^{-1} \cdot {}^{\tau^{m+1}}x$, so $h = {}^{\tau^m}h$ and therefore $h \in G(\mathbb{F}_{q^m})$. It is easy to see that $N(h) = g$. Suppose that $u \cdot h \cdot {}^{\tau}u^{-1} = h$, with $u \in G(\mathbb{F}_{q^m})$. Taking norms on both sides in this identity, $ugu^{-1} = g$. Now since $x \in A(K)$, $h \in A(K)$ which means that h is expressible as a polynomial in terms of g. Since u commutes with g, it commutes with h. Combining this fact with $uh^{\tau}u^{-1} = h$ we see that $u = {}^{\tau}u$, so $u \in G(\mathbb{F}_q)$. This proves one implication in (3), and the other implication is trivial.

With $g = N(h)$ and g, h satisfying (3), let us show that the functions f and f' which are the respective characteristic functions of the twisted conjugacy class of h and the conjugacy class of g have the same norm. The cardinality of the twisted conjugacy class of h is the index in $G(\mathbb{F}_{q^m})$ of the twisted centralizer of h, which by (3) is the same as $Z_{G(\mathbb{F}_q)}(g)$. Hence the L^2 norm of f is $1/|Z_{G(\mathbb{F}_q)}|$, and the L^2 norm of f' is the same. The result follows. □

2 Twisted Tensor representation

Lemma 1 *Let V be a finite-dimensional complex vector space, and let $A_1, \cdots, A_m$ be linear transformations of V. Then the trace of the linear transformation*

$$v_1 \otimes \cdots \otimes v_m \longrightarrow A_2v_2 \otimes \cdots \otimes A_mv_m \otimes A_1v_1 \tag{4}$$

is $\mathrm{tr}(A_1A_2\cdots A_m)$.

Proof If $x_1, \cdots, x_n$ is a basis of V, and if $A_i x_j = \sum_k a_{ikj} x_k$, then (4) takes $x_{i_1} \otimes \cdots \otimes x_{i_r}$ to

$$\sum_{j_1, \cdots, j_r} a_{2j_2i_2} a_{3j_3i_3} \cdots a_{rj_ri_r} a_{1j_1i_1} x_{j_2} \otimes \cdots \otimes x_{j_r} \otimes x_{j_1}.$$

The trace is the sum of the contributions with $j_2 = i_1, j_3 = i_2$, etc., that is,

$$\sum_{i_1, \cdots, i_r} a_{2i_1i_2} a_{3i_2i_3} \cdots a_{ri_{r-1}i_r} a_{1i_ri_1} = \mathrm{tr}(A_2 A_3 \cdots A_1) = \mathrm{tr}(A_1 A_2 \cdots A_r).$$

$\square$

Proposition 1 *Let G be a group with an automorphism τ of order m, and let $\hat{G}$ be the semidirect product $G \rtimes \langle t \rangle$, where $\langle t \rangle$ is a cyclic group with generator t of order m, acting on G by τ:*

$$tgt^{-1} = {}^\tau g, \qquad g \in G.$$

If (π, V) is a complex representation of G, with character χ, then there exists a representation (Π, W) of $\hat{G}$ with character Ξ such that

$$\Xi(tg) = \chi(N(g)), \qquad N(g) = g \cdot {}^\tau g \cdot \ldots \cdot {}^{\tau^{m-1}} g.$$

Proof Let $W = V \otimes \cdots \otimes V$. Define compatible actions of G and $\langle t \rangle$ on W by

$$\Pi(g) = \pi(g) \otimes \pi({}^\tau g) \otimes \cdots \otimes \pi({}^{\tau^{m-1}} g),$$

$$\Pi(t)(v_1 \otimes \ldots \otimes v_m) = v_2 \otimes v_3 \otimes \cdots \otimes v_m \otimes v_1.$$

Applying the Lemma with $A_i = \pi({}^{\tau^{i-1}} g)$ gives the advertised value of $\Xi(tg)$. $\square$

3 Compatibility with Parabolic Induction

If H is a subgroup of G and φ is a class function, we may induce it to G by the formla

$$\mathrm{Ind}(\varphi)(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\varphi}(xgx^{-1}),$$

where if $g \in H$,

$$\dot{\varphi}(g) = \begin{cases} \varphi(g) & \text{if } g \in H, \\ 0 & \text{otherwise}. \end{cases}$$

If φ is the character of a representation of H, then $\mathrm{Ind}(\varphi)$ is the character of the induced representation.

Lemma 2 *Suppose that $H \subset G$ and that φ is the characteristic function of a single conjugacy class. If h is a representative, then if g is conjugate to h*

$$\mathrm{Ind}(\varphi)(g) = \begin{cases} [C_G(h) : C_H(h)] & \text{if } g \text{ is conjugate to } h, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Clearly $\mathrm{Ind}(\varphi)(g)$ vanishes off the G-conjugacy class of h, so it suffices to prove this when $g = h$.

$$\mathrm{Ind}(\varphi)(h) = \frac{1}{|H|} \sum_{h' \text{ is } H\text{-conjugate to } h} \#\{x | xhx^{-1} = h'\}.$$

The number of conjugates h' of h in H is $[H : C_H(h)]$ and for each conjugate h' of h, the cardinality of $\{x | xhx^{-1} = h'\}$ is $|C_G(h)|$, and the statement follows. □

Let P be a parabolic subgroup of $\mathrm{GL}(n)$. We will denote by $\hat{P}(\mathbb{F}_{q^m})$ and $\widehat{\mathrm{GL}}(n, \mathbb{F}_{q^m})$ the semidirect products of $P(\mathbb{F}_{q^m})$ and $\mathrm{GL}(n, \mathbb{F}_{q^m})$ by a cyclic group $\langle t \rangle$ acting by conjugation, so that $t\,g\,t^{-1} = {}^\tau g$.

Let f' and f be functions on $P(\mathbb{F}_q)$ and $P(\mathbb{F}_{q^m})$, respectively, which match as in Section 1. Thus f' is a class function, f a twisted class function, and $f(g) = f'(h)$ when g is conjugate to $N(h)$. We can associate with f a class function $\tilde{f}$on $\hat{P}(\mathbb{F}_{q^m})$ by $\tilde{f}(gt) = f(g)$ when $g \in P(\mathbb{F}_{q^m})$, while $\tilde{f}(\hat{g}) = 0$ if $\hat{g} \notin P(\mathbb{F}_{q^m})$. We compare the inductions of f' and $\tilde{f}$ to $\mathrm{GL}(n, \mathbb{F}_{q^m})$ and $\widehat{\mathrm{GL}}(n, \mathbb{F}_{q^m})$.

Proposition 2 *With notations as above,*

$$\mathrm{Ind}(\tilde{f})(ht) = \mathrm{Ind}(f')(N(h)). \tag{5}$$

The function $\mathrm{Ind}(\tilde{f})$ vanishes off the coset $\mathrm{GL}(n, \mathbb{F}_{q^m})\,t$.

Proof The coset $\mathrm{GL}(n, \mathbb{F}_{q^m})\,t$ is stable under conjugation, and the support of $\tilde{f}$ is contained in this coset, so $\mathrm{Ind}(\tilde{f})$ vanishes off the coset $\mathrm{GL}(n, \mathbb{F}_{q^m})\,t$.

We may assume that the function f is supported on a single twisted conjugacy class, say that of h_0 where $h_0 \in P(\mathbb{F}_{q^m})$. Let $N(h_0) = g_0$. As we saw in the proof of Theorem 2, we may choose h_0 and g_0 so that $g_0 \in P(\mathbb{F}_q)$ and the twisted centralizer of h_0 in $P(\mathbb{F}_{q^m})$ is the ordinary centralizer of g_0 in $P(\mathbb{F}_q)$:

$$u \in P(\mathbb{F}_{q^m}) \text{ and } u \cdot h_0 \cdot {}^\tau u^{-1} = h_0 \iff u \in P(\mathbb{F}_q) \text{ and } u g_0 u^{-1} = g_0. \tag{6}$$

In fact, the proof of Theorem 2 constructs h_0 in the multiplicative group of the algebra generated by g_0, and if this is done, the argument there shows that

$$u \in G(\mathbb{F}_{q^m}) \text{ and } u \cdot h_0 \cdot {}^\tau u^{-1} = h_0 \iff u \in G(\mathbb{F}_q) \text{ and } u g_0 u^{-1} = g_0. \tag{7}$$

Now $\tilde{f}$ is supported on the $\tilde{P}(\mathbb{F}_{q^m})$ conjugacy class of $h_0 t$, and f' is supported on the $P(\mathbb{F}_q)$-conjugacy class of g_0.

The function on the left-hand side of (5) is constant on twisted conjugacy classes of $G(\mathbb{F}_{q^m})$ and supported on a single twisted conjugacy class, that of h_0. Now we claim that the following is true of the right-hand side. Indeed, if $\mathrm{Ind}(f')(N(h)) \neq 0$, then $N(h)$ is in the support of f'. This means that $N(h)$ is a $G(\mathbb{F}_q)$-conjugate of $g_0 = N(h_0)$, which as we know means that h is twisted-conjugate to h_0.

We now have to show that the left and right sides have the same values at $h = h_0$. We can use the Lemma to evaluate both sides. For the left side, we note that the ordinary centralizer of $h_0 t$ in $\tilde{P}(\mathbb{F}_{q^m})$ contains as a subgroup of index m the twisted centralizer $C^{\tau}_{P(\mathbb{F}_{q^m})}(h_0)$ of h_0 in $P(\mathbb{F}_{q^m})$, and similarly the ordinary centralizer of $h_0 t$ in G contains as a subgroup of index m the twisted centralizer $C^{\tau}_{G(\mathbb{F}_{q^m})}(h_0)$. Thus

$$\mathrm{Ind}(\tilde{f})(h_0 t) = [C^{\tau}_{G(\mathbb{F}_{q^m})}(h_0) : C^{\tau}_{P(\mathbb{F}_{q^m})}].$$

By (6) and (7), this equals

$$[C_{G(\mathbb{F}_q)}(g_0) : C_{P(\mathbb{F}_q)}(g_0)] = \mathrm{Ind}(f')(h_0).$$

□

4 Completeness of Green's generalized characters

Let θ be a character of $\bar{\mathbb{F}}_q^{\times}$, and if $k \leqslant n$ let

$$e_k(X_1, \cdots, X_n) = \sum_{i_1 < \cdots < i_k} X_{i_1} \cdots X_{i_k}$$

be the k-th elementary symmetric polynomial. Let

$$\Sigma_k^{\theta}(g) = e_k(\theta(\alpha_1), \cdots, \theta(\alpha_n)),$$

where the e_k are the eigenvalues of $g \in \mathrm{GL}(n, \mathbb{F}_q)$. We have proved already that the Σ_k^{θ} are generalized characters.

Theorem 3 *Every generalized character of* $\mathrm{GL}(n, \mathbb{F}_q)$ *is a linear combination with integer coefficients of the* Σ_k^{θ} *and characters parabolically induced from parabolic subgroups.*

We have not quite proven this yet. We have proven that if n is a prime, then any cuspidal character differs from such a character (with $k = 1$) and parabolically induced characters. If we prove this statement for the monatomic characters then the result follows. For the time being we are going to assume this and give the proof of Shintani's Theorem.

5 The Proof

Lemma 3 *Suppose that* $\alpha_1, \cdots, \alpha_h \in \mathbb{Z}[\zeta]$, *where* $\zeta = e^{2\pi i/m}$. *Suppose that* $\sum |\alpha_h|^2 = 1$. *Then all but one of the* α_i *are zero, and the unique nonzero* α_i *is a power of* ζ.

Proof Let $|\alpha_i|^2 = a_i$. These are elements of the ring R of integers in the maximal real subfield $E = \mathbb{Q}(\zeta+\bar{\zeta})$ of $\mathbb{Q}(\zeta)$. They are totally positive, because if $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, then σ commutes with complex conjugation, another element of this abelian Galois group, and so $\sigma(a_i) = \sigma(\alpha_i)\overline{\sigma(\alpha_i)} > 0$. Let $\mathrm{tr}, N : R \longrightarrow \mathbb{Z}$ be the trace and norm maps. By the inequality of the arithmetic and geometric means, we have

$$\frac{1}{[E:\mathbb{Q}]}\,\mathrm{tr}(a_i) \geqslant N(a_i)^{1/d} \geqslant 1, \qquad \text{if } a_i \neq 0.$$

Now

$$[E:\mathbb{Q}] = \sum_{\sigma\in\mathrm{Gal}(E/\mathbb{Q})} 1 = \sum_{\sigma\in\mathrm{Gal}(E/\mathbb{Q})} \sum_i \sigma(a_i) = \sum_i \mathrm{tr}(a_i).$$

Since all nonzero $\mathrm{tr}(a_i) \geqslant [E:\mathbb{Q}]$, exactly one must be nonzero. Moreover, noting that the inequality of the arithmetic and geometric means is strict unless the parameters are equal, we see that the nonzero a_i is equal to its Galois conjugates, so it is 1 and α_i is a root of unity. $\square$

Lemma 4 *Let* G *be a group and* K *a normal subgroup such that* G/K *is cyclic and generated by the coset of* $t \in G$.

(i) Suppose that χ *is an irreducible character of* G *whose restriction to* K *is reducible. Then* $\chi(t) = 0$.

(ii) Suppose that χ_1 *and* χ_2 *are irreducible characters of* G *whose restrictions to* K *are irreducible. Then*

$$\frac{1}{|K|}\sum_{g\in Kt} \chi_1(g)\overline{\chi_2(g)} = \begin{cases} 1 & \text{if } \chi_1|_K = \chi_2|_K; \\ 0 & \text{otherwise}. \end{cases}$$

Proof Let $(\chi_K)^G$ be the character obtained by first restricting χ to K, then inducing it back to G. Let $\eta : G \longrightarrow \mathbb{C}^\times$ be the linear character with kernel K such that $\eta(t) = \zeta_m = e^{2\pi i m}$, where $m = [G:K]$. We note that

$$(\chi_K)^G = \sum_{i=0}^{m-1} \chi \otimes \eta^i.$$

Indeed both sides have the value $\chi(g)$ if $g \in K$, and zero otherwise. By Frobenius reciprocity, for fixed j we have, noting that χ and $\chi\otimes\gamma^j$ have the same restriction to K,

$$\sum_{i=0}^{m-1} \left\langle \chi\otimes\gamma^j, \chi\otimes\gamma^i \right\rangle_G = \left\langle \chi\otimes\gamma^j, (\chi_K)^G \right\rangle_G = \langle \chi,\chi\rangle_K . \tag{8}$$

To prove (i) we assume that χ restricted to K is reducible. Then $\langle\chi,\chi\rangle > 1$. Each $\chi\otimes\gamma^i$ is irreducible as a representation of G, so the inner products on the left in (9) are each 0 or 1; thus at least one $\left\langle\chi\otimes\gamma^j,\chi\otimes\gamma^i\right\rangle_G = 1$ with $i\neq j$. Thus $\chi\otimes\gamma^i = \chi\otimes\gamma^j$. Evaluating at t gives $\chi(t)\zeta^i = \chi(t)\zeta^j$ so $\chi(t) = 0$.

To prove (ii), assume that χ restricted to K is irreducible. The terms on the left in (9) are all nonnegative integers and the right-hand side is 1. Thus this implies that exactly one term on the left is nonzero and we must have

$$\left\langle\chi\otimes\gamma^j,\chi\otimes\gamma^i\right\rangle_G = \begin{cases} 1 & \text{if } i=j; \\ 0 & \text{otherwise}. \end{cases}$$

Now χ times the characteristic function of the coset Kt is $\frac{1}{m}\sum_i \zeta^{-i}\chi\otimes\gamma^i$. Thus

$$\frac{1}{|K|}\sum_{g\in Kt}|\chi(g)|^2 = m\left\langle\frac{1}{m}\sum_i\zeta^{-i}\chi\otimes\gamma^i,\frac{1}{m}\sum_j\zeta^{-j}\chi\otimes\gamma^j\right\rangle.$$

Expanding, just the diagonal terms $i=j$ are nonzero and this is 1.

If χ_1 and χ_2 are characters with distinct restrictions to K, then

$$\sum_{i=0}^{m-1}\left\langle\chi_1\otimes\gamma^j,\chi_2\otimes\gamma^i\right\rangle_G = \left\langle(\chi_1\otimes\gamma^j)_K,(\chi_2|_K)^G\right\rangle_G = \langle\chi_1,\chi_2\rangle_K = 0. \tag{9}$$

This implies that the $\left\langle\chi_1\otimes\gamma^j,\chi_2\otimes\gamma^i\right\rangle = 0$ for all i,j, which implies that

$$\frac{1}{|K|}\sum_{g\in Kt}\chi_1(g)\overline{\chi_2(g)} = 0,$$

since χ_i restricted to Kt is a linear combination of the $\chi_i\otimes\gamma^j$. □

Theorem 4 (Shintani) *There exists a bijection between the irreducible representations of* $\mathrm{GL}(n,\mathbb{F}_q)$ *and the* $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$*-invariant irreducible representations of* $\mathrm{GL}(n,\mathbb{F}_{q^m})$*. In this bijection, let* χ *be the character of an irreducible representation of* $\mathrm{GL}(n,\mathbb{F}_q)$*, and let* (Π,V) *be the corresponding representation of* $\mathrm{GL}(n,\mathbb{F}_{q^m})$*. There exists an endomorphism* j *of order* m *of* V *such that*

$$j\circ\Pi(g) = \Pi({}^tg)\circ j.$$

There exists a constant $\varepsilon=\pm1$ *such that*

$$\chi(N(g)) = \varepsilon\ \mathrm{tr}(j\circ\Pi(g))$$

for $g\in\mathrm{GL}(n,\mathbb{F}_{q^m})$*, where* $\chi(N(g))$ *is interpreted as the value of* χ *on any element of* $\mathrm{GL}(n,\mathbb{F}_q)$ *conjugate to* $N(g)$*.*

Proof The first step is to show that for every generalized character χ of $\mathrm{GL}(n,\mathbb{F}_q)$ there exists a generalized character $\tilde{\chi}$ of $\widetilde{\mathrm{GL}}(n,\mathbb{F}_{q^m})$ such that

$$\chi(N(g)) = \tilde{\chi}(tg). \tag{10}$$

By Theorem 3 it is sufficient to check this separately for Green's Σ_k^θ and for characters obtained by parabolic induction. By its construction, Σ_k^θ can be extended to $\mathrm{GL}(n, \mathbb{F}_q)$. For such a character, the statement follows from Proposition 1. On the other hand for a character, parabolically induced from a character ξ of a Levi factor $M(\mathbb{F}_q)$ parabolic subgroup $P(\mathbb{F}_q) = M(\mathbb{F}_q)U(\mathbb{F}_q)$ with Levi factor $M(\mathbb{F}_q)$, by induction the existence of a generalized character $\tilde{\xi}$ of $\tilde{P}(\mathbb{F}_{q^m})$ may be assumed. Now by Proposition 2, inducing $\tilde{\xi}$ parabolically to $\widetilde{\mathrm{GL}}(n, \mathbb{F}_{q^m})$ produces a character $\tilde{\chi}$ satisfying (10).

Now let $\chi_1, \cdots, \chi_h$ be the distinct irreducible characters of $\mathrm{GL}(n, \mathbb{F}_q)$ and let $\tilde{\chi}_1, \cdots, \tilde{\chi}_h$ be generalized characters of $\widetilde{\mathrm{GL}}(n, \mathbb{F}_{q^m})$ corresponding by (10).

We have proved in *Twisted Conjugacy* that the number of Galois-invariant irreducible characters of $\mathrm{GL}(n, \mathbb{F}_{q^m})$ is equal to h. Let $\eta_1, \cdots, \eta_h$ be these characters. Each of these can be extended to $\widetilde{\mathrm{GL}}(n, \mathbb{F}_{q^m})$, and we fix such an extension, which we denote $\tilde{\eta}_i$. Let $\gamma : \widetilde{\mathrm{GL}}(n, \mathbb{F}_{q^m}) \longrightarrow \mathbb{C}^\times$ be the linear character of order m which is trivial on $\mathrm{GL}(n, \mathbb{F}_{q^m})$ and takes value $\zeta_m = e^{2\pi i/m}$ on t. Then η_i has m distinct extensions to $\widetilde{\mathrm{GL}}(n, \mathbb{F}_{q^m})$ and these are $\tilde{\eta}_i \otimes \gamma^j$, $0 \leqslant j < m$. Finally, let $\varphi_1, \cdots, \varphi_l$ be the irreducible characters of $\widetilde{\mathrm{GL}}(n, \mathbb{F}_{q^m})$ which are *not* Galois invariant.

We can write

$$\tilde{\chi}_i = \sum_{j=1}^{h} \sum_{k=0}^{m-1} c_{ijk} \eta_j \otimes \gamma^k + \sum_{k=0}^{l} d_{ik} \varphi_k, \qquad c_{ijk}, d_{ik} \in \mathbb{Z}.$$

We note that the φ_k are induced from subgroups of $\widetilde{\mathrm{GL}}(n, \mathbb{F}_{q^m})$ which contain $\mathrm{GL}(n, \mathbb{F}_{q^m})$ but do not contain t, so their value is zero on $t\,\mathrm{GL}(n, \mathbb{F}_{q^m})$. Thus writing $c_{ij} = \sum_k c_{ijk} \zeta_m^k$, we have

$$\chi_i(N(g)) = \tilde{\chi}(tg) = \sum c_{ij} \eta_j(tg), \qquad c_{ij} \in \mathbb{Z}[\zeta_m].$$

Now using Theorem 2, the functions $g \longmapsto \chi_i(g)$ are orthonormal on $\mathrm{GL}(n, \mathbb{F}_{q^m})$, and so are the functions $g \longmapsto \eta_j(tg)$. Thus

$$1 = \sum_j |c_{ij}|^2,$$

which implies that for fixed i the c_{ij} as j varies are all zero but one, which is a root of unity. After reordering the η_j we may thus denote $c_i = c_{ii}$ and write $\chi_i = c_i \eta_i$, where c_i is a root of unity.

We may replace η by $\eta \otimes \gamma^j$ for any j. Thus we may change c_i by ζ^j. Since the roots of unity in $\mathbb{Z}[\zeta]$ are $\pm\zeta^j$, we may arrange that $c_i = \pm 1$. Now taking Π to be the representation of $\mathrm{GL}(n, \mathbb{F}_{q^m})$ affording the character η_i, the result follows. $\square$