The Rankin-Selberg Method: An Introduction and Survey

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It is a great pleasure to dedicate this paper to Steve Rallis, from whom we have all learned so very much.

This paper has three aims: to explain some ideas, to survey some of the work of the last 15 years in the Rankin-Selberg method, and particularly to indicate the great contributions of Rallis in this field. The scope of this paper is limited to constructions of L-functions, omitting noneulerian integrals, special value results, p-adic theory, and applications to number theory. A previous survey of this topic was given in [5]. Due to the existence of this older report, we will not be systematic about describing older work.

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1 Langlands L-functions

We assume the reader to be at least somewhat familiar with Langlands L-functions and the functoriality conjecture. See Borel [4] for precise statements. Fairly recent expository discussions of the Langlands conjectures may be found in Bump [6], Cogdell [23] and Knapp [59], as well as other papers in the same two volumes as the last two mentioned. We summarize Langlands L-functions only to fix notations.

If G is an algebraic group defined over a field F we will denote by G_F or $G(F)$ the group of F-rational points of G. If F is a global field and v is a place of F we will denote $G(F_v)$ as G_v.

Let G be a quasisplit reductive affine algebraic group over a nonarchimedean local field F with residue cardinality q. There is associated with G a complex analytic group ${}^LG$. It is often taken to be the product of a complex analytic group with a Galois group, but we will mostly ignore the Galois group part.

If π is a spherical representation of G_F there is associated with π a conjugacy class $A_\pi \in {}^LG$. This follows from Langlands' interpretation of the Satake isomorphism, identifying the spherical Hecke algebra on G_F with respect to a special maximal compact subgroup with a ring of class functions on ${}^LG$. If $\rho : {}^LG \longrightarrow \mathrm{GL}(d, \mathbb{C})$ is an analytic representation, then

$$L(s, \pi, \rho) = \det \left(I - \rho(A_\pi) q^{-s}\right)^{-1} \tag{1}$$

is the local Langlands L-function.

If G is a reductive affine algebraic group over a global field F with center Z, and π is an automorphic cuspidal representation with unitary central character $\omega : Z_F \backslash Z_\mathbb{A} \longrightarrow \mathbb{C}$, then π acts by right translation on an irreducible invariant subspace of the space $L_0^2(Z_\mathbb{A} G_F \backslash G_\mathbb{A}, \omega)$ of square integrable cusp forms with central character ω. Replacing F with the completion F_v of a global field F at a nonarchimedean place v and π by the constituent π_v of a we will denote this local L-function (1) as $L_v(s, \pi_v, \rho)$. There exists a finite set of places S of F such that if v is a place of F not in S then v is nonarchimedean and the localization π_v is spherical. Let $L^S(s, \pi, \rho) = \prod_{v \notin S} L_v(s, \pi_v, \rho)$. Langlands conjectured that $L^S(s, \pi, \rho)$ has analytic continuation and a functional equation under $s \longrightarrow 1 - s$.

2 The $\mathrm{GL}(n) \times \mathrm{GL}(m)$ convolution

If π_1 and π_2 are automorphic representations of $\mathrm{GL}(n)$ and $\mathrm{GL}(m)$, respectively, the L-function $L(s, \pi_1 \times \pi_2)$ is well understood. If $n = m = 2$, this example was the original discovery of Rankin and Selberg. It is holomorphic, except that when $n = m$ and π_1 and π_2 are contragredient representations, it has a simple pole at $s = 1$. This important fact has many applications.

Cogdell [22] is an excellent reference for the $\mathrm{GL}(n) \times \mathrm{GL}(m)$ convolution. It was discussed in Bump [6] but in classical language which has not experienced any great revival. We will use this convolution to illustrate some points later so we now review it, referring to the Cogdell article for the reader who wants more detail, and references to the original papers, particularly those of Jacquet, Piatetski-Shapiro and Shalika.

There is a noted difference between the cases $n = m$ and $n \neq m$. If $n = m$, the construction involves an Eisenstein series. Let F be a global field, $\mathbb{A}$ its adele ring. Let P be the standard maximal parabolic subgroup of $\mathrm{GL}(n)$ with Levi factor $\mathrm{GL}(n-1) \times \mathrm{GL}(1)$, and let $\delta : P_{\mathbb{A}} \longrightarrow \mathbb{C}^{\times}$ be the modular quasicharacter:

$$\delta \begin{pmatrix} h & * \\ & a \end{pmatrix} = |\det(h)|a^{-(n-1)}, \qquad h \in \mathrm{GL}(n-1, \mathbb{A}), a \in \mathbb{A}^{\times}.$$

If $s \in \mathbb{C}$ let $f_s \in \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}(n,\mathbb{A})}(\delta_P^s)$. This means that f_s is a smooth function and $f_s(pg) = \delta_P(p)^s f_s(g)$. We assume that the restriction of f_s to a standard maximal compact subgroup of $\mathrm{GL}(n, \mathbb{A})$ is independent of s. There is no real loss of generality in assuming that f_s is a "pure tensor" in $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}(n,\mathbb{A})}(\delta_P^s) = \otimes_v \mathrm{Ind}_{P_v}^{\mathrm{GL}(n,F_v)}(\delta_P^s)$. This means that $f_s(g) = \prod_v f_{s,v}(g_v)$ where $f_v \in \mathrm{Ind}_{P_v}^{\mathrm{GL}(n,F_v)}(\delta_P^s)$. We have an Eisenstein series

$$E(g, s) = \zeta(ns) \sum_{P_F \backslash \mathrm{GL}(n,F)} \delta_P(\gamma g)^s. \tag{2}$$

This series is convergent if $\mathrm{re}(s)$ is sufficiently large, and has meromorphic continuation to all s.

Now if ϕ_1 and ϕ_2 are $\mathrm{GL}(n)$ cusp forms in automorphic representations π_1, π_2 we consider

$$\int_{\mathrm{GL}(n,F) Z_{\mathbb{A}} \backslash \mathrm{GL}(n,\mathbb{A})} \phi_1(g)\, \phi_2(g)\, E(g, s)\, dg, \tag{3}$$

where Z is the center of $\mathrm{GL}(n)$. (We will assume for simplicity that the product of the central characters of π_1 and π_2 is trivial.) After "unfolding" this equals

$$\zeta(ns) \int_{N_{\mathbb{A}} Z_{\mathbb{A}} \backslash \mathrm{GL}(n,\mathbb{A})} W_1(g)\, W_2(g)\, f_s(g)\, dg, \tag{4}$$

where W_1 and W_2 are "Whittaker functions" to be described as follows. Let $\psi : \mathbb{A}/F \longrightarrow \mathbb{C}$ be a nontrivial additive character (a notation we will use without comment throughout this paper). Let N be the algebraic subgroup of upper triangular unipotent matrices in $\mathrm{GL}(n)$, and define a character $\psi_N : N_{\mathbb{A}} \longrightarrow \mathbb{C}$ by

$$\psi_N(n) = \psi\left(\sum_{i=1}^{n-1} n_{i,i+1}\right).$$

Then

$$W_1(g) = \int_{N_F\backslash N_{\mathbb{A}}} \phi_1(ng)\,\psi(n)\,dn, \qquad W_2(g) = \int_{N_F\backslash N_{\mathbb{A}}} \phi_2(ng)\,\psi(n)^{-1}\,dn.$$

Note that we use inverse characters ψ and ψ^{-1} in the two Whittaker functions. Due to the well-known uniqueness of Whittaker models, these functions are Euler products. That is (assuming without much loss of generality that $\phi_i = \otimes\phi_{i,v}$ is a "pure tensor" in $\pi_i = \otimes_v \pi_{i,v}$) we have

$$W_i(g) = \prod_v W_{i,v}(g_v), \tag{5}$$

where $W_{i,v}$ is a function on $\mathrm{GL}(n, F_v)$ about which we will have more to say later. Immediately however (5) and (4) show that the integral is an Euler product; in fact (4) equals

$$\prod_v \zeta_v(ns) \int_{N_v Z_v\backslash \mathrm{GL}(n,F_v)} W_{1,v}(g_v)\,W_{2,v}(g_v)\,f_{s,v}(g_v)\,dg_v, \tag{6}$$

where ζ_v is the local Dedekind zeta function. Thus if v is a nonarchimedean place with residue cardinality q_v we have $\zeta_v(ns) = (1 - q_v^{-ns})^{-1}$.

If $m = n - 1$, then the global integral is of *Hecke type*. This means that it does not involve an Eisenstein series. Instead, it looks like:

$$\int_{\mathrm{GL}(n-1,F)\backslash \mathrm{GL}(n-1,\mathbb{A})} \phi_1\begin{pmatrix} g & \\ & 1 \end{pmatrix} \phi_2(g)\,|\det(g)|^{s-1/2}\,dg.$$

Rankin-Selberg integrals of Hecke type are fairly rare: most Rankin-Selberg integrals which have been discovered involve Eisenstein series. (Many authors do not use the term "Rankin-Selberg" unless there is an Eisenstein series, but this seems to me a questionable distinction since the the Hecke integrals which are employed to represent L-functions are more like Rankin-Selberg integrals than they are unlike them.)

If $m < n-1$, then it is necessary to "pad" the gap between the two GL's with a unipotent integration. For example if $m = n-2$ the integration looks like this:

$$\int_{\mathrm{GL}(n-2,F)\backslash\,\mathrm{GL}(n-2,\mathbb{A})} \int_{(\mathbb{A}/F)^{n-1}} \int_{\mathbb{A}/F} \phi_1 \begin{pmatrix} \boxed{g} & & \xi \\ & 1 & x \\ & & 1 \end{pmatrix} \phi_2(g)$$
$$|\det(g)|^{s-1}\,\psi(x)\,dx\,d\xi\,dg. \tag{7}$$

3 The Casselman-Shalika Formula

As we mentioned before, the fact that the integral (5) is an Euler product follows from the well-known uniqueness of Whittaker functions. If (π, V) is an irreducible representation of $\mathrm{GL}(n, F)$, with F local, there can be at most one (up to scalar multiple) linear functional $\Lambda : V \longrightarrow \mathbb{C}$ such that

$$\Lambda\big(\pi(n)\xi\big) = \psi_N(n)\Lambda(\xi), \qquad n \in N(F), \xi \in V. \tag{8}$$

If F is archimedean we require that Λ extends to a functional on the space of smooth vectors. The space of functions $W_\xi(g) = \Lambda\big(\pi(g)\xi\big)$ with $\xi \in V$ is called the *Whittaker model*. Uniqueness of Whittaker models was proved by Gelfand and Graev first over finite fields, later by Gelfand and Kazhdan, Piatetski-Shapiro, and Shalika over local fields.

The notion of a Whittaker model can be extended to arbitrary split reductive groups. The notion also extends naturally to representations of adele groups, particularly automorphic representations. Not all automorphic representations have Whittaker models. An automorphic representation with a Whittaker model is called *generic*. If $G = \mathrm{GL}(n)$, however, every automorphic cuspidal representation does have a Whittaker model.

Returning to the local situation, assume that π is unramified. This means that it has a *spherical vector* ξ° invariant under the maximal compact subgroup $K = \mathrm{GL}(n, \mathfrak{o})$, where $\mathfrak{o}$ is the maximal order in F. The vector ξ, like the Whittaker functional, is unique up to scalar multiple, and we choose Λ so that $W^\circ = W_{\xi^\circ}$ takes value 1 at the identity. Then W° is the *spherical Whittaker function* for which a remarkable formula exists. This formula was conjectured by Langlands, proved by Shintani (for $\mathrm{GL}(n)$) and Kato (for more general reductive groups). A particularly beautiful proof was found by Casselman and Shalika [21], and due to the importance of this proof it is habitually referred to as the Casselman-Shalika formula, though they did not discover or first prove it.

We begin by stating the formula precisely for $\mathrm{GL}(n)$, then describing it more impressionistically for an arbitrary split reductive group. Since W is left quasi-invariant under N_F and right invariant under K, its values are determined by $W(t)$ as t runs through a set of representatives of the double cosets $N_F\backslash\,\mathrm{GL}(n,F)/K$. We may take these to be the matrices

$$t_\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix},$$

where ϖ is a generator of the maximal ideal $\mathfrak{p}$ of $\mathfrak{o}$, and $\lambda = (\lambda_1, \cdots, \lambda_n)$ are a set of integers. $W(t_\lambda)$ vanishes unless λ is *dominant*, that is

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n. \tag{9}$$

Let $\alpha_1, \cdots, \alpha_n$ be the eigenvalues of the semisimple conjugacy class A_π parametrizing π, as in Section 1. The Casselman-Shalika formula asserts that

$$W^\circ(t_\lambda) = \delta_B(t_\lambda)^{1/2} s_\lambda(\alpha_1, \cdots, \alpha_n),$$

where $\delta_B(t_\lambda) = \prod_i |\varpi|^{\lambda_i(n+1-2i)}$ is the modular quasicharacter of the standard Borel subgroup B_F of $\mathrm{GL}(n,F)$, consisting of upper triangular matrices, and the *Schur polynomial*

$$s_\lambda(x_1, \cdots, x_n) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \cdots & x_n^{\lambda_2+n-2} \\ \vdots & & & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix}} \tag{10}$$

The irreducible analytic representations of $\mathrm{GL}(n,\mathbb{C})$ are parametrized by integer sequences λ satisfying (9). In fact, the Weyl character formula shows that the value of $s_\lambda(\alpha_1, \cdots, \alpha_n)$ is precisely equal to $\chi_\lambda(A_\pi)$, where χ_λ is the character of the irreducible representation π_λ of $\mathrm{GL}(n, \mathbb{C})$ with highest weight vector λ, where A_π is the semisimple conjugacy class in the L-group parametrizing π, as explained in Section 1.

Thus we may rewrite the Casselman-Shalika formula

$$W^\circ(t_\lambda) = \begin{cases} \delta_B(t_\lambda)^{1/2}\chi_\lambda(A_\pi) & \text{if } \lambda \text{ is dominant;} \\ 0 & \text{otherwise}. \end{cases} \tag{11}$$

Now suppose that G is an arbitrary split reductive group over a nonarchimedean local field. Let T be a split maximal torus of G. As in the case just considered, there exists a parametrization of the irreducible representations of ${}^LG$ by the dominant elements of $T_F/T_\mathfrak{o}$, and if $t \in T_F$ corresponds in this parametrization to the representation whose character χ_λ has highest weight vector λ determined by t, then (11) is true. This follows from combining the Casselman-Shalika formula (for G) with the Weyl character formula (for ${}^LG$).

This seems like magic, since the initial problem had nothing to do with the representation theory of ${}^LG$. Moreover it is highly applicable magic, as we will see in the next section. In an extraordinary way, the Casselman-Shalika is a window by which the representation theory of the L-group enters into the Rankin-Selberg method.

We can generalize the Casselman-Shalika formula as follows. If H is a subgroup of G, and if ψ is a character of H such that the induced representation $\mathrm{Ind}_H^G(\psi)$ is multiplicity free, and if π is an irreducible representation of G which occurs in $\mathrm{Ind}_H^G(\psi)$, then π can be realized in a unique space $\mathcal{W}_\pi$ of functions on G, invariant under right translation, satisfying $f(hg) = \psi(h)f(g)$ for $h \in H$. Now if K is another subgroup such that $\mathrm{Ind}_K^G(1)$ is multiplicity free, then $\mathcal{W}_\pi$ can contain at most one linearly independent K-fixed element W. In the examples we have in mind, G and H will be p-adic groups and K will be a maximal compact subgroup. The problem is then to parametrize the double cosets in $H\backslash G/K$ and to compute the values of W on a set of representatives. Any such formula potentially has applications to the Rankin-Selberg method. We will not attempt to catalog these, but we will encounter a number of examples of non-Whittaker models in our survey of the known constructions.

4 Symmetric Algebra Decompositions

Suppose that a Rankin-Selberg integral on a group G unfolds to an integral of the Whittaker function on G. Using the Casselman-Shalika formula, each unramified integral can be expressed in terms of the character values $\chi_\lambda(A_{\pi,v})$.

On the other hand, we are hoping that this is equal to a local Langlands L-function $L_v(s,\pi_v,\rho)$, where $\rho : {}^LG \longrightarrow \mathrm{GL}(d,\mathbb{C})$ is an analytic representation. Let us therefore reconsider (1).

If $\mu_1,\cdots,\mu_d$ are the eigenvalues of $\rho(A_\pi)$ the coefficient of q^{-ks} in (1) is the k-th elementary symmetric polynomial $h_k(\mu_1,\cdots,\mu_d)$, that is, the sum of all monomials of degree k in the μ_i. This is the trace of $\rho(A_\pi)$ on the k-th symmetric power $\vee^k\rho$ of ρ. Thus

$$L_v(s,\pi_v,\rho) = \sum_{k=0}^{\infty} \mathrm{tr}(A_\pi|\vee^k\rho)\, q^{-ks}.$$

Verifying that the local Whittaker integral equals $L_v(s,\pi_v,\rho)$ amounts to computing the decomposition of the symmetric algebra $\bigvee\rho = \bigoplus_k \vee^k\rho$ into irreducible representations of ${}^LG$.

Let us illustrate this with the $\mathrm{GL}(n)\times\mathrm{GL}(n)$ convolution. The integrand is K_v invariant, where $K_v = \mathrm{GL}(n,\mathfrak{o}_v)$, and T_v contains a complete set of coset representatives of $N_v\backslash\mathrm{GL}(n,F_v)/K$. The integral (6) therefore reduces to an integral over the torus:

$$(1-q^{-ns})^{-1}\int_{Z_v\backslash T_v} W_1(t)\,W_2(t)\,f_s(t)\,\delta_B(t)^{-1}\,dt.$$

Applying the Casselman-Shalika formula this equals

$$(1-q^{-ns})^{-1}\sum_{\lambda_1\geqslant\cdots\geqslant\lambda_n=0}\chi_\lambda(A_{\pi_1})\chi_\lambda(A_{\pi_2})q^{-|\lambda|s}. \tag{12}$$

where $|\lambda|$ denotes $\sum_i\lambda_i$. Since we are dividing by Z_v we may integrate over the subtorus of diagonal matrices whose last diagonal entry is trivial; thus $\lambda = (\lambda_1,\cdots,\lambda_{n-1},0)$. Let $\alpha_1,\cdots,\alpha_n$ and $\beta_1,\cdots,\beta_n$ be the eigenvalues of A_{π_1} and A_{π_2}, respectively. Recall that we are assuming that the product of the central characters of π_1 and π_2 is trivial. This means that $\alpha_1\cdots\alpha_n\beta_1\cdots\beta_n = 1$. The Schur polynomial satisfies

$$\chi_{(\lambda_1,\cdots,\lambda_n)}(\alpha_1,\cdots,\alpha_n) = \left(\prod\alpha_i\right)^{\lambda_n}\chi_{(\lambda_1-\lambda_n,\cdots,\lambda_{n-1}-\lambda_n,0)}(\alpha_1,\cdots,\alpha_n).$$

As a consequence of this we can rewrite (12) as

$$\sum_{\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0} \chi_\lambda(A_{\pi_1})\, \chi_\lambda(A_{\pi_2})\, q^{-|\lambda| s},$$

where now the summation is over all λ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. Indeed, the summation over λ_n produces a geometric series whose sum is $(1-q^{-ns})^{-1}$. Now we have a well-known formula of Cauchy, which was missattributed to a later writer in [5].

Proposition 1 (Cauchy) *We have*

$$\sum_{\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0} s_\lambda(\alpha_1, \cdots, \alpha_n)\, s_\lambda(\beta_1, \cdots, \beta_n) q^{-|\lambda| s} = \prod_{i,j} (1 - \alpha_i \beta_j q^{-s})^{-1}. \quad (13)$$

Cauchy's identity identifies (6) as the local L-function of the $\mathrm{GL}(n) \times \mathrm{GL}(n)$ convolution.

This is a typical example of how a symmetric algebra decomposition, combined with the Casselman-Shalika formula, underlies a local computation in the Rankin-Selberg method. Some irreducible representations have simple symmetric algebra decompositions, some have complicated ones. Both cases can appear in the Rankin-Selberg method.

Various proofs of Cauchy's identity may be found in the literature. I have not seen the following proof in print; it was told to me some years ago by Rallis.

Since s_λ is a homogeneous polynomial of degree $|\lambda|$ it is sufficient to prove the Cauchy identity with $s = 0$. We omit q^{-s} from the following considerations, and assume that $|\alpha_i|, |\beta_j| < 1$.

The group $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ acts on $\mathrm{Mat}_n(\mathbb{C})$ by left and right multiplication. There is a corresponding action on the ring $\mathcal{P}$ of all polynomial functions on $\mathrm{Mat}_n(\mathbb{C})$, thus:

$$(g_1, g_2) f(x) = f({}^t g_1 x g_2), \qquad g_i \in \mathrm{GL}(n, \mathbb{C}), \quad f \in \mathcal{P}, \quad x \in \mathrm{Mat}_n(\mathbb{C}).$$

This infinite-dimensional representation of $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ is the symmetric algebra on $\mathbb{C}^n \otimes \mathbb{C}^n \cong \mathrm{Mat}_n(\mathbb{C})$, where $\mathbb{C}^n$ is the standard module of $\mathrm{GL}(n, \mathbb{C})$.

To decompose it into irreducibles, let us first consider the corresponding action on the affine ring $\mathcal{A}$ of $\mathrm{GL}(n, \mathbb{C})$, which is generated by the subring

$\mathcal{P}$ together with the function $\det^{-1}$ which is polar on the determinant locus. We will show that as a $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ module

$$\mathcal{A} \cong \bigoplus_{\lambda_1 \geqslant \cdots \geqslant \lambda_n} \pi_\lambda \otimes \pi_\lambda. \tag{14}$$

Note that in this sum, unlike (13) where $\lambda_n \geqslant 0$, we allow λ_n to be negative. This is a sum over the characters χ_λ of *all* irreducible analytic representations of $\mathrm{GL}(n,\mathbb{C})$.

To prove (14) we begin by noting that the affine ring $\mathcal{A}$ can be identified with the ring of matrix coefficients of the maximal compact subgroup $U(n)$ of $\mathrm{GL}(n,\mathbb{C})$; a *matrix coefficient* of a compact group is a function whose left and right translates span a finite-dimensional vector space. Indeed, it is not hard to see that a polynomial function on $\mathrm{GL}(n,\mathbb{C})$ (which is allowed to have factors of $\det^{-1}$) is determined by its restriction to $U(n)$, and the restriction is a matrix coefficient. Using the Peter-Weyl theorem, the ring of matrix coefficients of the compact group $U(n)$ decomposes into irreducibles as

$$\mathcal{A} \cong \bigoplus \hat{\pi}_\lambda \otimes \pi_\lambda,$$

where the sum is over all irreducible representations, and the action of $U(n) \times U(n)$ is now

$$(g_1, g_2) f(x) = f(g_1^{-1} x g_2), \qquad g_i \in U(n), \quad f \in \mathcal{A}, \quad x \in U(n).$$

Such a decomposition of the ring of matrix coefficients is valid for any compact group. Now (14) follows as a decomposition over $U(n)$ on noting that the outer automorphism $g \longrightarrow {}^t g^{-1}$ interchanges a representation and its contragredient. As a decomposition over $\mathrm{GL}(n,\mathbb{C})$ it follows from the fact that restriction to $U(n)$ gives a bijection between the irreducible analytic representations of $\mathrm{GL}(n,\mathbb{C})$ and the irreducible representations of $U(n)$.

Not every element of $\mathcal{A}$ can be extended to an element of $\mathcal{P}$. The representations which occur in $\mathcal{P}$ are those whose matrix coefficients are polynomials (not involving $\det^{-1}$). In terms of the parametrization by highest weight vectors, this means that $\lambda_n \geqslant 0$, and the Cauchy identity is now proved.

5 Doubling Variables

Typically a Rankin-Selberg integral involves a pair of groups, $G \subset H$, and a pair of automorphic forms on G and H; one is a cusp form, the other an Eisenstein series. The Rankin Selberg method can involve:

I. a cusp form on H, an Eisenstein series on G;
II. a cusp form on G, Eisenstein series on H;
III. more complicated unipotent integrations, etc.

Case I is the easiest to analyze.

Case II leads to a double coset decomposition. Suppose that ϕ is a cusp form on G and $E(g,s) = \sum_{P_F\backslash H_F} f_s(\gamma g)$ is an Eisenstein series on H. We have

$$\int_{G(F)\backslash G(\mathbb{A})} \phi(g)\, E(g,s)\, dg =$$
$$\sum_{\gamma\in P_F\backslash H_F/G_F} \int_{{}^{\gamma}G(\mathbb{A})\backslash G(\mathbb{A})} f_s(\gamma k) \quad \left[\int_{{}^{\gamma}G(F)\backslash {}^{\gamma}G(\mathbb{A})} \phi(gk)\, dg\right] dk, \tag{15}$$

where ${}^{\gamma}G = G \cap \gamma^{-1}H\gamma$. If ${}^{\gamma}G$ contains the unipotent radical of a parabolic subgroup of G then the inner integral vanishes by cuspidality. If all goes well, only one particular γ will contribute, and that integral can be reduced to a Whittaker one, or to another unique model.

A type II example is worked out in Bump [6] Section 3.10.

As an example of this type of integration, Piatetski-Shapiro and Rallis [70] gave a construction which included the standard L-functions of all classical groups. The symplectic case of the same construction was found independently by Böcherer [3]. This construction is known as the "doubling variables" method.

For definiteness let us assume that $G = O(n) \times O(n)$. Let $H = O(2n)$. Take an Eisenstein series on H for the parabolic P whose Levi factor is $\mathrm{GL}(1) \times O(2n-2)$, and consider the integral

$$\int_{G(F)\backslash G(\mathbb{A})} \phi_1(g_1)\, \overline{\phi_2(g_2)}\, E\big((g_1,g_2),s\big)\, dg_1 \times dg_2.$$

To use (15) one must compute the double cosets $P\backslash H/G$, and show that only one double coset contributes to the integral. The integral unfolds to

$$\int_{O(n,F)\backslash O(n,\mathbb{A})} \langle \phi_1, \pi(g)\, \phi_2\rangle\, f_s\big(\gamma(g,g)\big)\, dg,$$

for suitable $\gamma \in H_F$. Here

$$\langle \phi_1, \pi(g)\phi_2\rangle = \int_{O(n,F)\backslash O(n,\mathbb{A})} \phi_1(x)\overline{\phi_2(xg)}\, dx$$

is the usual L^2 inner product, with $\pi(g)\,\phi_2(x) = \phi_2(xg)$ since the group acts by right translation. We see that we must take ϕ_1 and ϕ_2 in the same automorphic representation of $O(n)$, or the integral is nonzero. The matrix coefficient $\langle \phi_1, \pi(g)\phi_2 \rangle$ is Eulerian, and its local constituent is the spherical function at almost all places.

This construction was quite different from anything seen before. The appearance of matrix coefficients is not quite unprecedented since they also appear in Godement and Jacquet [49]. It works equally well with symplectic or unitary groups.

6 The Rankin-Selberg and Langlands-Shahidi methods contrasted

The Rankin-Selberg method seeks to construct Langlands L-functions as integrals of automorphic forms. An alternative approach to this problem is the Langlands-Shahidi method, in which the Langlands L-function is realized as the Whittaker coefficient of an Eisenstein series on a larger group. See Shahidi [78] for a survey of the recent spectacular successes of the Langlands-Shahidi method.

Over the years, there has been a change in how these two methods are viewed, as we will now explain.

It should be born in mind that the Langlands-Shahidi method constructs a fixed and known list of L-functions. To construct a Langlands L-function of an automorphic form on G by this method, one needs to find a maximal parabolic subgroup of a larger group H whose Levi subgroup is isomorphic to G. One may read off from the Dynkin diagrams of the various semisimple Lie groups the Levi factors of their parabolic subgroups, and then a bit of additional work is needed to determine what Langlands L-function is obtained. Those with connected Levi subgroup were listed in Langlands [62], and those with disconnected factors are listed in [77].

By contrast, the Rankin-Selberg method does not construct a fixed and known list of L-functions. There is no classification theorem for L-functions constructible by the Rankin-Selberg method, only a set of examples and heuristics. By now, Rankin-Selberg constructions are becoming harder to find, but there remains the tantalizing possibility that new ideas will bring new examples.

Both the Rankin-Selberg and Langlands-Shahidi methods fall far short of the goal of constructing all Langlands L-functions. Both construct enough L-functions for nontrivial applications.

Until around 1985, comparatively few Rankin-Selberg constructions were known. The $\mathrm{GL}(n) \times \mathrm{GL}(m)$ construction and the Jacquet-Godement construction of the standard L-function were known. The exterior square construction of Jacquet and Shalika [55] was known but not yet published. There were also some constructions for classical groups due to Novodvorsky, Piatetski-Shapiro, Gelbart and Piatetski-Shapiro, and others, though the most important constructions for the classical groups were not yet available. There were some important symplectic constructions of Andrianov, Shimura's integrals using theta functions, and Asai's constructions. The survey of Bump [6], written shortly after this time, gives an indication of this state of affairs.

In 1985 it was suspected by many that all Rankin-Selberg L-functions were also constructible by the Langlands-Shahidi method, and it was possible to believe that the Rankin-Selberg list was a proper subset of the Langlands-Shahidi list. Nevertheless it was thought that when the Rankin-Selberg method did work it gave more information.

These opinions seemed reasonable in 1985 but they did not stand the test of time. In the spring of 1988 Piatetski-Shapiro, Rallis and Schiffman [73] found a Rankin-Selberg construction for the standard L-function of $G_2 \times \mathrm{GL}(2)$. (Surprisingly, the standard L-function of $G_2 \times \mathrm{GL}(2)$ proved a little easier than the standard L-function of G_2 by itself, which was constructed much later by Ginzburg [39].) Since $G_2 \times \mathrm{GL}(2)$ is not the Levi factor of a parabolic subgroup, this L-function cannot be constructed as a Langlands-Shahidi L-function. After this example, the Rankin-Selberg list was no longer a subset of the Shahidi list. Moreover, since this time, the Rankin-Selberg list has expanded so that there are now only a few Langlands-Shahidi L-functions for which Rankin-Selberg constructions are not known.

The second belief, that the Rankin-Selberg gives more precise information than the Langlands-Shahidi method, has also proved unfounded. A strong motivation for developing both methods was found in the converse theorem, developed by Piatetski-Shapiro and Cogdell, and applied by them and other collaborators [25]. If G is a classical group, its (connected) L-group is also a classical group and has a "standard" representation ${}^LG \longrightarrow \mathrm{GL}(r)$, so there is a standard L-function. We may tensor this standard representation with the standard representation of the (connected) L-group $\mathrm{GL}(n, \mathbb{C})$ of $\mathrm{GL}(n)$, and obtain the standard L-functions of $G \times \mathrm{GL}(n)$, an Euler product of degree rn associated with a cusp form on G and a cusp form on $\mathrm{GL}(n)$. The converse theorem shows that if functional equations for such L-functions can be proved, together with analyticity and boundedness in vertical strips, then the cusp form on G can be lifted to a cusp form on $\mathrm{GL}(r)$, in accordance with the functoriality conjecture.

Happily, these $G \times \mathrm{GL}(n)$ L-functions are quite approachable by both the Rankin-Selberg and Langlands-Shahidi methods, though the Rankin-Selberg constructions were only known in special cases in 1985. The $G \times \mathrm{GL}(n)$ constructions are much easier than, say, $G \times G$. I believe that the only direct Rankin-Selberg construction for $G \times G$ which is known where G is a classical group of rank > 1 is the case of the spin $\times$ spin L-function for $G \times G$ when $G = \mathrm{GSp}(4)$, in Jiang [56]. (Of course if the lifting from classical groups is known, these L-functions can be constructed by first lifting to $\mathrm{GL}(r)$, so *indirect* constructions are possible.) The same is true for the Langlands-Shahidi method—if G is a classical group, the $G \times \mathrm{GL}(n)$ L-functions can be constructed, but the $G \times G$ constructions cannot be.

Once Rankin-Selberg constructions were found for the $G \times \mathrm{GL}(n)$ L-functions it was believed for a time that these would be better suited for obtaining control of the poles needed for the converse theorem. During the early 1990's, much work was devoted toward developing the Rankin-Selberg method for $G \times \mathrm{GL}(n)$.

But it was shown by Moeglin and Waldspurger [64] for the $\mathrm{GL}(n) \times \mathrm{GL}(n)$ convolution that Shahidi's method could precisely determine the poles, filling a gap in the literature, since although the Rankin-Selberg construction was well known, but the details in print were quite sketchy. After this, the Langlands-Shahidi method was developed by Kim and Shahidi, culminating in the work of Cogdell, Kim, Piatetski-Shapiro and Shahidi [24] proving liftings from classical groups to $\mathrm{GL}(r)$. Around this time and in the same current Ramakrishnan [75] proved the automorphicity of the Rankin-Selberg L-function on $\mathrm{GL}(2) \times \mathrm{GL}(2)$.

Probably either the Rankin-Selberg method or the Langlands-Shahidi method would have proved sufficient for the lifting program, though ultimately it was the Langlands-Shahidi method that was used.

Although the above history presents the Rankin-Selberg method and the Langlands-Shahidi method as competing programs, they are related to each other. Certain definitions, ultimately applied in the Langlands-Shahidi method were inspired by corresponding definitions first found in the Rankin-

Selberg method.

7 $G \times \mathrm{GL}(n)$, G orthogonal

The importance of the $G \times \mathrm{GL}(n)$ constructions, with G a classical group, was explained in the previous section. The completion of the case where G is orthogonal and π generic, begun by Piatetski-Shapiro, Novodvorsky and Gelbart, was completed by Ginzburg and Soudry.

After the early work of Novodvorsky and Piatetski-Shapiro there came the work of Gelbart and Piatetski-Shapiro [36] which treated $G \times \mathrm{GL}(n)$ in the special case where the rank of G is n. In a nutshell, this construction is as follows. If $G = \mathrm{SO}(2n+1)$, one constructs with the given cusp form on $\mathrm{GL}(n)$ an Eisenstein series on $\mathrm{SO}(2n)$, restricts the cusp form from $\mathrm{SO}(2n+1)$ to $\mathrm{SO}(2n)$ and integrates against this Eisenstein series. Or if $G = \mathrm{SO}(2n-1)$, one can still integrate against the same Eisenstein series on $\mathrm{SO}(2n)$, restricting it to $\mathrm{SO}(2n-1)$.

Alternatively, if $G = \mathrm{SO}(2n)$, one forms an Eisenstein series on $\mathrm{SO}(2n+1)$ with the $\mathrm{GL}(n)$ cusp form, restricts to $\mathrm{SO}(2n)$, and integrates against the cusp form there. One can do the same thing if $G = \mathrm{SO}(2n+2)$.

This is a special case of a more general construction for $\mathrm{SO}(2r+1)\times\mathrm{GL}(n)$ and $\mathrm{SO}(2r) \times \mathrm{GL}(n)$. We see that in 1985 the construction was known when r and n are equal or nearly so.

In the general case one proceeds in the same way—for $\mathrm{SO}(2r+1)\times\mathrm{GL}(n)$ one makes an Eisenstein series with the $\mathrm{GL}(n)$ cusp form on $\mathrm{SO}(2n)$, and for $\mathrm{SO}(2r)\times\mathrm{GL}(n)$ one makes an Eisenstein series on $\mathrm{SO}(2n+1)$. In either case, one has an Eisenstein series and a cusp form, one on an odd orthogonal group and one on an even orthogonal group. If the ranks differ, however, one must "pad" the difference between the two orthogonal groups with a unipotent integration. We saw this already with the $\mathrm{GL}(n) \times \mathrm{GL}(m)$ convolutions in (7). This is not a serious complication but may have delayed the discovery of these integrals by a few years. If $r > n$ this construction was found by Ginzburg [37].

Let us illustrate the need for a unipotent integration with an example. We describe the Rankin-Selberg convolution for $\mathrm{SO}(7) \times \mathrm{GL}(2)$. Make an Eisenstein series on $\mathrm{SO}(4)$ with the $\mathrm{GL}(2)$ cusp form τ. One may embed

SO(4) into SO(7) and integrate against the SO(7) cusp for ϕ as follows:

$$\int_{O(4,F)\backslash O(4,\mathbb{A})} \int_{\mathbb{A}/F} \int_{(\mathbb{A}/F)^4} E_\tau(h,s)\,\psi(r)$$
$$\phi\left(\begin{pmatrix} 1 & & & & x_1 & & \\ & 1 & & & x_2 & & \\ x_4 & x_3 & 1 & r & * & * & * \\ & & & 1 & -r & & \\ & & & & 1 & & \\ & & & & * & 1 & \\ & & & & * & & 1 \end{pmatrix}\begin{pmatrix} \boxed{h} & & \boxed{h} \\ & I_3 & \\ \boxed{h} & & \boxed{h} \end{pmatrix}\right) dx_i\,dr\,dh$$

here $\psi : \mathbb{A}/F \longrightarrow \mathbb{C}$ is a nontrivial additive character. The need to pad the integration with a unipotent group explains the fact that the equal rank case was discovered historically earlier. (Although the case $n = 1$ was also discovered early, by Novodvorsky.)

The SO(4)$\times$GL(3) construction is outwardly similar: one makes an Eisenstein series on SO(7) with the GL(3) cusp form and considers the same integral, with the same unipotent subgroup, except that the roles of the cusp form and Eisenstein series are interchanged.

However there is an important difference between the cases n small and n large. If $n \leqslant r$ (where $G = \mathrm{SO}(2r)$ or $\mathrm{SO}(2r+1)$) then the decomposition of the symmetric algebra of the L-group is simple and easily comprehensible. In the classification of Section 5, this is a Type I integral, which unfolds to a straightforward integral of the Whittaker function. The unramified computation is therefore simple.

If $n > r$, then the decomposition of the symmetric algebra is chaotic. The integration is of Type II, and the unramified computation seems intractable.

Luckily Soudry [82] p. 97 was able to solve this problem very ingeniously. We have already observed a similarity between the constructions for $\mathrm{SO}(7) \times \mathrm{GL}(2)$ and $\mathrm{SO}(4) \times \mathrm{GL}(3)$. More generally, we consider the constructions for $\mathrm{SO}(2r+1) \times \mathrm{GL}(n)$ and $\mathrm{SO}(2n) \times \mathrm{GL}(r)$ to be in duality. Note that not only are the global integrals similar, with the roles of the cusp form and Eisenstein series interchanged, but the local integrals are Euler products of the same degree $2rn$. (The L-groups are $\mathrm{Sp}(2r) \times \mathrm{GL}(n)$ and $\mathrm{SO}(2n) \times \mathrm{GL}(r)$.) Soudry was able to find a method of transforming the (hard) unramified computation for $\mathrm{SO}(2r+1) \times \mathrm{GL}(n)$ with n large into the (easy) unramified computation for $\mathrm{SO}(2n) \times \mathrm{GL}(r)$.

For automorphic cuspidal representations which are not generic, including those over orthogonal groups which may not be split or quasisplit, a complete and extensive theory was given by Ginzburg, Piatetski-Shapiro and Rallis [44]. The Whittaker model is replaced by more general unique models (Bessel model). Modifying (8) for the Bessel model the unipotent group N is replaced by another which may involve a smaller orthogonal group as well as a normal unipotent subgroup. An earlier such construction, which is a special case of the general one of Ginzburg, Piatetski-Shapiro and Rallis, but which may still be useful as an example is Furusawa [32]. In this paper Furusawa constructs the L-function for $\mathrm{SO}(4) \times \mathrm{GL}(2)$

Yet another construction was found by Murase and Sugano [66], who consider the restriction of an Eisenstein series induced from the parabolic subgroup with Levi factor $\mathrm{GL}(1) \times O(n-1)$ to $O(n+1)$, restricted to $O(n)$ and integrated against a cusp form there. This construction thus involves cusp forms on $O(n-1)$ and $O(n)$.

This example illustrates an observation, made by Bernstein, that there is a strong tendency for any integral of an $O(n)$ automorphic form against an $O(n-1)$ automorphic form, if nonzero, to be an L-function. One may take a cusp form on the larger group and an Eisenstein series on the smaller one or conversely. The explanation for this is contained in a uniqueness principle, to be discussed in the next section. Example of this type are Murase and Sugano [67] and the Appendix by Furusawa to Bump, Friedberg and Furusawa [8]. This phenomenon means that orthogonal constructions may be fairly abundant and difficult to classify.

8 Uniqueness Principles

We have already seen one uniqueness principle underlying the $\mathrm{GL}(n) \times \mathrm{GL}(n)$ convolution, namely that the global Rankin-Selberg integral (3) unfolds to a unique model, the Whittaker model. Yet the global integral (3) itself reflects a uniqueness theorem: this integral may be regarded as an invariant trilinear form on the product of three representations spaces, π_1, π_2 and (corresponding to the Eisenstein series) $\mathrm{Ind}_P^{\mathrm{GL}(n)}(\delta_P^s)$. There is a uniqueness for such trilinear forms, which we may state locally, though it is true also globally.

Theorem 1 *Let F be a local field, let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of $\mathrm{GL}(n, F)$, and let $V_3(s) = \mathrm{Ind}_{P(F)}^{\mathrm{GL}(n,F)}(\delta_P^s)$. If s*

is in general position, the space of invariant trilinear forms $T : V_1 \times V_2 \times V_3(s) \longrightarrow \mathbb{C}$ *is at most one dimensional.*

See Jacquet, Piatetski-Shapiro and Shalika [53] for this result, and Gelbart and Piatetski-Shapiro [36] for a discussion of how this sort of result is proved, using the Bernstein-Zelevinsky theory of derivatives. Although this is a strictly local statement, the proof is parallel to the unfolding of the global integral (3).

An important paradigm is that uniqueness theorems in representation theory correspond to constructions of L-functions, or sometimes special values of L-functions.

To explain what this means suppose that G is a reductive algebraic group over a global field F and H a subgroup, and suppose that (τ, V_τ) is an automorphic representation of H such that the induction of τ to G is multiplicity free. Let (π, V_π) be an automorphic representation of G. We can consider the integral

$$\int_{H(F)\backslash H(\mathbb{A})} f(h)\phi(h)\, dh, \qquad f \in V_\tau, \phi \in V_\pi.$$

This integral is an H-invariant bilinear form $V_\tau \times V_\pi \longrightarrow \mathbb{C}$, and by Frobenius reciprocity, the fact that $\mathrm{Ind}(\tau)$ is multiplicity free means that this functional is unique. Therefore we expect that unless the integral vanishes identically (which often happens) it may represent a value of an L-function. If f is an Eisenstein series, we may get an L-function of π, or if ϕ is an Eisenstein series, we may get an L-function of τ. If neither automorphic form is an Eisenstein series, we may get a special value or residue of an L-function.

As an example, we may take $G = \mathrm{GL}(n) \times \mathrm{GL}(n)$ and $H = \mathrm{GL}(n)$ embedded diagonally in G, and take $\tau = \mathrm{Ind}_P^G(\delta_P^s)$. The uniqueness theorem quoted above may be paraphrased by saying that inducing τ to G gives a multiplicity free representation. Therefore we expect the Rankin-Selberg integral (3) to be Eulerian.

We may now explain the remark at the end of the last section. In an influential lecture of May, 1989 at the Piatetski-Shapiro symposium Bernstein pointed out, among other things, that $O(n-1)$ inside $O(n)$ is a full multiplicity one subgroup, namely if τ is any irreducible representation of $O(n-1)$ its induction to $O(n)$ is multiplicity free. As a consequence, orthogonal integrals of the type described at the end of the last section have a strong tendency to be Eulerian.

9 The Symmetric and Exterior Squares

It is expected that a self-dual automorphic representation π of $\mathrm{GL}(n)$ is a lift from a classical group. To see this, note that $L(s,\pi\times\pi)=L(s,\pi\times\hat{\pi})$ has a simple pole from the general properties of the Rankin-Selberg convolution. We can factor

$$L(s,\pi\times\pi)=L(s,\pi,\vee^2)L(s,\pi,\wedge^2),$$

into the symmetric and exterior square L-functions, corresponding to the representations

$$\vee^2:\mathrm{GL}(n,\mathbb{C})\longrightarrow\mathrm{GL}\left(\frac{n(n+1)}{2},\mathbb{C}\right),\qquad\wedge^2:\mathrm{GL}(n,\mathbb{C})\longrightarrow\mathrm{GL}\left(\frac{n(n-1)}{2},\mathbb{C}\right).$$

Thus one of these L-functions has a pole, the other doesn't. It follows from a Rankin-Selberg representation that $L(s,\pi,\wedge^2)$ can have a pole only of n is even. A heuristic based on Galois representations (explained in the introduction to [15]) shows that the possibilities should be:

	Polar L-function	Lift From
$n=2m$	$L(s,\pi,\vee^2)$	$\mathrm{SO}(2m)$
$n=2m$	$L(s,\pi,\wedge^2)$	$\mathrm{SO}(2m+1)$
$n=2m+1$	$L(s,\pi,\vee^2)$	$\mathrm{Sp}(2m)$

In Ginzburg, Rallis and Soudry [48] it is shown that indeed, the existence of a pole of $L(s,\pi,\wedge^2)$ implies that π is a lift from $\mathrm{SO}(2m+1)$, and the other predicted liftings can be similarly realized.

Shimura [79] and [80] gave two Rankin-Selberg constructions involving the Jacobi theta function

$$\theta(z)=\sum_{n=-\infty}^{\infty}e^{\pi i n^2 z},\qquad z=x+iy,y>0.\tag{16}$$

Both constructions have many important generalizations, and they should be mentioned together because they are in some sense dual to each other.

This θ is a modular form of weight $1/2$, and in our present view is actually an automorphic form on the metaplectic double cover $\widetilde{\mathrm{GL}}(2)$ of $\mathrm{GL}(2)$. This is a central extension:

$$1\longrightarrow\{\pm1\}\longrightarrow\widetilde{\mathrm{GL}}(2)\longrightarrow\mathrm{GL}(2)\longrightarrow1.\tag{17}$$

Given an automorphic representation of $\widetilde{\mathrm{GL}}(2)$, either $\{\pm 1\}$ acts trivially, in which case the automorphic representation is actually a representation of GL(2), or it acts nontrivially, in which case we say the representation is *genuine*. The automorphic form θ, like all automorphic forms of half integral weight, belongs to a genuine automorphic representation of $\widetilde{\mathrm{GL}}(2)$.

One may consider a convolution varying (3) of the form:

$$\int_{Z_{\mathbb{A}}\,\mathrm{GL}(2,F)\backslash\,\mathrm{GL}(2,\mathbb{A})} \theta(g)\,\phi(g)\,E(g,s)\,dg. \tag{18}$$

The integrand must be invariant under $\{\pm 1\}$ (see (17)), so two of these forms must be genuine, and one must be nongenuine. We can, if we prefer, just integrate over SL(2) since the theta function is essentially supported there modulo the center.

If ϕ is a cusp form of half integral weight $k+\frac{1}{2}$, this is an Euler product of degree two, and the functional equation is identical to that of a modular form of weight $2k$. This leads to the Shimura correspondence.

If ϕ is an ordinary automorphic form (not genuine) this is an Euler product of degree 3, corresponding to the symmetric square representation ${}^L\mathrm{GL}(2) = \mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(3,\mathbb{C})$. This construction was generalized to GL(3) by Patterson and Piatetski-Shapiro [69] and to GL(n) by Bump and Ginzburg [15].

For the exterior square L-function, there are two constructions known, related to each other and both important. We will describe these constructions for GL($2n$). Let $\phi \in V$ where (π, V) is an automorphic cuspidal representation of GL($2n$).

The first construction, due to Jacquet and Shalika [55] involves the following integral:

$$\int_{Z_{\mathbb{A}}\,\mathrm{GL}(n,F)\backslash\,\mathrm{GL}(n,\mathbb{A})} \int_{\mathrm{Mat}_n(F)\backslash\,\mathrm{Mat}_n(\mathbb{A})} \phi\left(\begin{pmatrix} I & X \\ & I \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)$$
$$\overline{\psi(\mathrm{tr}\ X)}E(g,s)\,dX\,dg,$$

where the Eisenstein series $E(g,s)$ is the same one which appeared in (3). From the fact that this represents the exterior square L-function, and from the fact that the residue at $s=1$ of $E(g,s)$ is constant, we see that the period:

$$\int_{Z_{\mathbb{A}}\,\mathrm{GL}(n,F)\backslash\,\mathrm{GL}(n,\mathbb{A})} \int_{\mathrm{Mat}_n(F)\backslash\,\mathrm{Mat}_n(\mathbb{A})} \phi\left(\begin{pmatrix} I & X \\ & I \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\overline{\psi(\mathrm{tr}\ X)}dX\,dg$$

is nonzero if and only if $L(s,\pi,\wedge^2)$ has a pole at $s=1$. Conjecturally, we have already observed, this is true if and only if π is a functorial lift from $\mathrm{SO}(2n+1)$. This is the *Shalika period*. It corresponds to a functional proved unique by Jacquet and Rallis [54], namely the *Shalika functional*. The corresponding model is the *Shalika model*.

Specifically, let (π,V) be an irreducible representation of $\mathrm{GL}(n,F)$ over a local field F. A Shalika functional Λ is one which satisfies

$$\Lambda\left(\pi\begin{pmatrix} I & X \\ & I\end{pmatrix}\pi\begin{pmatrix} g & \\ & g\end{pmatrix}v\right)=\psi(\mathrm{tr}\,X)\Lambda(v).$$

By the result of Jacquet and Rallis, if it exists, it is unique; and our expectation is that it exists if π is self-dual, and a local lifting from $\mathrm{SO}(2n+1)$.

The second Rankin-Selberg integral was found by Bump and Friedberg [7]. It involves two parameters and represents a product of two L-functions. It is:

$$\int_{Z_{\mathbb{A}}(\mathrm{GL}(n,F)\times\mathrm{GL}(n,F))\backslash(\mathrm{GL}(n,\mathbb{A})\times\mathrm{GL}(n,\mathbb{A}))}\phi\begin{pmatrix} g_1 & \\ & g_2\end{pmatrix}$$
$$\left|\frac{\det(g_1)}{\det(g_2)}\right|^{s_1-1/2}E(g_1,s_2)\,d(g_1\times g_2).$$

It represents $L(s_1,\pi)L(s_2,\pi,\wedge^2)$. From this (taking the residue at $s=1$) we learn that the integral

$$\int_{Z_{\mathbb{A}}\left(\mathrm{GL}(n,F)\times\mathrm{GL}(n,F)\right)\backslash\left(\mathrm{GL}(n,\mathbb{A})\times\mathrm{GL}(n,\mathbb{A})\right)}\phi\begin{pmatrix} g_1 & \\ & g_2\end{pmatrix}\left|\frac{\det(g_1)}{\det(g_2)}\right|^{s-1/2}d(g_1\times g_2)$$

is nonvanishing only if $L(s,\pi,\wedge^2)$ has a pole, in which case it represents the standard L-function. Friedberg and Jacquet [30] studied this integral directly without making use of the Bump-Friedberg construction. They showed that it unfolds to the Shalika model and represents the standard L-function.

Another construction unfolding to the Shalika model may be found in Bump, Friedberg and Ginzburg [9]. This is a construction on $\mathrm{SO}(2n+2)$. However if $n=2$, making use of the relationship between $\mathrm{SO}(6)$ and $\mathrm{GL}(4)$, it may be interpreted as a Rankin-Selberg integral on $\mathrm{GL}(4)$ which vanishes unless the cuspidal automorphic form π on $\mathrm{GL}(4)$ is such that $L(s,\pi,\wedge^2)$ has a pole at $s=1$. It represents a degree 5 L-function. This is not a Langlands L-function attached to π. However assuming that $L(s,\pi,\wedge^2)$ has a pole, it is expected that π is the lift of an automorphic represenation π' of $\mathrm{SO}(5)$, and this is the degree 5 L-function of π'.

10 $G \times \mathrm{GL}(n)$, G symplectic

Rankin-Selberg constructions for symplectic groups tend to look a little different from constructions for orthogonal groups. Typically they are Shimura integals like (18) involving a theta function. It should be remembered that $\mathrm{GL}(2) = \mathrm{GSp}(2)$ and $\mathrm{SL}(2) = \mathrm{Sp}(2)$, so (18) can be regarded as an archetypal symplectic integral. The phenomenon of a "See-Saw," described in Kudla [61] and developed in Rallis [74] should be borne in mind in approaching symplectic integrals, and their relationship with orthogonal ones. A typical see-saw looks like this:

$$\begin{tikzcd}
Sp(2n) \times Sp(2n) \arrow[d, dash] \arrow[dr, dash] & O(r+s) \arrow[d, dash] \arrow[dl, dash] \\
Sp(2n) & O(r) \times O(s)
\end{tikzcd}$$

All groups in the picture fit into one large metaplectic group $\widetilde{\mathrm{Sp}}(2n(r+s))$ in such a way that the centralizer of $\mathrm{Sp}(2n)$ is $O(r+s)$, and the centralizer of $\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$ is $O(r) \times O(s)$. If an $\mathrm{Sp}(2n)$ in this diagram is paired with $O(k)$, k odd, we should write it as $\widetilde{\mathrm{Sp}}(2n)$ since the metaplectic cover does not split over it in this case. Two out of three of the numbers r, s and $r+s$ will be odd, the other even (unless they are all even). This means that two out of three of the symplectic groups on the left will be metaplectic groups $\widetilde{\mathrm{Sp}}(2n)$, the third a symplectic group $\mathrm{Sp}(2n)$.

The vertical lines are embeddings; on the left we have the diagonal embedding of $\mathrm{Sp}(2n)$ in $\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$. The diagonal lines are theta lifts.

By the general theory of dual reductive pairs and Howe duality, restricting the automorphic Weil representation of $\widetilde{\mathrm{Sp}}(2n(r+s))$ produces automorphic theta kernels for theta lifts of automorphic forms from $\mathrm{Sp}(2n)$ to $O(r+s)$, and from $O(r) \times O(s)$. This allows us to transfer Rankin-Selberg constructions from orthogonal groups to symplectic groups. The symplectic integrals appear as trilinear expressions like (18), involving three automorphic forms, typically one of them (say the lift from $O(s)$) a recognizable theta series, another an Eisenstein series, and yet another a cusp form.

We note that depending on how it is embedded in $\widetilde{\mathrm{Sp}}(2n(r+s))$, the metaplectic cover may or may not split over $\mathrm{Sp}(2n)$. In the see-saw, one of the three $\mathrm{Sp}(2n)$'s is paired with $O(k)$ with k even, the cover splits; otherwise it does not split.

As an example, take $r = 2n$ and $s = 1$. The see-saw looks as follows:

$$\begin{tikzcd}
Sp(2n)\times\widetilde{Sp}(2n) \arrow[d, dash] \arrow[dr, dash] & O(2n+1) \arrow[d, dash] \arrow[dl, dash] \\
\widetilde{Sp}(2n) & O(2n)\times O(1)
\end{tikzcd}$$

On the top $\widetilde{\mathrm{Sp}}(2n)$ we have the theta lift from $O(1)$, which is a theta function generalizing Jacobi's theta function (16). We will take a cusp form ϕ on $\mathrm{Sp}(2n)$ and an Eisenstein series on the bottom $\widetilde{\mathrm{Sp}}(2n)$. For the Eisenstein series, we can take a cusp form τ on $\mathrm{GL}(n)$ (Levi factor of the Siegel parabolic) and make an Eisenstein series on $\widetilde{\mathrm{Sp}}(2n)$. This is possible because the metaplectic double cover splits over $\mathrm{GL}(n)$. The integral on the left side is:

$$\int_{\mathrm{Sp}(2n,F)\backslash\,\mathrm{Sp}(2n,\mathbb{A})} \phi(g)\, E_\tau(g,s)\, \theta(g)\, dg. \tag{19}$$

The corresponding integral on the orthogonal side is the $O(2n)\times\mathrm{GL}(n)$ convolution of Gelbart and Piatetski-Shapiro [36], already explained in Section 7. The symplectic integral just explained is also in Gelbart and Piatetski-Shapiro [36].

If the lift from $\mathrm{Sp}(2n)$ to $O(2n)$ in this diagram is zero, we enlarge the orthogonal groups on the right and lift to $O(2n+2)$ instead.

For generic cusp forms, a complete theory of Rankin-Selberg integrals for $\mathrm{Sp}(2r)\times\mathrm{GL}(n)$ was given by Ginzburg, Rallis and Soudry [47]. This has many features in common with the orthogonal theory. First, the "equal rank" case, just discussed, is simpler, in avoiding any unipotent integration. In the unequal rank case, the cases $r < n$ and $r > n$ are different, but in duality. The case where $r < n$ is the hardest, and requires Soudry's idea of transforming the hard case into the easy case. All these features are perfectly parallel with the orthogonal case.

As we explained at the end of Section 7, orthogonal examples may be abundant and difficult to classify. By duality, the same may be true with symplectic constructions. One fairly unique construction is that of Murase and Sugano [65], involving a Heisenberg model that does not fit into the above description.

11 Metaplectic groups

Many symplectic integrals will have variants for genuine cusp forms on the metaplectic group. Just as with Shimura's integral, we may interchange the roles in (19) of the Eisenstein series and the cusp form. This means that the cusp form will be genuine and the Eisenstein series will not. As a consequence we obtain an Eulerian Rankin-Selberg associated with a genuine cusp form on $\widetilde{\mathrm{Sp}}(2n)$ and a $\mathrm{GL}(n)$ cusp form. This L-function has degree $2n \times n$ instead of degree $(2n+1) \times n$, a phenomenon already noted with the Shimura integral (18) when $n = 1$. Similarly, the metaplectic integrals of Zhuravlev [88] resemble those of Andrianov [1] but give an L-function of degree $2n$ instead of $2n+1$. This shows that the standard L-function on $\mathrm{Sp}(2n)$ has degree $2n+1$, but the standard L-function on the double cover $\widetilde{\mathrm{Sp}}(2n)$ has degree $2n$.

Matsumoto [63] defined a metaplectic cover of degree r generalizing (17) over G_F (F local) or $G_{\mathbb{A}}$ (F global) if F contains the r-th roots of unity and G is simply connected. If G is not simply connected you may still define a cover but you may require more roots of unity. For example the double cover of $O(n)$ is defined if F contains the fourth roots of unity.

Although Langlands' definition of the L-group does not include higher metaplectic covers, the Shimura correspondence and its generalizations show that an L-group can be defined. Savin [76] computed Iwahori Hecke algebras for covering groups and the answer can be inferred from his results. In general, Savin's computations show that the L-group of the metaplectic r-fold cover of $\mathrm{Sp}(2n)$ should be defined as $O(2n+1,\mathbb{C})$ if r is odd, and $\mathrm{Sp}(2n,\mathbb{C})$ if r is even. This is why the standard L-function on $\mathrm{Sp}(2n)$ has degree $2n+1$, but the standard L-function on the double cover $\widetilde{\mathrm{Sp}}(2n)$ has degree $2n$.

Generally Rankin-Selberg integrals on covering groups resemble (as global integrals) known Rankin-Selberg integrals on non-metaplectic groups. The Kazhdan-Patterson generalized theta series on the r-fold cover [58] of $\mathrm{GL}(r)$

often appears as a *deus ex machina*, convolution with which will produce an Euler product. Examples of this type are described in the next section.

12 Nonunique Models

Most known examples of Rankin-Selberg integrals unfold to unique models such as the Whittaker model. Among the more exotic constructions, the Whittaker model is predominant. In a few known cases, the integral unfolds to a model which is not unique, but for which the Rankin-Selberg integral nevertheless represents an Euler product. Andrianov [1] considered a Rankin-Selberg integral which superficially resembles (19), but in which there is no cusp form in the Eisenstein series, and in which the theta function is a lift from $O(n)$ rather than $O(1)$. In order to prove that the integral represented the standard L-function on $\mathrm{Sp}(2n)$, of degree $2n+1$, a difficult procedure involving finding recursion relations between Fourier coefficients was required. Zhuravlev [88] considered the corresponding problem on $\widetilde{\mathrm{Sp}}(2n)$, obtaining the standard L-function (of degree $2n$).

Meanwhile Bump and Hoffstein [20], [19] found Euler products for cusp forms on the metaplectic n-fold cover of $\mathrm{GL}(r)$ analogous to Shimura [80] making use of generalized theta series on the n-fold of $\mathrm{GL}(n)$, constructed by Kazhdan and Patterson. This construction, later extended by T. Suzuki [84], was proved by a difficult procedure analogous to that used by Andrianov. (Alas, no corresponding analog of Shimura [79] is known, though see Bump, Ginzburg and Hoffstein [18] for a construction of the symmetric cube L-functions on $\mathrm{GL}(2)$ using the generalized theta series on the 3-fold cover of $\mathrm{GL}(3)$.)

Goetze [50] showed that the Rankin-Selberg convolution $O(5) \times \mathrm{GL}(3)$, using the global integral of Gelbart and Piatetski-Shapiro [36], but working on the 3-fold metaplectic cover, taking a cusp form on $O(5) \cong \mathrm{PGSp}(4)$ and the Kazhdan-Patterson theta function on the 3-fold cover of $\mathrm{GL}(3)$, gives an Euler product. In general it is expected that the Ginzburg-Soudry construction for $O(n)\times\mathrm{GL}(r)$ described in Section 7 can be used to construct Euler products associated with automorphic forms on the metaplectic covers of all orthogonal groups.

Piatetski-Shapiro and Rallis [72] reconsidered this situation, and reproved Andrianov's integrals by representation theoretic considerations. They found that the integral unfolded to a nonunique model, but that the local integral, applied to a spherical vector in any model of the representation, produced the same local factor.

Bump, Furusawa and Ginzburg [12] gave further examples of constructions of L-functions involving nonunique models, and recently I was shown another very good example which is apparently still unpublished and which I will therefore not describe here. It is an important question whether these examples are common or rare. It is conceivable that they are common but hard to discover because adequate heuristics are not yet known.

13 Spin L-functions

If $G = \mathrm{PGSp}(2n)$, the L-group is the topological double cover $\mathrm{spin}(2n+1)$ of $\mathrm{SO}(2n+1)$, which has an irreducible representation of degree 2^n, the spin representation. In a few cases if (π, V) is an automorphic cuspidal representation of G a construction of $L(s, \pi, \mathrm{spin})$ is known. We include the possibility that $G = \mathrm{PGSp}(2n_1) \times \mathrm{PGSp}(2n_2)$, which has a spin L-function of degree $2^{n_1+n_2}$. Note that $\mathrm{GSp}(2) = \mathrm{GL}(2)$.

For GSp(4), there is a Hecke construction, due to Novodvorsky [68]. A useful current reference is Takloo-Bighash [85]. Novodvorsky gave a Rankin-Selberg construction for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ in the same article. Furusawa [31] gave a Rankin-Selberg construction for this same L-function which (unlike Novodvorsky's) unfolds not to a Whittaker model but to a Bessel model, and is therefore applicable to holomorphic Siegel modular forms (which are not generic). He used this construction to prove a special value result.

For GSp(6), $\mathrm{GSp}(6) \times \mathrm{GL}(2)$, GSp(8) and GSp(10), Rankin-Selberg constructions were found by Bump and Ginzburg [17], [14]. The case of GSp(6) is the subject of Vo [87]. Finally the spin L-function for $\mathrm{GSp}(4) \times \mathrm{GSp}(4)$ is constructed by Jiang [56].

Similarly if $G = \mathrm{PGO}(2n)$, the L-group is $\mathrm{spin}(2n)$. This group admits two spin representations both of degree 2^{n-1}. Rankin-Selberg constructions were found in a few cases by Ginzburg [40].

Bump, Friedberg and Ginzburg [10] gave examples of constructions producing the product of two L-functions in separate variables, one of which is the spin L-function. These are thus reminiscent of the construction of Bump and Friedberg of the exterior square L-function.

Except for Furusawa's construction, all integrals discussed above apply only to generic cusp form, hence do not apply to Siegel modular forms. Other constructions of the (degree 4) spin L-function for GSp(4) are known, but we omit mention of the older ones surveyed in [5]. One novel construction of the degree four L-function which was found later is that of Kohnen and Skoruppa [60], which involves two cusp forms F and G. If G is in the Maass space the Dirichlet series produces the degree 4 L-function of F. Extensions of this work can be found in Gritsenko [51] and Heim [52].

14 The Towers and the Magic Square

Ginzburg and Rallis [45] found four Rankin-Selberg integrals which involve the same G_2 Eisenstein series, and which can be given a uniform construction. There is an underlying heuristic behind their paper from Kac [57], which contains a list (Tables II and III, with imporant corrections at the end of [26]) that allows one to read off the normalizing factors of the Eisenstein series appearing in many Rankin-Selberg convolutions. Kac' table can be regarded as a list of L-group representations for which one seeks Rankin-Selberg constructions. It is remarkable how many important constructions can be found in this list. The invariant degrees $d_1, \cdots, d_l$ which appear in the table will be the degrees of the L-functions which appear as normalizing factors of the Eisenstein series, giving a clue which will allow some constructions to be guessed. For example, with $\mathrm{SL}_n \otimes \mathrm{SL}_n$ Kac gives a single degree $d = n$, from which we infer the normalizing factor $\zeta(ns)$ in (2).

Ginzburg and Rallis [45] began with the observation that three familiar Rankin-Selberg convolutions can be regarded as forming a tower. These are:

G	${}^L G$	representation
$\mathrm{GL}(n)$	$\mathrm{GL}(n, \mathbb{C})$	symmetric square
$\mathrm{GL}(n) \times \mathrm{GL}(n)$	$\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$	Rankin-Selberg
$\mathrm{GL}(2n)$	$\mathrm{GL}(2n, \mathbb{C})$	exterior square

These three constructions form a "tower" meaning that they should be viewed together. From their heuristic based on Kac' paper this is true. An aspect of this is that the invariant degree d in Kac' Table II is the same for these three cases, just n, reflecting the fact that the Rankin-Selberg constructions all involve the same "mirabolic" Eisenstein series on $\mathrm{GL}(n)$, with normalizing factor $\zeta(ns)$. (It is true that in the case of the symmetric square, the Eisenstein series is on the double cover but in their view this is not an important distinction.)

Next they give another tower:

G	${}^LG$	representation
GL(2)	GL$(2,\mathbb{C})$	symmetric fourth power
GL(3)	GL$(3,\mathbb{C})$	adjoint (degree 15)
SO(7)	Sp$(4,\mathbb{C})$	second fundamental
F_4	F_4	standard

The first construction, of the fourth symmetric power of GL(2), is described in Ginzburg [42]. The second of these constructions was previously found by Ginzburg [38]. The remaining two are new in this paper. They remark that the degree 14 L-function on SO(7) is dual to the construction of Piatetski-Shapiro, Rallis and Schiffmann [73] of a degree 14 L-function on $G_2 \times \mathrm{GL}(2)$. The duality phenomenon was explained in Section 7.

Ginzburg [41] gave two more towers. First, a tower of constructions involving a GL(3) mirabolic tower, which is an extension of the GL(n) mirabolic tower already described, adding a fourth entry when $n = 3$:

G	${}^LG$	representation
GL(3)	GL$(3,\mathbb{C})$	symmetric square
GL(3) $\times$ GL(3)	GL$(3,\mathbb{C}) \times$ GL$(3,\mathbb{C})$	Rankin-Selberg
GL(6)	GL$(6,\mathbb{C})$	exterior square
GE_6	GE_6	standard

Ginzburg gives another tower involving an Eisenstein series on the Siegel parabolic subgroup of GSp(6). This tower, as he decribes it, has five levels. I will assume, for reasons which will presently become clear, that the first two levels in this tower as he presents it are red herrings. The correct first level of the tower, I think, must be the construction of Bump, Friedberg and Ginzburg [11] of the degree 14 L-function of SO(7), corresponding to the *third* fundamental representation of the L-group Sp$(6,\mathbb{C})$. This appears in Kac' table II as $\wedge_0^3\mathfrak{sp}(6)$ and as with the other cases for this tower, $d = 4$ in Kac' list. *Caution*: both the second and third fundamental representations of Sp$(6,\mathbb{C})$ have degree 14, and both appear in this discussion. With this modification the third tower is:

G	${}^LG$	representation
SO(7)	Sp(6)	third fundamental
GL(6)	GL(6, $\mathbb{C}$)	symmetric cube
GSO(12)	spin(12)	spin
GE_7	GE_7	standard

The symmetric cube L-function on GL(6) was constructed by Ginzburg and Rallis [46], the spin L-function on GSO(12) was constructed by Ginzburg [40], and the E_7 construction is in Ginzburg [41].

In the exceptional groups one encounters the *Magic Square* of Freudenthal [29].

Geometry	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
2-dimensional elliptic	B_1	A_2	C_3	F_4
2-dimensional projective	A_2	$A_2 \times A_2$	A_5	E_6
5-dimensional symplectic	C_3	A_5	D_6	E_7
hypersymplectic	F_4	E_6	E_7	E_8

The magic square has appeared in many different ways. From Freudenthal's viewpoint, four different geometries, one of them (hypersymplectic) being an exotic one which does not fit into an infinite family, were considered over four different fields. It will be noted that the L-groups in the first tower of Ginzburg and Rallis, and the second and third towers found by Ginzburg (with the third tower modified as explained above) coincide exactly with the first three columns of the magic square.

Does this mean that we should expect Rankin-Selberg constructions to be found for the fourth tower? We do not know the answer to this question, but there is some reason to be pessimistic. For one thing, the heuristic based on Kac' list does not predict any such tower. In the next section we will explain another "numerological" heuristic and investigate what it predicts for the fourth tower.

15 Numerology

Suppose that some Rankin-Selberg integral, involving a cusp form and an Eisenstein series, unfolds to a Whittaker model. Let $\boldsymbol{W}$ be the dimension of the unipotent subgroup in the Whittaker model, let $\boldsymbol{E}$ be the dimension of the unipotent radical in the Eisenstein series, and let $\boldsymbol{D}$ be the dimension of the entire integration in the Rankin-Selberg integral. Then

$$\boldsymbol{D} = \boldsymbol{W} + \boldsymbol{E}.$$

This numerical indentity can be quite handy in searching for Rankin-Selberg integrals. We will apply it in a case where it fails, but it was used with success to find some of the integrals in Section 12. A different dimensional heuristic is found in Garrett [34].

As an example, let us consider the $\mathrm{GL}(n)\times\mathrm{GL}(n)$ Rankin-Selberg integral (3). The integration is over $Z\backslash\,\mathrm{GL}(n)$, or n^2-1 dimensions. There are two Whittaker functions, each contributing $\frac{1}{2}n(n-1)$ to $\boldsymbol{W}$, so $\boldsymbol{W} = n^2 - n$. Finally, the Eisenstein series is over the parabolic subgroup with Levi factor $\mathrm{GL}(n-1)\times\mathrm{GL}(1)$, and its unipotent radical has dimension $\boldsymbol{E} = n-1$.

Now let us consider second, third and fourth examples from the third tower. In each case, there is a parabolic subgroup of G with Levi decomposition MU_k, and a character ψ_U of U_k whose stabilizer in M is $\mathrm{Sp}(6)$. The dimension of integration $\boldsymbol{D}$ is therefore known, as is the dimension of the Whittaker model. Subtracting, $\boldsymbol{E} = 6$ so we are looking in each case for an integration over a parabolic subgroup of $\mathrm{Sp}(6)$ with a unipotent radical of dimension 6. This is the Siegel parabolic.

	ρ of L-gp	Integration	$\boldsymbol{D}$	$\boldsymbol{W}$	$\boldsymbol{D}-\boldsymbol{W}$
A_5	ω_3 of GL(6)	Sp(6)	21	15	6
D_6	spin_{12}	$\mathrm{Sp}(6)\times U_{15}$	36	30	6
E_7	standard	$\mathrm{Sp}(6)\times U_{48}$	69	63	6

This heuristic is quite handy in searching for Rankin-Selberg integrals. Can a similar scenario pertain for a hypothetical fourth tower?

	Integration	$\boldsymbol{D}$	$\boldsymbol{W}$	$\boldsymbol{D}-\boldsymbol{W}$
E_6	F_4	52	36	16
E_7	$F_4\times U_{27}$	79	63	16
E_8	$F_4\times U_{84}$	136	120	16

Note that $\boldsymbol{D}-\boldsymbol{W}$ is constant. So far so good, but if the scenario is to be identical to that of the third tower we would be looking for a parabolic subgroup of F_4 with a unipotent radical of dimension 16. Such a parabolic does not exist.

16 Miscellany

An early Rankin-Selberg construction unfolding to a non-Whittaker model is Sugano [83].

A construction on $\mathrm{SU}(2,1)$ was found independently by Shintani [81] and Gelbart and Piatetski-Shapiro [35]. Further unitary constructions were found by Tamir [86].

Bump and Ginzburg [16] and [13] have two related constructions of the adjoint L-function on GL(4), of degree 15, and a degree 27 L-function on split spin(9), corresponding to the second fundamental representation of the L-group PGSp(8). Both these constructions use the same Eisenstein series on the exceptional group F_4.

Ginzburg [43] has a construction for the adjoint L-function of Sp(4), an Euler product of degree 10.

In Section 9 we saw that the symmetric and exterior square L-functions could be used to distinguish the images of two different lifts from $O(2n)$ and $O(2n+1)$ to $\mathrm{GL}(2n)$. The Asai L-function [2], which is a Galois twisted version of the $\mathrm{GL}(2) \times \mathrm{GL}(2)$ convolution plays a role similar in distinguishing the images of two different types of base change lift, and showing that their images are disjoint. Flicker [27], [28] gave a similar Galois twisted analog of the $\mathrm{GL}(n) \times \mathrm{GL}(n)$ and applied it to quadratic base change. Piatetski-Shapiro and Rallis [71] gave an analog of the Asai L-function for GL(2) and *cubic* Galois extensions. This is a twisted version of the important construction Garrett [33] of a $\mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$ Rankin-Selberg construction based on the restriction of an Eisenstein series of Sp(6). We note that a $\mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$ can also be obtained from the Gelbart-Piatetski-Shapiro $O(4) \times \mathrm{GL}(2)$ construction since $O(4)$ is isogenous to $\mathrm{GL}(2) \times \mathrm{GL}(2)$, but this alternative method does not give the twisted L-function.

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