Schubert Eisenstein Series

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Abstract. We define Schubert Eisenstein series as sums like usual Eisenstein series but with the summation restricted to elements of a particular Schubert cell, indexed by an element of the Weyl group. They are generally not fully automorphic. We will develop some results and methods for GL_3 that may be suggestive about the general case. The six Schubert Eisenstein series are shown to have meromorphic continuation and some functional equations. The Schubert Eisenstein series $E_{s_1 s_2}$ and $E_{s_2 s_1}$ corresponding to the Weyl group elements of order three are particularly interesting: at the point where the full Eisenstein series is maximally polar, they unexpectedly become (with minor correction terms added) fully automorphic and related to each other.

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We define *Schubert Eisenstein series* as sums like usual Eisenstein series but with the summation restricted to elements coming from a particular Schubert cell. More precisely, let G be a split semisimple algebraic group over a global field F, and let B be a Borel subgroup. The usual Eisenstein series are sums over $B(F)\backslash G(F)$, that is, over the integer points in the flag variety $X = B\backslash G$. Given a Weyl group element w, one may alternatively consider the sum restricted to a single Schubert cell X_w. This is the closure of the image in X of the double coset BwB. If $w = w_0$, the long Weyl group element, then $X_w = X$ so this contains the usual Eisenstein series as a special case. The notion of Schubert Eisenstein series seems a natural one, but little studied. The purpose of this paper is to look closely at the special case where $G = \mathrm{GL}(3)$ that suggest general lines of research for the general case.

The Schubert Eisenstein series is not automorphic, so its place in the spectral theory is less obvious. An immediate question is whether the Schubert Eisenstein series, like the classical ones have analytic continuation. We will prove this when $G = \mathrm{GL}(3)$ and we hope that it is true in general. We

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will observe some other interesting phenomena on GL(3), to be described below.

We will begin by supplying some motivation for this investigation. Recently it has been observed that Fourier-Whittaker coefficients of some Eisenstein series, such as the Borel Eisenstein series on GL_{r+1}, are multiple Dirichlet series which may often be expressed as sums over Kashiwara crystals. See the survey article Bump [5] for discussion of this this phenomenon and its history. An analysis of the proof of one particular case, in Brubaker, Bump and Friedberg [3] shows the mechanism behind this phenomenon makes use of *Bott-Samelson varieties*. In this connection, we call attention to one particular point: that such a representation of the Whittaker coefficient of an Eisenstein series as a sum over a crystal requires a choice of a *reduced word*, by which we mean a decomposition of the long Weyl group element w_0 into a product of simple reflections of shortest possible length.

Bott-Samelson varieties have important applications to the study of Schubert varieties. First, they give a desingularization. Also, they are used in the analyzing the cohomology of the flag variety, and also the cohomology of line bundles on Schubert varieties, that is, the Demazure character formula. See Demazure [10] and Andersen [1].

To define the Bott-Samelson variety, one chooses reduced word $\mathfrak{w}$ for w, after which one may define $Z_{\mathfrak{w}}$, the so-called *Bott-Samelson variety*, together with a birational morphism to X_w. (The definition is given below.) The variety $Z_{\mathfrak{w}}$ is always nonsingular, and may be built up by successive fiberings by $\mathbb{P}^1$, which corresponds to the procedure in representation theory of reducing a computation on G to a series of SL_2 computations. And this is what was done (for the full Eisenstein series, that is, for the case where $w = w_0$) in Brubaker, Bump and Friedberg [3].

Once one accepts the idea of studying Eisenstein series by means of the Bott-Samelson variety for the full flag variety, one is led to consider Schubert Eisenstein series. Even if one only cares about the full Eisenstein series (which is the sum over the integer points in the full flag variety X_{w_0}) the Bott-Samelson varieties for other Schubert cells appear naturally. This is because Bott-Samelson varieties are built up from one another by successive fiberings. So a calculation that involves Bott-Samelson varieties will usually be an inductive one involving Bott-Samelson varieties for lower-dimensional Schubert cells.

We turn now to a more detailed discussion of what is in this paper.

Let G be a split reductive algebraic group over a global field F. Let $\hat{T}$

be the maximal torus of the group $\hat{G}$ with opposite root data, so that $\hat{G}(\mathbb{C})$ is the connected Langlands L-group. Let $\nu \in \hat{T}(\mathbb{C})$. Then ν parametrizes a character χ_ν of $T(\mathbb{A})/T(F)$, where $\mathbb{A}$ is the adele ring of F. Extending χ_ν to the Borel subgroup $B(\mathbb{A})$, let f_ν be an element of the corresponding induced representation, so that

$$f_\nu(bg) = (\delta^{1/2}\chi_\nu)(b)\, f(g), \qquad b \in B(\mathbb{A}). \tag{1}$$

Here δ is the modular quasicharacter of the Borel subgroup. The usual Eisenstein series is defined to be

$$E(g,\nu) = \sum_{\gamma \in B(F)\backslash G(F)} f_\nu(\gamma g) = \sum_{\gamma \in X(F)} f_\nu(\gamma g).$$

In the last expression, we are observing that the sum is actually over the integer points of $X = B\backslash G$, which is the flag variety.

The Bruhat decomposition of G gives the decomposition of the flag variety into Schubert cells

$$X = \bigcup_{w \in W} Y_w$$

where W is the Weyl group and Y_w is the image of BwB in $B\backslash G$. The closure of Y_w is the closed Schubert variety

$$X_w = \bigcup_{u \leqslant w} Y_u$$

where $\leqslant$ is the Bruhat order. It seems a natural question to consider the *Schubert Eisenstein series*

$$E_w(g,\nu) = \sum_{\gamma \in X_w(F)} f_\nu(\gamma g). \tag{2}$$

This is no longer an automorphic form, but we may ask whether it has analytic continuation and at least some functional equations.

In order to see how this could be useful, let us recall the very useful Bott-Samelson varieties and their relationship with Schubert varieties. (See Bott and Samelson [2] and Demazure [10].) We will denote by α_i and s_i the simple roots and corresponding simple reflections. Let $w \in W$ and let $\mathfrak{w} = (s_{i_1}, s_{i_2}, \cdots, s_{i_k})$ be a reduced decomposition of w into a product of simple reflections: $w = s_{i_1} \cdots s_{i_k}$. Let P_j be the minimal parabolic subgroup, which is rank one parabolic subgroup, generated by B and s_j. We define a left action of B^k on $P_{i_1} \times \cdots P_{i_k}$ by

$$(b_1, \cdots, b_k) \cdot (p_{i_1}, \cdots, p_{i_k}) = (b_1 p_{i_1} b_2^{-1}, b_2 p_{i_2} b_3^{-1}, \cdots, b_k p_{i_k}). \tag{3}$$

The quotient $B^k \backslash (P_{i_1} \times \cdots \times P_{i_k})$ is the *Bott-Samelson variety* $Z_{\mathfrak{w}}$. There is a morphism $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ induced by the multiplication map that sends

$$(p_{i_1}, \cdots, p_{i_k}) \longmapsto p_{i_1} \cdots p_{i_k}.$$

This map is a surjective birational morphism.

Unlike the Schubert varieties, Bott-Samelson varieties are always nonsingular, so this gives a resolution of the singularities of X_w. The map $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ may not be an isomorphism. In special cases where it is an isomorphism, every element of X_w has a unique representation as a product $i_{\alpha_1}(\gamma_1) \cdots \iota_{\alpha_k}(\gamma_k)$, where if α is a root (in this case a simple root) ι_α is the Chevalley embedding of $\mathrm{SL}(2)$ into G corresponding to α, so the image of ι_{α_i} lies in the Levi subgroup of P_i. Beyond these special cases where $\mathrm{BS}_{\mathfrak{w}}$ is an isomorphism, in every case each element of X_w has such a factorization, and if the element is in general position, it is unique, since $\mathrm{BS}_{\mathfrak{w}}$ is birational. Let us call this a *Bott-Samelson factorization*. (See Lemma 2 for a precise statement.) This means that we may write

$$E_{s_1 \cdots s_k}(g, \nu) = \sum_{\gamma_k \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} E_{s_1 \cdots s_{k-1}}(\iota_{\alpha_k}(\gamma_k) g, \nu), \tag{4}$$

building up the Schubert Eisenstein series by repeated SL_2 summations. If $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ is not an isomorphism, a modification of this method should be applicable. (Proposition 13.)

This method of representing the Eisenstein series $E(g, \nu) = E_{w_0}(g, \nu)$, with w_0 the long Weyl group element, is implicit in the method used by Brubaker, Bump and Friedberg [3] in order to prove that the Whittaker function of Eisenstein series on the metaplectic cover of $\mathrm{GL}_{r+1}(F)$ had a representation as a sum over a crystal basis of a representation of GL_{r+1}. The proof depends on a parametrization, described in Section 5 of the paper, of an element of $P \backslash G$, where P is a maximal parabolic subgroup, by choosing the representative factored over such a product of SL_2. Although P is a maximal parabolic subgroup, the process is an inductive one, and one could equally well avoid the induction and take the summation over $B \backslash G$. The mechanism underlying this proof therefore is the Bott-Samelson factorization.

This suggests looking more closely at the Schubert Eisenstein series E_w. Even though E_w is not automorphic, and not accessible by the usual methods of automorphic forms, one may hope that it has analytic continuation and functional equations by some subgroup. If w is the long element of the Weyl group of the Levi subgroup M of some parabolic subgroup, then this is true. The first cases where w is not the long element of a Levi subgroup are $w = s_1 s_2$ and $s_2 s_1$, in the case where $G = \mathrm{GL}_3$. Therefore we will look at these Schubert Eisenstein series in detail. As it turns out, these had occurred previously in Bump and Goldfeld [7] and in Vinogradov and Takhtajan [15], in disguised forms.

We will take a close look at $E_{s_1 s_2}$. We have described it here by means of the definition (2) and by the recursive formula (4), but we will also see that it emerges naturally when one works out the Piatetski-Shapiro [14] Fourier-Whittaker expansion of the Eisenstein series. For a cusp form ϕ on GL_n with Whittaker function W, this Fourier expansion appears as

$$\phi(g) = \sum_{\gamma \in U_{\mathrm{GL}_{n-1}}(F) \backslash \mathrm{GL}_{n-1}(F)} W\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right),$$

where $U_{\mathrm{GL}_{n-1}}$ is the unipotent radical of the standard Borel subgroup of GL_{n-1}. If ϕ is not cuspidal, then one must include other degenerate terms, and then the summation over γ may produce Schubert Eisenstein series. We will see this for GL_3.

An extremely interesting phenomenon occurs in this GL_3 case at the point where the Eisenstein series has its pole. We will choose coordinates ν_1, ν_2 for the Langlands parameters such that the poles of the Eisenstein series are on the six lines ν_1, ν_2 or $1 - \nu_1 - \nu_2$ equals 0 or $\frac{2}{3}$, and we will look at the pole at $\nu_1 = \nu_2 = 0$. In the Laurent expansion of the Eisenstein series $E(g; \nu_1, \nu_2)$ the coefficient of $\nu_1^{N_1} \nu_2^{N_2}$ is nonzero if $N_1, N_2 \geqslant -1$. If $N_1 = N_2 = -1$, the coefficent is constant. Following Bump and Goldfeld, the coefficient $\kappa(g)$ of ν_1^{-1} is then interesting.

Bump and Goldfeld [7] proved the following result. If $K/\mathbb{Q}$ is a cubic field, and $\mathfrak{a}$ is an ideal class of K one may associate with $\mathfrak{a}$ a compact torus of GL_3, and if $L_\mathfrak{a}$ is the period of $\kappa(g)$ over this torus, then the Taylor expansion of the L-function $L(s, \mathfrak{a})$ has the form $\rho s^{-1} + L_\mathfrak{a} + \cdots$. Therefore if θ is a nontrivial character of the ideal class group then $L(s, \theta) = \sum \theta(\mathfrak{a}) L_\mathfrak{a}$. The proof involves showing that the torus period of the Eisenstein series equals a Rankin-Selberg integral of a Hilbert modular Eisenstein series.

An analysis of this situation reveals that $\kappa(g)$ may be expressed in terms of the Schubert Eisenstein series. There are two ways to do this, giving expressions involving either $E_{s_1s_2}$ or $E_{s_2s_1}$ at a special value. Thus at the point where the residue is taken, the Schubert Eisenstein series (with some correction terms) is "promoted" to full GL_3 automorphicity! It is also surprising that $E_{s_1s_2}$ and $E_{s_2s_1}$, which are presumably unrelated in general, develop an unexpected relationship at $\nu_1 = \nu_2 = 0$.

Now let us indicate a few questions about Schubert Eisenstein series in general. As we will see, these questions have interesting affirmative answers in the case of GL_3.

- Does the Schubert Eisenstein series always have meromorphic continuation to all values of the parameters?

- Although they will not have the full group of functional equations that the complete Eisenstein series has, they should have some functional equations.

- In Theorems 4 and 5 we will give examples of linear combinations of Schubert Eisenstein series for GL_3 that are entire, that is, have no poles in the parameters. It would be desirable to have a general theory of such linear combinations.

- In Proposition 13 we give an example of how to represent a Schubert Eisenstein series recursively in a case where the Bott-Samelson map $\mathrm{BS}_{\mathfrak{w}}$ is not an isomorphism. It would be good to work this out for more complicated examples.

- We find that for GL_3 Schubert Eisenstein series occur naturally in the context of the Piatetski-Shapiro Fourier-Whittaker expansion when one takes degenerate terms into account. It would be good to see generalizations of this phenomenon.

- We may speculate that it is possible to associate a Whittaker function with E_w. This would be an Euler product whose p-part may be expressed in terms of Demazure characters. Such an expression follows from the Casselman-Shalika formula if w is the long element in a parabolic subgroup of the Weyl group, so the first test case of this hypothesis is when $w = s_1s_2$ (or s_2s_1) on $GL(3)$. In this case, we have checked that a suitably defined Whittaker function may indeed be expressed in terms of the Demazure character corresponding to $s_1 s_2$. For reasons of space, we are not including these computations. Brubaker, Bump and Licata [4] have local results relating Iwahori Whittaker functions to Demazure characters, but we do not know how to relate those formulas to Schubert Eisenstein series.

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1 Review of Eisenstein series

If G is an algebraic group defined over a field contained in a commutative ring R, we will use $G(R)$ or G_R interchangeably to denote the group of R-rational points of G.

Let F be a global field, and $\mathbb{A}$ its adele ring. Let G be a split semisimple algebraic group over F, with Borel subgroup $B = TU$, where T is its maximal split torus and U the unipotent radical. Let $W = N(T)/T$ be the Weyl group, where $N(T)$ is the normalizer of T. If v is a place of F, we will denote by $G_v = G(F_v)$, and similarly for algebraic subgroups of G. We will denote by Φ the root system of G, divided as usual into positive and negative roots Φ^+ and Φ^-. If α_i is a simple root, we will denote by s_i the corresponding simple reflection in W.

If v is a place of F, let K_v be a maximal compact subgroup of $G_v = G(F_v)$. We assume that $K_v = G(\mathfrak{o}_v)$ for all nonarchimedean places v. We assume that $G_v = B_v K_v$. Then $K = \prod_v K_v$ is a maximal compact subgroup of $G(\mathbb{A})$. If $w \in W$ we will choose a representative of W that is in K; by abuse of notation we will denote this representative by the same letter w.

We review the definition of the usual Eisenstein series. Let χ be a quasicharacter of $T(\mathbb{A})/T(F)$. We may extend χ_v to a quasicharacter of B_v by letting U_v be in the kernel.

Let $(\pi_v(\chi_v), V_v(\chi_v))$ be the corresponding principal series representation. Thus $V_v(\chi_v)$ is the space of functions $f_v : G_v \longrightarrow \mathbb{C}$ that satisfy

$$f_v(bg) = (\delta^{1/2}\chi_v)(b)\, f_v(g)$$

for $b \in B_v = B(F_v)$, and which are K_v-finite. Here δ is the modular quasicharacter. If v is nonarchimedean the group G_v acts by right-translation:

$$\pi_v(g_v)f_v(x) = f_v(xg_v).$$

If v is archimedean, this definition is wrong since $\pi_v(g_v)f_v$ may not be K_v-finite, but the K_v-finite vectors are invariant under the corresponding representation of the Lie algebra $\mathfrak{g}_v$ and so at an archimedean place v, $V_v(\chi_v)$ is a $(\mathfrak{g}_v, K_v)$-module.

For simplicity we assume that $\chi = \otimes_v \chi_v$ where χ_v is unramified at every nonarchimedean place. This means that the space of K_v-fixed vectors is nonzero. The vector space $V_v(\chi_v)$ has a K_v-fixed vector $f_v^\circ = f_{\chi_v}^\circ$ that is unique up to scalar multiple. We will normalize it so that $f_v^\circ(1) = 1$.

Let $V(\chi)$ be the space of finite linear combinations of functions of the form $\prod_v f_v(g_v)$ where $f_v \in V_v(\chi_v)$ and $f_v = f_v^\circ$ for all but finitely many v. If the function f is of this form (rather than a finite linear combination of such functions) then we will write $f = \otimes_v f_v$. The space $V(\chi)$ is thus the restricted tensor product of the local modules $V_v(\chi_v)$.

Then we may consider the Eisenstein series

$$E(g, f, \chi) = \sum_{\gamma \in B_F \backslash G_F} f(\gamma g), \qquad f \in V(\chi).$$

This will be convergent for particular χ. Indeed, for every simple positive root α there is a Chevalley embedding $\iota_\alpha : \mathrm{SL}_2 \longrightarrow G$ such that $\iota_\alpha(\mathrm{SL}_2(\mathfrak{o}_v)) \subset K_v$ for v nonarchimedean, where $\mathfrak{o}_v$ is the ring of integers of F_v. Then

$$\left| \chi\left(\iota_\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right) \right| = |t|^{\nu(\alpha)}, \tag{5}$$

for some $\nu(\alpha) \in \mathbb{C}$. Indeed, since χ is trivial on $T(F)$, the left-hand side of (5) is 1 when $t \in F^\times$; then if $\mathbb{A}_1^\times$ is the group of ideles of norm 1, the left-hand side of (5) defines a homomorphism of $\mathbb{A}_1^\times / F^\times$ into the multiplicative group of positive reals. But $\mathbb{A}_1^\times / F^\times$ is compact, so the left-hand side of (5) is trivial on $\mathbb{A}_1^\times$ and thus must be a power of $|t|$. The Eisenstein series will be absolutely convergent provided every $\mathrm{re}(\nu(\alpha)) > \frac{1}{2}$. For χ not satisfying this inequality, we may make sense of the Eisenstein series by meromorphic continuation, with the exception of χ corresponding to poles of the Eisenstein series.

In order to state the functional equations of the Eisenstein series, one considers the standard intertwining integrals. If $w \in W$, define a map

$$\mathcal{M}_v(w) : V_v(\chi_v) \longrightarrow V_v(\chi_v^w),$$

where W acts on the right on quasicharacters by

$$\chi_v^w(t) = \chi_v(wtw^{-1}).$$

If $\mathrm{re}(\nu(\alpha)) > 0$, then $\mathcal{M}_v(w)$ may be defined by the integral

$$\mathcal{M}_v(w) f_v(g) = \int_{(U_v \cap w^{-1} U_v w) \backslash U_v} f_v(wug)\, du = \int_{U_v \cap w^{-1} U_v^- w} f_v(wug)\, du,$$

where U_v^- is the unipotent radical of the opposite Borel subgroup of B. It may be checked that $\mathcal{M}_v(w) V_v(\chi_v) \subseteq V_v(\chi_v^w)$, and that $\mathcal{M}_v(w)$ is an intertwining operator. The map $\mathcal{M}_v(w)$ may then be extended by meromorphic continuation to other values of χ and ν.

The formula of Gindikin and Karpelevich computes $\mathcal{M}_v(w) f_v^\circ$. First assume that v is nonarchimedean. If α is a positive root, let us denote by a_α the element

$$\iota_\alpha \begin{pmatrix} \varpi_v & \\ & \varpi_v^{-1} \end{pmatrix},$$

where ϖ_v is a generator of the maximal ideal $\mathfrak{p}_v$ of $\mathfrak{o}_v$. Let $q_v = |\mathfrak{o}_v / \mathfrak{p}_v|$. We choose the volume element dx_v on F_v so that $\mathfrak{o}_v$ has volume 1.

Proposition 1 *If v is nonarchimedean then*

$$\mathcal{M}_v(w) f_{\chi_v}^\circ = \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}(\alpha) \in \Phi^-}} \frac{1 - q_v^{-1} \chi_v(a_\alpha)}{1 - \chi_v(a_\alpha)} f_{\chi_v^w}^\circ.$$

This is called the formula of Gindikin and Karpelevich, but in this nonarchimedean case, it is due to Langlands.

Proof See Casselman [8], Theorem 3.1. □

Next assume that v is archimedean. Let Γ be the usual gamma function and let

$$\Gamma_v(s) = \begin{cases} \pi^{-s/2} \Gamma(s/2) & \text{if } v \text{ is real,} \\ (2\pi)^{-s} \Gamma(s) & \text{if } v \text{ is complex.} \end{cases}$$

Since χ_v is unramified, χ_v is trivial on $T_v \cap K_v$, and it follows that

$$\chi\left(\iota_\alpha\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\right) = |t|^{\nu(\alpha)}.$$

Proposition 2 *If v is archimedean then*

$$\mathcal{M}_v(w) f^\circ_{\chi_v} = \prod_{\substack{\alpha\in\Phi^+\\ w^{-1}(\alpha)\in\Phi^-}} \frac{\Gamma_v(\nu(\alpha))}{\Gamma_v(\nu(\alpha)+1)} f^\circ_{\chi_v^w}. \tag{6}$$

Proof This is the original formula of Gindikin and Karpelevich [11]. We are choosing the volume element on F_v to be the one that makes this formula true. □

We have choosen dx_v for every v to be the volume element that makes the formula of Gindikin and Karpelevich true. On the adele group $\mathbb{A}$ there is a natural volume element dx, which is self-dual for the Fourier transform determined by an additive character ψ on $\mathbb{A}$ that is trivial on F. Equivalently, dx is the volume element that gives $\mathbb{A}/F$ volume 1. The local and global volumes are related by the formula

$$dx = |D_F|^{-1/2}\prod_v dx_v, \tag{7}$$

where D_F is the discriminant of F.

There is also a global intertwining integral $\mathcal{M}(w) : V(\chi) \longrightarrow V(\chi^w)$, defined by

$$\mathcal{M}(w)f(g) = \int_{(U_\mathbb{A}\cap w^{-1}U_\mathbb{A}w)\backslash U_\mathbb{A}} f(wug)\,du = \int_{U_\mathbb{A}\cap w^{-1}U_\mathbb{A}^- w} f(wug)\,du$$

We are normalizing the Haar measure so that the volume $U_\mathbb{A}/U_F$ is 1, and similarly for its unipotent algebraic subgroups such as $U_\mathbb{A}\cap w^{-1}U_\mathbb{A}w$ and $U_\mathbb{A}\cap w^{-1}U_\mathbb{A}^- w$.

If α is a positive root, let

$$\zeta_v(\chi_v,\alpha) = \begin{cases} (1-\chi_v(a_\alpha))^{-1} & \text{if } v \text{ is nonarchimedean}\\ \Gamma_v(\nu(\alpha)) & \text{if } v \text{ is archimedean.}\end{cases}$$

We will also denote

$$\zeta_v(|\cdot|\chi_v,\alpha) = \begin{cases} (1-q_v^{-1}\chi_v(a_\alpha))^{-1} & \text{if } v \text{ is nonarchimedean,}\\ \Gamma_v(\nu(\alpha)+1) & \text{if } v \text{ is archimedean.}\end{cases}$$

Then let

$$\zeta(\chi,\alpha)=\prod_v \zeta_v(\chi_v,\alpha), \qquad \zeta(|\cdot|\chi,\alpha)=\prod_v \zeta_v(|\cdot|\chi_v,\alpha).$$

Proposition 3 *Suppose that χ is unramified at every place, and define $f_\chi^\circ \in V(\chi)$ to be $\prod_v f_{\chi_v}^\circ(g_v)$. Then*

$$\mathcal{M}(w)f_\chi^\circ = |D_F|^{l(w)/2} \prod_{\substack{\alpha\in\Phi^+\\ w^{-1}(\alpha)\in\Phi^-}} \frac{\zeta(\chi,\alpha)}{\zeta(|\cdot|\chi,\alpha)} f_{\chi^w}^\circ,$$

where $l(w)$ is the length function on the Weyl group.

Proof Because the dimension of $U\cap w^{-1}Uw$ is $l(w)$, (7) implies that, when du and du_v are the Haar measures on $U_{\mathbb{A}}\cap w^{-1}U_{\mathbb{A}}^- w$ and $U_v\cap w^{-1}U_v^- w$ with our normalizations we have

$$du = |D_F|^{l(w)/2}\prod_v du_v.$$

The statement then follows on combining (1) and (6). □

2 Induction and restriction

Mackey's theorem for finite groups and their representations may be formulated in different ways, but one statement is as follows. Let H_1 and H_2 be subgroups of G and let π_1 be representations of H_1 and H_2. We want to determine the restriction of $\mathrm{Ind}_{H_1}^G(\pi_1)$ to H_2. To answer this question we consider the double cosets $H_2\backslash G/H_1$. If w is a double coset representative, let $H_w = H_1\cap w^{-1}H_2 w$. Then we may restrict π_1 to H_w, and conjugating by w we obtain a representation π_1^w of $wH_w w^{-1} = wH_1w^{-1}\cap H_2$. This is a subspace of H_2, and Mackey's theorem states that

$$\mathrm{Ind}_{H_1}^G(\pi_1)|_{H_2} = \bigoplus_{w\in H_2\backslash G/H_1} \mathrm{Ind}_{wH_ww^{-1}}^{H_2}(\pi_1^w).$$

There is an analogous property of Eisenstein series. The induction and restriction functors between finite groups and subgroups will be replaced by

Eisenstein series and constant term functors for Levi subgroups. Let P and Q be parabolic subgroups of G containing B. Let $P = M_P U_P$ and $Q = M_Q U_Q$ be the Levi decompositions, with unipotent radicals U_P and U_Q contained in U. Given an automorphic form on M_Q, one may consider the corresponding Eisenstein series on G and its constant term with respect to U_P, which is an automorphic form on M_P. The problem is to describe its spectral expansion.

Using the Bruhat decomposition $G = \bigcup BwB$, representatives of double cosets $P\backslash G/Q$ may be chosen in W, and thus $P\backslash G/Q$ is in bijection with $W_P\backslash W/W_Q$, where W_P and W_Q are the Weyl groups of the Levi subgroups of P and Q. If w is such a representative, $M_Q \cap w^{-1}M_P w$ is a Levi subgroup of M_Q, so we may take the constant term along the unipotent radical of the corresponding parabolic subgroup $Q \cap w^{-1}Pw$ and obtain an automorphic form for $M_Q \cap w^{-1}M_P w$. Then conjugate this to $wM_Q w^{-1} \cap M_P$ which is an Eisenstein series on M_P. Summing over w should give an identity with the automorphic form obtained previously.

Let us prove this in the special case where $Q = B$. In this case, $M_B = T$ is the maximal torus. We will denote $M = M_P$, and $B_M = B \cap M$. We will denote by $\Phi_M \subset \Phi$ the root system of M. We will also denote by W_M the Weyl group of M, which was previously denoted W_P.

Lemma 1 *Every coset in W/W_M has a representative w such that if $\alpha \in \Phi_M$ then $\alpha \in \Phi_M^+$ if and only if $w(\alpha) \in \Phi^+$. For this w, we have*

$$P \cap w^{-1}B\,w = U^w B_M, \qquad U^w = U_P \cap w^{-1}Bw.$$

Proof We leave this to the reader. □

Let Σ_M be the particular set of representatives for W/W_M given by Lemma 1. If $g \in M(\mathbb{A})$ we will denote

$$E_M(g, f, \chi) = \sum_{B_M(F)\backslash M(F)} f(\gamma g),$$

which is an Eisenstein series for the Levi subgroup M.

Theorem 1 *Let $g \in M(\mathbb{A})$.*

$$\int_{U_P(F)\backslash U_P(\mathbb{A})} E(ug, f, \chi)\, du = \sum_{w \in \Sigma_M} E_M(g, \mathcal{M}(w)f, \chi^w) \tag{8}$$

Proof We may enumerate coset representatives for $B_F\backslash G_F$ as follows. Let w run through a set of coset representatives for $B_F\backslash G_F/P_F$, and for each w let γ run through a set of coset representatives for $H^w_F\backslash P_F$, where $H^w = P\cap w^{-1}Bw$. Then $w\gamma$ runs through a complete set of coset representatives for $B_F\backslash G_F$.

Using the Bruhat decomposition, we know that we may choose the representatives for w from a set of coset representatives of W/W_M, and we choose these as in Lemma 1. Therefore $H^w = U^w B_M$ where $U^w = U_P \cap w^{-1}Bw$. Then we may further analyze $\gamma \in H^w_F\backslash P_F$ as $\gamma_U\gamma_1$ where $\gamma_1 \in B_M(F)\backslash M_F$ and $\gamma_U \in U^w_F\backslash U_F$.

We may write the left-hand side in (8) as

$$\sum_{w\in\Sigma_M}\int_{U_P(F)\backslash U_P(\mathbb{A})}\sum_{\gamma_1\in B_M(F)\backslash M(F)}\sum_{\gamma_U\in U^w_F\backslash U_F} f(w\gamma_U\gamma_1 ug)\,du.$$

Since M normalizes U_P, we may interchange u and γ_1 in this expression, then telescope the integration with the summation over γ_U. After this we will write γ instead of γ_1, and obtain

$$\sum_{w\in\Sigma_M}\int_{U^w(F)\backslash U_P(\mathbb{A})}\sum_{\gamma\in B_M(F)\backslash M(F)} f(wu\gamma g)\,du.$$

We may write the integral as

$$\sum_{w\in\Sigma_M}\int_{U^w(F)\backslash U^w(\mathbb{A})}\int_{U^w(\mathbb{A})\backslash U_P(\mathbb{A})}\sum_{\gamma\in B_M(F)\backslash M(F)} f(wu_1u\gamma g)\,du\,du_1,$$

but the integration over the compact quotient $\int_{U^w(F)\backslash U^w(\mathbb{A})}$ may be discarded since $f(wu_1g) = f(wg)$ independent of $u_1 \in U^w(\mathbb{A})$. Hence we obtain

$$\sum_{w\in\Sigma_M}\sum_{\gamma\in B_M(F)\backslash M(F)} (\mathcal{M}(w)f)(\gamma g)\,du\,,$$

and (8) is proved. □

3 Schubert Eisenstein series

The flag variety $X = B\backslash G$ is a projective variety. We recall its decomposition into Schubert cells. We have the Bruhat decomposition $G = \bigcup BwB$, a disjoint union over $w \in W$, and let Y_w be the image of BwB in X. The Schubert cell X_w is the Zariski closure of Y_w. It equals

$$\bigcup_{\substack{u \in W \\ u \leqslant w}} Y_u,$$

where $u \leqslant w$ is the Bruhat order. Let G_w be the subset of G that is the union of BuB for $u \leqslant w$. It is not a subgroup in general. Let $X_w(F)$ be the set of $\gamma \in B_F \backslash G_F$ belonging to X_w. Thus $X_w(F) = B_F \backslash G_w(F)$. We may now define the *Schubert Eisenstein series*

$$E_w(g, f, \chi) = \sum_{\gamma \in X_w(F)} f(\gamma g).$$

As we explained in the introduction, the Bott-Samelson map is a useful tool for studying Schubert Eisenstein series. We recall that we defined a smooth variety $Z_{\mathfrak{w}}$ for every reduced word $\mathfrak{w} = (s_{i_1}, \cdots, s_{i_k})$ representing the Weyl group element w, with a birational morphism $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$.

Lemma 2 *If* $\mathrm{BS}_{\mathfrak{w}}$ *is an isomorphism then we may enumerate* $X_w(F)$ *as follows. Let* γ_i *run through* $B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)$ *for* $i = 1, \cdots, k$*. Then*

$$\iota_{\alpha_{i_1}}(\gamma_1) \cdots \iota_{\alpha_{i_k}}(\gamma_k) \tag{9}$$

runs through $X_w(F)$ *(without repetition).*

If $\mathrm{BS}_{\mathfrak{w}}$ is not an isomorphism, then every element of $X_w(F)$ can still be written as in (9), but the representation will not necessarily be unique. (It will be unique if the element is in general position.) See Proposition 13.

Proof If $\mathrm{BS}_{\mathfrak{w}}$ is an isomorphism, then we may choose the representatives for $Z_{\mathfrak{w}}$ as follows. First choose $p_{i_k} \in B \backslash P_{i_k}$. We are allowed to choose this in the Levi subgroup $M_{i_k} \cong \mathrm{SL}_2$, and so we may choose this representative to be $\iota_{\alpha_{i_k}}(\gamma_k)$ with γ_k chosen from $B_{\mathrm{SL}_2} \backslash \mathrm{SL}_2$, where B_{SL_2} is the Borel subgroup of upper triangular matrices in SL_2. Then we may choose $p_{i_{k-1}}$ from $B \backslash P_{i_{k-1}}$, and again we may choose it from the Levi subgroup of $P_{i_{k-1}}$. Continuing this way, the statement is clear. $\square$

4 GL_3 Schubert Eisenstein series

Let

$$\zeta^*(s) = |D_F|^{\frac{s}{2}} \prod_v \zeta_v(s), \qquad \zeta_v(s) = \begin{cases} (1-q_v^{-s})^{-1} & \text{if } v \text{ is nonarchimedean,} \\ \Gamma_v(s) & \text{if } v \text{ is archimedean} \end{cases}$$

where we recall that D_F is the discriminant of F. With this normalization of the Dedekind zeta function the functional equation is

$$\zeta^*(s) = \zeta^*(1-s).$$

For simplicity we will assume that the character χ is unramified at every place. Find $\nu_1, \nu_2 \in \mathbb{C}$ such that

$$(\delta^{1/2}\chi)\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} = |y_1|^{2\nu_1+\nu_2}|y_2|^{\nu_2-\nu_1}|y_3|^{-\nu_1-2\nu_2}.$$

We will denote this character χ_{ν_1,ν_2}. Also, take $f = f^\circ$ where

$$f^\circ(g) = f^\circ_{\nu_1,\nu_2}(g) = \prod_v f^\circ_v(g_v).$$

Thus if $k \in K$

$$f^\circ_{\nu_1,\nu_2}\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k\right) = |y_1|^{2\nu_1+\nu_2}|y_2|^{\nu_2-\nu_1}|y_3|^{-\nu_1-2\nu_2}.$$

Then we will denote

$$E(g;\nu_1,\nu_2) = E(g, f^\circ; \chi_{\nu_1,\nu_2}).$$

Due to the fact that the K-finite vectors are not invariant under right translation, we will sometimes restrict ourselves to g in the GL_3 of the finite adeles.

Denoting by α_1 and α_2 the simple positive roots we have

$$\zeta_v(|\cdot|\chi, \alpha_1) = \zeta_v(3\nu_1), \qquad \zeta_v(|\cdot|\chi, \alpha_2) = \zeta_v(3\nu_2), \qquad \zeta_v(|\cdot|\chi, \alpha_1+\alpha_2) = \zeta_v(3\nu_1+3\nu_2-1).$$

The product of these three factors is the local normalizing factor for the Eisenstein series at the place v. However we wish to include a power of the discriminant in the global normalizing factor, so we use $\zeta^*(s)$ which includes gamma factors and a power of the discriminant, and define

$$E^*(g;\nu_1,\nu_2)=\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)E(g;\nu_1,\nu_2).$$

The normalized Eisenstein series E^* is analytic except at poles where ν_1,ν_2 or $1-\nu_1-\nu_2$ equals 0 or $\frac{2}{3}$. It satisfies the functional equations

$$E^*(g;\nu_1,\nu_2)=E^*(g;w(\nu_1,\nu_2))$$

Here the action of $w\in W$ on the parameters ν_1,ν_2 is as follows. The simple reflections s_1 and s_2 send (ν_1,ν_2) to $\left(\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right)$ and $\left(\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2\right)$ respectively. We will similarly normalize the Schubert Eisenstein series and denote

$$E_w^*(g;\nu_1,\nu_2)=\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)E_w(g;\nu_1,\nu_2).$$

If $w=1$, then

$$E_1^*(g;\nu_1,\nu_2)=\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)f^\circ_{\nu_1,\nu_2}(g). \tag{10}$$

For particular w, we will also define E_w^{**} with only some of the normalizing zeta functions. We will omit g from the notation.

$$E_{s_1}^{**}(\nu_1,\nu_2)=\zeta^*(3\nu_1)E_{s_1}(\nu_1,\nu_2),\qquad E_{s_2}^{**}(\nu_1,\nu_2)=\zeta^*(3\nu_2)E_{s_2}(\nu_1,\nu_2),$$

$$E_{s_1s_2}^{**}(\nu_1,\nu_2)=\zeta^*(3\nu_1)E_{s_1s_2}(\nu_1,\nu_2),\qquad E_{s_2s_1}^{**}(\nu_1,\nu_2)=\zeta^*(3\nu_2)E_{s_2s_1}(\nu_1,\nu_2).$$

We will also consider some linear combinations denoted $\hat{E}_w^*$ or $\hat{E}_w^{**}$ that have better decay properties. These are

$$\hat{E}_{s_1}^*(\nu_1,\nu_2)=E_{s_1}^*(\nu_1,\nu_2)-E_1^*(\nu_1,\nu_2)-E_1^*\left(\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right),$$

$$\hat{E}_{s_1}^{**}(\nu_1,\nu_2)=E_{s_1}^{**}(\nu_1,\nu_2)-\zeta^*(3\nu_1)f^\circ_{\nu_1,\nu_2}(g)-\zeta^*(3\nu_1-1)f^\circ_{\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}}(g),$$

$$\hat{E}_{s_2}^*(\nu_1,\nu_2)=E_{s_2}^*(\nu_1,\nu_2)-E_2^*(\nu_1,\nu_2)-E_2^*\left(\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2\right),$$

$$\hat{E}_{s_2}^{**}(\nu_1,\nu_2)=E_{s_2}^{**}(\nu_1,\nu_2)-\zeta^*(3\nu_2)f^\circ_{\nu_1,\nu_2}(g)-\zeta^*(3\nu_2-1)f^\circ_{\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2}(g),$$

$$\hat{E}_{s_1s_2}^*(\nu_1,\nu_2)=E_{s_1s_2}^*(\nu_1,\nu_2)-E_{s_2}^*(\nu_1,\nu_2)-E_{s_2}^*\left(\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right),$$

$$\hat{E}^*_{s_2s_1}(\nu_1,\nu_2) = E^*_{s_2s_1}(\nu_1,\nu_2) - E^*_{s_1}(\nu_1,\nu_2) - E^*_{s_1}\left(\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2\right),$$

$$\hat{E}^{**}_{s_1s_2}(\nu_1,\nu_2) =$$
$$E^{**}_{s_1s_2}(\nu_1,\nu_2) - \zeta^*(3\nu_1)E_{s_2}(\nu_1,\nu_2) - \zeta^*(3\nu_1-1)E_{s_2}\left(\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right),$$

$$\hat{E}^{**}_{s_2s_1}(\nu_1,\nu_2) =$$
$$E^{**}_{s_2s_1}(\nu_1,\nu_2) - \zeta^*(3\nu_2)E_{s_1}(\nu_1,\nu_2) - \zeta^*(3\nu_2-1)E_{s_1}\left(\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2\right).$$

Proposition 4 *We have*

$$\int_{U_F\backslash U_{\mathbb{A}}} E(ug;\nu_1,\nu_2)\,du = \sum_{w\in W}\mathcal{M}(w)f^\circ_{\nu_1,\nu_2}(g).$$

Moreover

$$\int_{U_F\backslash U_{\mathbb{A}}} E^*(ug;\nu_1,\nu_2)\,du = \sum_{w\in W}E^*_1(g;w(\nu_1,\nu_2)). \tag{11}$$

Here E_1 is the Schubert Eisenstein series corresponding to the identity $1\in W$. Thus $E_1=f^\circ$ and $E^*_1=\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)f^\circ$.

Proof The first formula the special case of Theorem 1 where $P=B$. For the second we need to know that

$$\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)\,\mathcal{M}(w)f^\circ_{\nu_1,\nu_2}(g) = E^*_1(g;w(\nu_1,\nu_2)). \tag{12}$$

Using the fact that $\mathcal{M}(ww')=\mathcal{M}(w)\circ\mathcal{M}(w')$ when the length $l(ww')=l(w)+l(w')$, we are reduced to the case where w is a simple reflection. For example, if $w=s_1$, Proposition 3 implies that

$$\mathcal{M}(w)f^\circ_{\nu_1,\nu_2}(g) = \frac{\zeta^*(3\nu_1-1)}{\zeta^*(3\nu_1)}f^\circ_{\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}}(g).$$

Now using the functional equation $\zeta^*(3\nu_1-1)=\zeta^*(2-3\nu_1)$, the left-hand side of (12) equals

$$\zeta^*(2-3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)f^\circ_{\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}}(g),$$

as required. $\square$

First we study E_{s_1}. This is essentially a GL_2 Eisenstein series. To see this, let $P = P_1$ be the parabolic with Levi factor $M_1 = \iota_{\alpha_1}(\mathrm{SL}_2)T$. Then provided $g \in M_1(\mathbb{A})$ we have

$$E_{s_1}(g;\nu_1,\nu_2) = \sum_{\gamma\in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} f^{\circ}_{\nu_1,\nu_2}(\iota_{\alpha_1}(\gamma)g) = E_{M_1}(g;\nu_1,\nu_2). \tag{13}$$

Proposition 5 *The normalized Schubert Eisenstein series $E^*_{s_1}$ has meromorphic continuation to all ν_1,ν_2, and satisfies*

$$E^*_{s_1}(g;\nu_1,\nu_2) = E^*_{s_1}\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right). \tag{14}$$

Furthermore

$$E^{**}_{s_1}(g;\nu_1,\nu_2) = E^{**}_{s_1}\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right). \tag{15}$$

We have

$$\int_{\mathbb{A}/F} E^*_{s_1}\left(\begin{pmatrix}1 & x & \\ & 1 & \\ & & 1\end{pmatrix} g;\nu_1,\nu_2\right)dx \;=$$
$$E^*_1(g;\nu_1,\nu_2) + E^*_1\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right). \tag{16}$$

Proof For $h \in \mathrm{GL}_2(\mathbb{A})$,

$$h \mapsto E_{M_1}\left(\begin{pmatrix}h & \\ & 1\end{pmatrix} g;\nu_1,\nu_2\right)$$

is a GL_2 Eisenstein series, and $\zeta^*(3\nu_1)$ is its normalizing factor. The analytic continuation and functional equation (15) follows from the well-known GL_2 theory. The two factors $\zeta^*(3\nu_2)$ and $\zeta^*(3\nu_1+3\nu_2-1)$ are interchanged by the transformation $(\nu_1,\nu_2) \longmapsto \left(\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right)$. Therefore the functional equation (14) follows. The GL_2 constant term is

$$\int_{\mathbb{A}/F} E^{**}_{s_1}\left(\begin{pmatrix}1 & x & \\ & 1 & \\ & & 1\end{pmatrix} g;\nu_1,\nu_2\right)dx \;=$$
$$\zeta^*(3\nu_1)E_1(g;\nu_1,\nu_2) + \zeta^*(3\nu_1-1)E_1\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right),$$

which is equivalent to (16). □

Proposition 6 *The truncated Eisenstein series* $\hat{E}^{**}_{s_1}(g;\nu_1,\nu_2)$ *is entire and of rapid decay in the the* α_1 *direction.*

By this we mean that

$$\hat{E}^{**}_{s_1}\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} g;\nu_1,\nu_2\right)$$

is analytic for all ν_1 and ν_2, and is of faster than polynomial decay as $|y_1/y_2| \longrightarrow \infty$, uniformly if g is in a compact set.

Proof This again follows from the theory of GL_2 Eisenstein series. We have the Fourier expansion

$$E^{**}_{s_1}(g) = \sum_{\alpha\in F}\int_{\mathbb{A}/F} E^{**}_{s_1}\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)\psi(\alpha x)\,dx,$$

where ψ is an additive character of $\mathbb{A}/F$. Using (16) the pieces that are subtracted to give $\hat{E}^{**}_{s_1}$ are the contribution of $\alpha = 0$. On the other hand if $\alpha \neq 0$

$$\int_{\mathbb{A}/F} E^{**}_{s_1}\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)\psi(\alpha x)\,dx = W\left(\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)$$

where

$$W(g) = \int_{\mathbb{A}/F} E^{**}_{s_1}\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)\psi(x)\,dx$$

is essentially a GL_2 Whittaker function. The analytic continuation of W to all ν_1, ν_2 is Théorème 1.9 of Jacquet [12], and its decay properties guarantee that

$$\hat{E}^{**}_{s_1}(g) = \sum_{\alpha\in F^\times} W\left(\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)$$

is entire and of rapid decay in the α_1 direction. □

Similarly

Proposition 7 *The normalized Schubert Eisenstein series $E^*_{s_2}$ has meromorphic continuation to all ν_1, ν_2, and satisfies*

$$E^*_{s_2}(g;\nu_1,\nu_2) = E^*_{s_2}\left(g;\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2\right). \tag{17}$$

Moreover $\hat{E}^{**}_{s_2}(g;\nu_1,\nu_2)$ is entire and is of rapid decay in the α_2 direction.

We turn now to the Schubert Eisenstein series $E_{s_1 s_2}$ and $E_{s_2 s_1}$. These are important examples since $s_1 s_2$ and $s_2 s_1$ are not long elements in Levi subgroups of the Weyl group, so their analytic properties do not follow from the usual theory of Eisenstein series.

Using (17) we have

$$\hat{E}^*_{s_1 s_2}(\nu_1,\nu_2) = E^*_{s_1 s_2}(\nu_1,\nu_2) - E^*_{s_2}(\nu_1,\nu_2) - E^*_{s_2}\left(\nu_2, 1-\nu_1+\nu_2\right). \tag{18}$$

Similarly

$$\hat{E}^*_{s_2 s_1}(\nu_1,\nu_2) = E^*_{s_2 s_1}(\nu_1,\nu_2) - E^*_{s_1}(\nu_1,\nu_2) - E^*_{s_1}\left(1-\nu_1+\nu_2, \nu_1\right). \tag{19}$$

Lemma 3 *Let $g \in G$. Let $f = f^\circ_{\nu_1,\nu_2}$. Then there exists a constant C depending only on g such that*

$$|f(hg)| < C|f(h)|.$$

Proof We write $h = bk$ where $b \in B(F)$ and $k \in K$. Then since $f = f^\circ$

$$|f(hg)| = |(\delta^{1/2}\chi)(b)||f(kg)| = |f(h)|\,|f(kg)|.$$

Since K is compact, $C = \max_K |f(kg)| < \infty$. □

Proposition 8 *The function*

$$\sum_{\gamma \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} \hat{E}^{**}_{s_1}(\iota_{\alpha_2}(\gamma)g;\nu_1,\nu_2) \tag{20}$$

is entire in ν_1 and ν_2.

Proof We know that $\hat{E}^{**}_{s_1}$ is entire but we need to show that the sum over γ is convergent for all ν_1 and ν_2. If $\gamma \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)$ consider

$$\begin{pmatrix} 1 & \\ & \gamma \end{pmatrix} g = \begin{pmatrix} y_1(\gamma) & * & * \\ & y_2(\gamma) & * \\ & & y_3(\gamma) \end{pmatrix} k, \qquad k \in K.$$

We will show that if $\sigma > 1$ then

$$\sum_\gamma \left| \frac{y_1(\gamma)}{y_2(\gamma)} \right|^{-2\sigma} < \infty. \tag{21}$$

Applying the Lemma to the function

$$f\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k\right) = \left|\frac{y_1}{y_2}\right|^{-2\sigma},$$

we may assume $g = 1$ in order to prove (21). Then we note that since $\gamma \in \mathrm{SL}_2$, we have $y_1(\gamma) = 1$ and $y_2(\gamma)y_3(\gamma) = 1$. Thus $y_1(\gamma)/y_2(\gamma) = \sqrt{y_3(\gamma)/y_2(\gamma)}$, and so we must show

$$\sum_\gamma \left| \frac{y_2(\gamma)}{y_3(\gamma)} \right|^{\sigma} < \infty.$$

This however is a GL_2 Eisenstein series and converges if $\sigma > 1$. Now due to the rapid decay of $\hat{E}^{**}_{s_1}$ in the α_1 direction, we have

$$\hat{E}^{**}_{s_1}\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} \ll \left|\frac{y_1}{y_2}\right|^{-2\sigma}$$

as $|y_1/y_2| \longrightarrow \infty$ for any σ. Thus the estimate (21) implies the convergence of (20). □

For $w = s_1s_2$, the Schubert variety $X_{s_1s_2}$ coincides with the Bott-Samelson variety $Z_{(s_1,s_2)}$, since the rational map $Z_{(s_1,s_2)} \longrightarrow X_{s_1s_2}$ is an isomorphism.

Theorem 2 *$E^*_{s_1s_2}(g;\nu_1,\nu_2)$ has meromorphic continuation to all ν_1,ν_2. It has a functional equation*

$$E^*_{s_1s_2}(g;\nu_1,\nu_2) = E^*_{s_1s_2}\left(g; \frac{2}{3} - \nu_1, \nu_1 + \nu_2 - \frac{1}{3}\right).$$

Moreover $\hat{E}^{**}_{s_1s_2}(g;\nu_1,\nu_2)$ is an entire function.

Proof When $w = s_1 s_2$ and $\mathfrak{w} = (s_1, s_2)$ the Bott-Samelson homomorphism $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ is an isomorphism and so by Lemma 2 we may write

$$E^*_{s_1 s_2}(g; \nu_1, \nu_2) = \sum_{\gamma \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} E^*_{s_1}(\iota_{\alpha_2}(\gamma) g; \nu_1, \nu_2). \tag{22}$$

Write this

$$\begin{aligned} \zeta^*(3\nu_2)\zeta^*(3\nu_1 + 3\nu_2 - 1) \sum_{\gamma \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} \hat{E}^{**}_{s_1}(\iota_{\alpha_2}(\gamma) g; \nu_1, \nu_2) \\ + \sum_{\gamma \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} E^*_1(\iota_{\alpha_2}(\gamma) g; \nu_1, \nu_2) \\ + \sum_{\gamma \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} E^*_1\left(\iota_{\alpha_2}(\gamma) g; \frac{2}{3} - \nu_1, \nu_1 + \nu_2 - \frac{1}{3}\right). \end{aligned}$$

The meromorphic continuation of each term is known; for the first term this is by Proposition 8. Moreover, dividing by $\zeta^*(3\nu_2)\zeta^*(3\nu_1 + 3\nu_2 - 1)$ and rearranging gives

$$\hat{E}^{**}_{s_1 s_2}(g; \nu_1, \nu_2) = \sum_{\gamma \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} \hat{E}^{**}_{s_1}(\iota_{\alpha_2}(\gamma) g; \nu_1, \nu_2),$$

so it follows from Proposition 8 that $\hat{E}^{**}_{s_1 s_2}(g; \nu_1, \nu_2)$ is entire. □

5 Fourier-Whittaker expansion

The Fourier-Whittaker expansion of a GL_n cusp form was described by Piatetski-Shapiro [14] and is standard. For forms which are not cuspidal, the Fourier expansion is slightly more complicated, and we recall it here. Before specializing to the Eisenstein series, let $E(g)$ denote an arbitrary automorphic form on GL_3. If $c, d \in F$, let

$$E^c_d(g) = \int_{(\mathbb{A}/F)^2} E\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right) \psi(c x_3 + d x_2)\, dx_2\, dx_3$$

and

$$E_{c,d}(g) = \int_{(\mathbb{A}/F)^3} E\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right) \psi(c x_1 + d x_2)\, dx_1\, dx_2\, dx_3.$$

We recall that ψ is a nontrivial additive character on $\mathbb{A}/F$.

Theorem 3 *We have*

$$E(g) = E_0^0(g) + \sum_{\gamma\in U_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} E_{0,1}(\iota_{\alpha_1}(\gamma)g) + \sum_{\gamma\in U_{\mathrm{GL}_2}(F)\backslash \mathrm{GL}_2(F)} W\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \tag{23}$$

Here $U_{\mathrm{GL}_2} = U_{\mathrm{SL}_2}$ is the one parameter subgroup $\iota_{\alpha_1}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$.

Proof The proof is in Chapter IV of Bump [6]. We leave it to the reader to translate it to the adelic setting. □

Now let us consider the case where $E(g) = E^*(g;\nu_1,\nu_2)$.

Proposition 9 *We have*

$$\int_{(\mathbb{A}/F)^2} E^*\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g;\nu_1,\nu_2\right) dx_2\, dx_3 =$$
$$E_{s_1}^*(g;\nu_1,\nu_2) + E_{s_1}^*(g;1-\nu_1-\nu_2,\nu_1) + E_{s_1}^*(g;\nu_2,1-\nu_1-\nu_2).$$

This is $E_0^0(g)$ when $E(g) = E^*(g;\nu_1,\nu_2)$.

Proof This is a special case of Theorem 1. The three double coset representatives in Σ_M are

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}.$$

Using (13) the corresponding GL_2 Eisenstein series may be written as

$$E_{s_1}^*(g;\nu_1,\nu_2), \qquad E_{s_1}^*\left(g;\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2\right), \qquad E_{s_1}^*\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right),$$

and using the functional equations these are the three terms in the statement. □

Proposition 10 *We have*

$$\int_{(\mathbb{A}/F)^3} E^*\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & \\ & & 1 \end{pmatrix} g; \nu_1, \nu_2\right) dx_1\, dx_3 \quad = $$
$$E^*_{s_2}(g;\nu_1,\nu_2) + E^*_{s_2}(g;1-\nu_1-\nu_2,\nu_1) + E^*_{s_2}(g;\nu_2,1-\nu_1-\nu_2).$$

Proof This is similar to Proposition 9 except that we use the other maximal parabolic subgroup. □

Proposition 11 *If* $E(g) = E^*(g;\nu_1,\nu_2)$ *then*

$$\sum_{\gamma \in U_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} E_{0,1}(\iota_{\alpha_1}(\gamma)g) \quad = $$
$$E^*_{s_2 s_1}(g;\nu_1,\nu_2) + E^*_{s_2 s_1}(g;1-\nu_1-\nu_2,\nu_1) + E^*_{s_2 s_1}(g;\nu_2,1-\nu_1-\nu_2)$$
$$-2(E^*_{s_1}(g;\nu_1,\nu_2) + E^*_{s_1}(g;1-\nu_1-\nu_2,\nu_1) + E^*_{s_1}(g;\nu_2,1-\nu_1-\nu_2))$$

Proof We may write the left-hand side as

$$\sum_{\gamma \in B_{\mathrm{SL}_2(F)}\backslash \mathrm{SL}_2(F)} \sum_{n \in F^*} E_{0,1}\left(\begin{pmatrix} n^{-1} & & \\ & n & \\ & & 1 \end{pmatrix} \iota_{\alpha_1}(\gamma)g\right).$$

A simple change of variables shows that

$$E_{0,1}\left(\begin{pmatrix} n^{-1} & & \\ & n & \\ & & 1 \end{pmatrix} g\right) = E_{0,n}(g)$$

so the left-hand side equals

$$\sum_{\gamma \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} \sum_{n \in F^\times} E_{0,n}(\iota_{\alpha_1}(\gamma)g).$$

We will show that

$$\sum_{\gamma \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} \sum_{n \in F} E_{0,n}(\iota_{\alpha_1}(\gamma)g) \quad = $$
$$E^*_{s_2 s_1}(g;\nu_1,\nu_2) + E^*_{s_2 s_1}(g;1-\nu_1-\nu_2,\nu_1) + E^*_{s_2 s_1}(g;\nu_2,1-\nu_1-\nu_2) \qquad (24)$$

and that

$$\sum_{\gamma\in B_{\mathrm{SL}_2}(F)\backslash\,\mathrm{SL}_2(F)} E_{0,0}(\iota_{\alpha_1}(\gamma)g) = \sum_{w\in W} E^*_{s_1}(g; w(\nu_1,\nu_2)). \tag{25}$$

Combining these two identities gives the statement. Observe that

$$\sum_{n\in F} E_{0,n}(g) = \sum_{n\in F}\int_{(\mathbb{A}/F)^3} E\left(\begin{pmatrix} 1 & x_1 & x_3\\ & 1 & x_2\\ & & 1\end{pmatrix} g\right)\psi(nx_2)\,dx_1\,dx_2\,dx_3 \;=\; \int_{(\mathbb{A}/F)^3} E\left(\begin{pmatrix} 1 & x_1 & x_3\\ & 1 & \\ & & 1\end{pmatrix} g\right)dx_1\,dx_3,$$

which is evaluated in Proposition 10. Thus (24) is the sum of three terms, a typical one being

$$\sum_{\gamma\in B_{\mathrm{SL}_2}(F)\backslash\,\mathrm{SL}_2(F)} E^*_{s_2}(\iota_{\alpha_1}(\gamma)g;\nu_1,\nu_2).$$

This is $E^*_{s_2s_1}(g;\nu_1,\nu_2)$, similarly to (22), whence (24). Also note that $E_{0,0}(g)$ is evaluated above in (11), and summing over $\iota_{\alpha_1}(\gamma)$ gives

$$\sum_{w\in W} E^*_{s_1}(g; w(\nu_1,\nu_2)).$$

We note that this may be written as

$$2(E^*_{s_1}(g;\nu_1,\nu_2) + E^*_{s_1}(g;1-\nu_1-\nu_2,\nu_1) + E^*_{s_1}(g;\nu_2,1-\nu_1-\nu_2))$$

because of the functional equation (14). □

Let

$$H(g;\nu_1,\nu_2) = \sum_{\gamma\in U_{\mathrm{GL}_2}(F)\backslash\,\mathrm{GL}_2(F)} W\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right), \tag{26}$$

where

$$W(g) = \int_{(\mathbb{A}/F)^3} E^*\left(\begin{pmatrix} 1 & x_1 & x_3\\ & 1 & x_2\\ & & 1\end{pmatrix} g;\nu_1,\nu_2\right)\psi(x_1+x_2)\,dx_1\,dx_2\,dx_3.$$

Theorem 4 *The function $H(g;\nu_1,\nu_2)$ is entire as a function of ν_1 and ν_2. We have*

$$\begin{aligned} E^*(g;\nu_1,\nu_2) &= \\ H(g;\nu_1\nu_2)+ & \\ E^*_{s_2s_1}(g;\nu_1,\nu_2)+E^*_{s_2s_1}(g;1-\nu_1-\nu_2,\nu_1)+E^*_{s_2s_1}(g;\nu_2,1-\nu_1-\nu_2) & \\ -E^*_{s_1}(g;\nu_1,\nu_2)-E^*_{s_1}(g;1-\nu_1-\nu_2,\nu_1)-E^*_{s_1}(g;\nu_2,1-\nu_1-\nu_2) &= \\ \hat{E}^*_{s_2s_1}(g;\nu_1,\nu_2)+\hat{E}^*_{s_2s_1}(g;1-\nu_1-\nu_2,\nu_1)+\hat{E}^*_{s_2s_1}(g;\nu_2,1-\nu_1-\nu_2) & \\ +E^*_{s_1}(g;\nu_1,\nu_2)+E^*_{s_1}(g;1-\nu_1-\nu_2,\nu_1)+E^*_{s_1}(g;\nu_2,1-\nu_1-\nu_2) & \end{aligned}$$

Proof We have

$$W(g)=\prod_v W_v(g_v)$$

where the Jacquet-Whittaker function W_v has analytic continuation for every place v by Jacquet [12], Corollaire 3.5, and the convergence of the sum in (26) follows from the decay properties of the Whittaker function (Proposition 2.2 in Jacquet, Piatetski-Shapiro and Shalika [13]. Therefore H is entire.

We note that $H(g;\nu_1,\nu_2)$ is one of the three terms in (23). The remaining terms are evaluated in Proposition 9 and Proposition 11. Combining these gives first expression. The second expression follows by using the definition of $\hat{E}^*_{s_2s_1}$. □

Similarly, one may prove that if

$$H'(g;\nu_1,\nu_2)=\sum_{\gamma\in U_{\mathrm{GL}_2}(F)\backslash\,\mathrm{GL}_2(F)} W\left(\begin{pmatrix}1 & \\ & \gamma\end{pmatrix}g\right)$$

then the following is true.

Theorem 5 *The function $H'(g;\nu_1,\nu_2)$ is entire as a function of ν_1 and ν_2. We have*

$$\begin{aligned} E^*(g;\nu_1,\nu_2) &= \\ H'(g;\nu_1\nu_2)+ & \\ E^*_{s_1s_2}(g;\nu_1,\nu_2)+E^*_{s_1s_2}(g;1-\nu_1-\nu_2,\nu_1)+E^*_{s_1s_2}(g;\nu_2,1-\nu_1-\nu_2) & \\ -E^*_{s_2}(g;\nu_1,\nu_2)-E^*_{s_2}(g;1-\nu_1-\nu_2,\nu_1)-E^*_{s_2}(g;\nu_2,1-\nu_1-\nu_2) &= \\ \hat{E}^*_{s_1s_2}(g;\nu_1,\nu_2)+\hat{E}^*_{s_1s_2}(g;1-\nu_1-\nu_2,\nu_1)+\hat{E}^*_{s_1s_2}(g;\nu_2,1-\nu_1-\nu_2) & \\ +E^*_{s_2}(g;\nu_1,\nu_2)+E^*_{s_2}(g;1-\nu_1-\nu_2,\nu_1)+E^*_{s_2}(g;\nu_2,1-\nu_1-\nu_2) & \end{aligned}$$

6 Kronecker Limit Formula

The poles of the Eisenstein series are on the six lines where ν_1, ν_2 or $1-\nu_1-\nu_2$ equals 0 or $\frac{2}{3}$. We will consider the Taylor expansions of E_w for various w at $\nu_1 = \nu_2 = 0$. In particular, the coefficient of ν_1^{-1} is interesting. If ϕ is a function of g and ν_1, ν_2, let $\mathfrak{R}\phi$ be the coefficient of ν_1^{-1} in the Taylor expansion of ϕ at $\nu_1 = \nu_2 = 0$. Let

$$\kappa(g) = \mathfrak{R}E(g; \nu_1, \nu_2).$$

Bump and Goldfeld [7] proved the following result. If $K/\mathbb{Q}$ is a cubic field, and $\mathfrak{a}$ is an ideal class of K one may associate with $\mathfrak{a}$ a compact torus of GL_3, and if $L_\mathfrak{a}$ is the period of $\kappa(g)$ over this torus, then the Taylor expansion of the L-function $L(s, \mathfrak{a})$ has the form $\rho s^{-1} + L_\mathfrak{a} + \cdots$. Therefore if θ is a character of the ideal class group then $L(s, \theta) = \sum \theta(\mathfrak{a}) L_\mathfrak{a}$. The proof involves showing that the torus period of the Eisenstein series equals a Rankin-Selberg integral of a Hilbert modular Eisenstein series.

An analysis of this situation reveals that $\kappa(g)$ may be expressed in terms of the Schubert Eisenstein series. There are two ways to do this, giving expressions involving either $E_{s_1 s_2}$ or $E_{s_2 s_1}$ at a special value. Thus at the point where the residue is taken, the Schubert Eisenstein series (with some correction terms) is "promoted" to full GL_3 automorphicity!

Let us write

$$\zeta^*(s) = \frac{\rho}{s} + \delta + O(s).$$

Then

$$E_{s_1}^{**}(g; \nu_1, \nu_2) = \frac{\rho}{3\nu_1} + \phi_{s_1}(g; \nu_2) + O(\nu_1)$$

where ϕ_{s_1} satisfies

$$\phi_{s_1}\left(i_{\alpha_1}(\gamma) g; \nu_2\right) = \phi_{s_1}(g; \nu_2),$$

since E_{s_1} has the same automorphicity. Similarly

$$E_{s_2}^{**}(g; \nu_1, \nu_2) = \frac{\rho}{3\nu_2} + \phi_{s_2}(g; \nu_1) + O(\nu_2).$$

We will write

$$\phi_{s_1}(g) = \phi_{s_1}(g; 0), \qquad \phi_{s_2}(g) = \phi_{s_2}(g; 0).$$

The automorphic forms ϕ_{s_1} and ϕ_{s_2} are essentially GL_2 automorphic forms, similar to the function $\log|\eta(z)|$ that appears in the classical Kronecker Limit Formula.

Let

$$c_0 = \frac{\rho}{3}\left[\delta\zeta^*(-1) + \rho(\zeta^*)'(-1)\right], \qquad c_0' = \frac{\rho}{3}\left[\zeta^*(3)\zeta^*(-1) + \rho\frac{d}{ds}(\zeta^*)'(-1)\right].$$

These are absolute constants depending only on the field.

Theorem 6 *We have*

$$\kappa(g) = \frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{s_2 s_1}(g;0,0) + E^{**}_{s_1}(g;1,0)\right] + c_0.$$

Furthermore

$$\kappa(g) = \frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{s_1 s_2}(g;1,0) + \phi_{s_2}(g)\right] + c_0'.$$

Proof The points $(\nu_1,\nu_2) = (0,0)$ and $(1,0)$ are related by a functional equation of the total Eisenstein series $E(g;\nu_1,\nu_2)$, but not of the Schubert Eisenstein series. We could alternatively take the Taylor coefficient of ν_2^{-1} and obtain a similar pair of identities.

By Theorem 4 we have

$$\kappa(g) = \sum_{i=1}^{6} \Re X_i$$

where X_i runs through the following six terms.

X_i	long form	$\Re X_i$
$\hat{E}^*_{s_2 s_1}(g;\nu_1,\nu_2)$	$\zeta^*(3\nu_1)\zeta^*(3\nu_1+3\nu_2-1)$ $\hat{E}^{**}_{s_2 s_1}(g;\nu_1,\nu_2)$	$\frac{\rho}{3}\zeta^*(-1)\hat{E}^{**}_{s_2 s_1}(g;0,0)$
$\hat{E}^*_{s_2 s_1}(g;1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3-3\nu_1-3\nu_2)\zeta^*(2-3\nu_2)$ $\hat{E}^{**}_{s_2 s_1}(g;1-\nu_1-\nu_2,\nu_1)$	0
$\hat{E}^*_{s_2 s_1}(g;\nu_2,1-\nu_1-\nu_2)$	$\zeta^*(3\nu_2)\zeta^*(2-3\nu_1)$ $\hat{E}^{**}_{s_2 s_1}(g;\nu_2,1-\nu_1-\nu_2)$	0
$E^*_{s_1}(g;\nu_1,\nu_2)$	$\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)$ $E^{**}_{s_1}(g;\nu_1,\nu_2)$	c_0
$E^*_{s_1}(g;1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3\nu_1)\zeta^*(2-3\nu_2)$ $E^{**}_{s_1}(g;1-\nu_1-\nu_2,\nu_1)$	$\frac{\rho}{3}\zeta^*(-1)E^{**}_{s_1}(g;1,0)$.
$E^*_{s_1}(g;\nu_2,1-\nu_1-\nu_2)$	$\zeta^*(3-3\nu_1-3\nu_2)\zeta^*(2-3\nu_1)$ $E^{**}_{s_1}(g;\nu_2,1-\nu_1-\nu_2)$	0

Alternatively, by Theorem 5 we may use the following six terms:

X_i	long form	$\Re X_i$
$\hat{E}^*_{s_1s_2}(\nu_1,\nu_2)$	$\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)$ $\hat{E}^{**}_{s_1s_2}(g;\nu_1,\nu_2)$	0
$\hat{E}^*_{s_1s_2}(1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3\nu_1)\zeta^*(2-3\nu_2)$ $\hat{E}^{**}_{s_1s_2}(g;1-\nu_1-\nu_2,\nu_1)$	$\frac{\rho}{3}\zeta^*(-1)\hat{E}^{**}_{s_1s_2}(1,0)$
$\hat{E}^*_{s_1s_2}(g;\nu_2,1-\nu_1-\nu_2)$	$\zeta^*(3-3\nu_1-3\nu_2)\zeta^*(2-3\nu_1)$ $\hat{E}^{**}_{s_1s_2}(g;\nu_2,1-\nu_1-\nu_2)$	0
$E^*_{s_2}(g;\nu_1,\nu_2)$	$\zeta^*(3\nu_1)\zeta^*(3\nu_1+3\nu_2-1)$ $E^{**}_{s_2}(g;\nu_1,\nu_2)$	$\frac{\rho}{3}\zeta^*(-1)\phi_{s_2}(g)$ $+\frac{\rho^2}{3}(\zeta^*)'(-1)$
$E^*_{s_2}(g;1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3-3\nu_1-3\nu_2)\zeta^*(2-3\nu_2)$ $E^{**}_{s_2}(g;1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3)\zeta^*(-1)\frac{\rho}{3}$.
$E^*_{s_2}(g;\nu_2,1-\nu_1-\nu_2)$	$\zeta(3\nu_2)\zeta^*(2-3\nu_1)$ $E^{**}_{s_2}(g;\nu_2,1-\nu_1-\nu_2)$	0

□

7 When $\mathrm{BS}_{\mathfrak{w}}$ is not an isomorphism

Let w_0 be the long Weyl group element. The Schubert Eisenstein series E_{w_0} is then just the full Eisenstein series, which is well understood. Nevertheless, we may try to understand it as a Schubert Eisenstein series.

For GL_3, there are two reduced words $\mathfrak{w} = (s_1, s_2, s_1)$ or (s_2, s_1, s_2) representing w_0. If $\mathfrak{w}$ is either of these, the Bott-Samelson homomorphism $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_{w_0} = X$ is not an isomorphism. However, since it is birational, it is a local isomorphism on the complement of a closed subvariety, which may be described as follows. The space X may be identified with the space of full flags in a 3-dimensional vector subspace V. Let $V_0 \subset V_1 \subset V_2 \subset V_3$ be the standard flag, where V_i is the span of $e_1, \cdots, e_i$, in terms of the standard basis vectors e_i of V.

Proposition 12 *With $\mathfrak{w} = (s_1, s_2, s_1)$, $Z_{\mathfrak{w}}$ may be identified with the space of flags $V_0 \subset U_1 \subset U_2 \subset V_3$ with an auxiliary piece of data, namely a one-dimensional vector space W_1 such that $W_1 \subset V_2 \cap U_2$.*

Proof To see this, consider the sequence of flags:

$$\begin{array}{ccccccc} V_3 & & V_3 & & V_3 & & V_3 \\ | & & | & & | & & | \\ V_2 & & V_2 & & U_2 & & U_2 \\ | & \xleftarrow{\theta_1} & | & \xleftarrow{\theta_2} & | & \xleftarrow{\theta_3} & | \\ V_1 & & W_1 & & W_1 & & U_1 \\ | & & | & & | & & | \\ V_0 & & V_0 & & V_0 & & V_0 \end{array} \tag{27}$$

We select elements θ_1, θ_2 and θ_3 of GL_3 such that θ_1 takes the second flag to the first, θ_2 takes the third to the second, and θ_3 takes the last to the third. Then θ_1 is in the parabolic subgroup P_1 that fixes the partial flag $V_0 \subset V_2 \subset V_3$, θ_2 stabilizes the partial flag $V_0 \subset W_1 \subset V_3$ and θ_3 fixes the partial flag $V_0 \subset U_2 \subset V_3$. This means that $\theta_1\theta_2^{-1}\theta_1^{-1}$ is in the parabolic subgroup P_2 that fixes the partial flag $V_0 \subset V_1 \subset V_3$ and similarly $\theta_1\theta_2\theta_3^{-1}\theta_2^{-1}\theta_1^{-1}$ is in P_1. Let us consider $(p_1, p_2, p_3) = (\theta_1^{-1}, \theta_1\theta_2^{-1}\theta_1^{-1}, \theta_1\theta_2\theta_3^{-1}\theta_2^{-1}\theta_1^{-1}) \in P_1 \times P_2 \times P_1$. It is easy to see that (p_1, p_2, p_3) is determined modulo the left action of $B \times B \times B$ on (p_1, p_2, p_3) defined in (3). The the coset of (p_1, p_2, p_3) is determined by the data in (27). In addition to the standard flag $V_0 \subset V_1 \subset V_2 \subset V_3$ (which is fixed throughout the discussion) this data consists of the flag $V_0 \subset U_1 \subset U_2 \subset V_3$ together with W_1, which can be any one-dimensional vector space contained in $V_2 \cap U_2$. □

Regarding X_{w_0} as the parameter space for the flag $V_0 \subset U_1 \subset U_2 \subset V_2$, the Bott-Samelson map $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_{w_0}$ consists of discarding the auxiliary piece of data W_1. We may now compute the *exceptional* subvariety of X_{w_0} where $\mathrm{BS}_{\mathfrak{w}}$ has a fiber that consists of more than one point. Clearly given the flag $V_0 \subset U_1 \subset U_2 \subset V_2$, the vector space W_1 satisfying $W_1 \subset V_2 \cap U_2$ will be determined except for the case where $U_2 = V_2$.

Because $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_{w_0}$ is not an isomorphism, Lemma 2 fails, but since we understand the exceptional set, we may understand how to remedy it and to express E_{w_0} in terms of $E_{s_1 s_2}$.

Proposition 13 *We have*

$$E_{w_0}(g; \nu_1, \nu_2) = E_{s_1}(g; \nu_1, \nu_2) + \sum_{\gamma_3 \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} (E_{s_1 s_2} - E_{s_1})(\iota_{\alpha_1}(\gamma_3) g; \nu_1, \nu_2).$$

Proof The element $\gamma = \theta_1\theta_2\theta_3$ has a unique factorization

$$\iota_{\alpha_1}(\gamma_1)\iota_{\alpha_2}(\gamma_2)\iota_{\alpha_1}(\gamma_3)$$

as in Lemma 2 with $\gamma_i \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)$ except when γ lies in the exceptional subvariety. This means that $\gamma(U_2) = V_2$, that is, when $\gamma \in G_{s_1} = B \cup Bs_1B$. These correspond to the terms where $\gamma_2 \in B_{\mathrm{SL}_2}$.

These exceptional terms contribute exactly E_{s_1}. For the remaining terms, we note that

$$\sum_{\substack{\gamma_1 \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F) \\ \gamma_2 \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F) \\ \gamma_2 \notin B_{\mathrm{SL}_2}}} f(\iota_{\alpha_1}(\gamma_1)\iota_{\alpha_2}(\gamma_2)g) = E_{s_1s_2} - E_{s_1},$$

and these terms therefore contribute the second term. □

This type of analysis would in principle allow one to represent more complicated Schubert Eisenstein series by an analog of the procedure we used for $E_{s_1s_2}$.

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