Weyl Group Multiple Dirichlet Series: Some Open Problems

Notes for the Workshop on Multiple Dirichlet Series,
Stanford University, July 9–16, 2006

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July 7, 2006

1 Introduction

The problems described here concern Weyl group multiple Dirichlet series (WMDs). Let F be a totally complex algebraic number field containing the group μ_n of n-th roots of unity. We will assume that -1 is an n-th power in F. Let $\Phi \subset \mathbb{R}^r$ be a reduced root system. Then one can attach to Φ and n a family of multiple Dirichlet series whose coefficients involve products of n-th order Gauss sums. These series are the Whittaker coefficients of the metaplectic Eisenstein series obtained by inducing from the Borel subgroup. (See [10, 11, 13] for aspects of the metaplectic group.) As such, they should have meromorphic continuation in all variables and satisfy a finite group of functional equations.

It turns out to be difficult to study these Whittaker coefficients directly. However, we believe that one can write down the correct objects and prove their functional equations by means of a reduction to $\widetilde{SL}_2$, using Bochner's theorem on continuation to a tube domain (see [2] or [12] for the $\widetilde{SL}_2$ theory, and [1] or [9] for Bochner's theorem). The study of such "Weyl group multiple Dirichlet series" using this method was initiated by Brubaker, Bump, Chinta, Friedberg and Hoffstein, and has been described in a series of papers ([3], [4], [5], [6]). If n is sufficiently large (the 'stable case') these series are completely understood, i.e. the meromorphic continuation and functional equations are proved. However, if n is not large (the 'non-stable case'), the situation is much more complicated. The problem is to understand it fully.

Let us next explain what is known in the non-stable case. The multiple Dirichlet series of concern here are of the form

$$Z_\Psi(s_1, \cdots, s_r) = \sum_{\mathfrak{c}_1, \cdots, \mathfrak{c}_r} H\Psi(\mathfrak{c}_1, \cdots, \mathfrak{c}_r) \ \mathbb{N}\mathfrak{c}_1^{-2s_1} \cdots \mathbb{N}\mathfrak{c}_r^{-2s_r}, \tag{1}$$

where the sum is over nonzero ideals $\mathfrak{c}_1, \cdots, \mathfrak{c}_r$ of the ring $\mathfrak{o}_S$ of S-integers of F, where S is a finite set of places chosen so that $\mathfrak{o}_S$ is a principal ideal domain. It is assumed that S contains all archimedean places, and those ramified over $\mathbb{Q}$.

The coefficients in Z thus have two parts, denoted $H(C_1, \cdots, C_r)$ and $\Psi(C_1, \cdots, C_r)$, defined for nonzero $C_i \in \mathfrak{o}_S$. The product $H\Psi$ is unchanged if C_i is multiplied by a unit, and so is a function of r-tuples of ideals in the principal ideal domain $\mathfrak{o}_S$. This fact is implicit in the notation (1), where use is made of the fact that $H\Psi(C_1, \cdots, C_r)$ depends only on the ideals $\mathfrak{c}_i = C_i\mathfrak{o}_S$. The factor Ψ is the less important of the two, and we will not define it here; it is described in [4]. Suffice it to say that Ψ is chosen from a finite-dimensional vector space of functions on $F_S = \prod_{v \in S} F_v$, and that these functions are constant on cosets of an open subgroup. If one changes the setup slightly, the function Ψ can be suppressed using congruence conditions.

The function H is more interesting and is the focus of discussion. Let us describe the function H which should correspond to the first non-trivial Whittaker coefficient (that is, is the $(1, \cdots, 1)$-coefficient, rather than the more general $(m_1, \cdots, m_r)$-coefficient). For simplicity we assume that Φ is simply-laced (recall this means that all roots have the same length, so that the associated Dynkin diagram has only single-line connections), and that all roots are normalized to have length 1; see [4] for the general case. Let $\alpha_1, \cdots, \alpha_r$ be the simple positive roots of Φ in some fixed order. The coefficients H have the following "twisted" multiplicativity. If $\gcd(C_1 \cdots C_r, C'_1 \cdots C'_r) = 1$, then

$$\frac{H(C_1C'_1, \cdots, C_rC'_r)}{H(C_1, \cdots, C_r)H(C'_1, \cdots, C'_r)} = \prod_{i=1}^{r} \left(\frac{C_i}{C'_i}\right)\left(\frac{C'_i}{C_i}\right) \prod_{\substack{i < j \\ \alpha_i, \alpha_j \text{ not orthogonal}}} \left(\frac{C_i}{C'_j}\right)^{-1} \left(\frac{C'_i}{C_j}\right)^{-1}. \tag{2}$$

The condition that α_i, α_j not be orthogonal means that these simple roots correspond to adjacent nodes in the Dynkin diagram. In this formula $\left(\frac{C}{D}\right)$ is the n-th order power-residue symbol, defined for nonzero coprime elements of $\mathfrak{o}_S$. It satisfies the reciprocity law

$$\left(\frac{C}{D}\right) = (D, C)_S \left(\frac{D}{C}\right),$$

where $(a, b)_S = \prod_{v \in S}(a, b)_v$ is the S-Hilbert symbol, defined for $a, b \in F_S^\times$. See [4] for further information.

Knowing the twisted multiplicativity of H, we may reduce the description of H to the case where the C_i are all powers of the same prime p. *The problem is to determine H when each C_i is a power of p.*

In the stable case, this is done in [4], and in greater generality in [6], where the series that correspond to general Whittaker coefficients are described. (We call the latter case the 'twisted case' as it roughly corresponds to twisting the series by characters.) For each series, it is found that there are exactly $|W|$ values of $(k_1, \cdots, k_r)$ such that $H(p^{k_1}, \cdots, p^{k_r}) \neq 0$, where W is the Weyl group of Φ, and there is a precise bijection between the Weyl group W and the set $\text{Supp}_{\text{stable}}(H)$ of possible $(k_1, \cdots, k_r)$. See [4, 6] for details.

The paragraph above describes the *stable* coefficients H. We turn now to the more difficult case where n is *not* assumed to be large, and discuss what modifications we expect. The set

$$\text{Supp}(H) = \{(k_1, \cdots, k_r) \,|\, H(p^{k_1}, \cdots, p^{k_r}) \neq 0\}$$

will still be finite, and will contain $\text{Supp}_{\text{stable}}(H)$. Moreover, the values of $H(p^{k_1}, \cdots, p^{k_r})$ when $(k_1, \cdots, k_r) \in \text{Supp}_{\text{stable}}(H)$ will still be given as in [4, 6]. However, there will be *other* values of $(k_1, \cdots, k_r)$ in $\text{Supp}(H)$. These will lie in the convex hull of $\text{Supp}_{\text{stable}}(H)$.

If we specialize to the case $\Phi = A_r$, then Brubaker, Bump, Friedberg and Hoffstein [5] have found a conjectural description of the coefficients valid for all n. In fact, the description gives the $(p^{l_1}, \cdots, p^{l_r})$ Whittaker coefficient for any $l_i \geq 0$. We describe this next.

2 The Gelfand-Tsetlin pattern conjecture

A Gelfand-Tsetlin pattern is a triangular array of integers

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{array} \right\} \tag{3}$$

where the rows interleave; that is, $a_{i-1,j-1} \geqslant a_{i,j} \geqslant a_{i-1,j}$. The pattern is *strict* if each row is strictly decreasing. The strict Gelfand-Tsetlin pattern $\mathfrak{T}$ in (3) is *left-leaning* at (i,j) if $a_{i,j} = a_{i-1,j-1}$, *right-leaning* at (i,j) if $a_{i,j} = a_{i-1,j}$, and *special* at (i,j) if $a_{i-1,j-1} > a_{i,j} > a_{i-1,j}$.

Given a strict Gelfand-Tsetlin pattern, for $j \geq i$ let

$$s_{ij} = \sum_{k=j}^{r} a_{ik} - \sum_{k=j}^{r} a_{i-1,k}, \tag{4}$$

and define

$$\gamma(i,j) = \begin{cases} \mathbb{N}p^{s_{ij}} & \text{if } \mathfrak{T} \text{ is right-leaning at } (i,j), \\ g(p^{s_{ij}-1}, p^{s_{ij}}) & \text{if } \mathfrak{T} \text{ is left-leaning at } (i,j), \\ \mathbb{N}p^{s_{ij}}(1-\mathbb{N}p^{-1}) & \text{if } (i,j) \text{ is special and } n|s_{ij}; \\ 0 & \text{if } (i,j) \text{ is special and } n \nmid s_{ij}, \end{cases}$$

where

$$g(a,c) = \sum_{d \bmod c} \left(\frac{d}{c\mathfrak{o}_S}\right) \psi\left(\frac{ad}{c}\right) \tag{5}$$

is an n-th order Gauss sum.

Also, define

$$G(\mathfrak{T}) = \prod_{j \geqslant i \geqslant 1} \gamma(i,j). \tag{6}$$

Given non-negative integers k_i, l_i, $1 \leq i \leq r$, and a prime p, we define the p-th contribution to the coefficient of a multiple Dirichlet series by

$$H_{GT}(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\mathfrak{T}} G(\mathfrak{T}) \tag{7}$$

where the sum is over all strict Gelfand-Tsetlin patterns $\mathfrak{T}$ with top row

$$l_1 + \ldots + l_r + r, l_2 + \ldots + l_r + r - 1, \cdots, l_r + 1, 0$$

such that for each i, $1 \leq i \leq r$,

$$\sum_{j=i}^{r} (a_{ij} - a_{0,j}) = k_i. \tag{8}$$

Note that $(k_1, \cdots, k_r) = k(\mathfrak{T})$ in the notation of [5]. The general coefficient of the multiple Dirichlet series $H_{GT}(C_1, \cdots, C_r; m_1, \cdots, m_r)$ – which conjecturally corresponds to the $(m_1, \cdots, m_r)$ Whittaker coefficient of the metaplectic Eisenstein series on the n-fold cover of GL_{r+1} induced from the Borel – is then defined by means of a suitable twisted multiplicativity. In [5] we conjecture that these multiple Dirichlet series have meromorphic continuation and satisfy functional equations, and prove this for $n = 1$ or $r = 2$; we also prove it for $n = 2$ and $r \leq 5$ by establishing compatibility with Chinta's description [7]. We prove this in general in [6] for n sufficiently large.

3 The Chinta-Gunnells Approach

Gautam Chinta and Paul Gunnells have a recent preprint [8] in which they find a WMDs valid for all Φ when $n = 2$. I will not type the description of this, but rather refer you to their elegant paper and to Gautam's write-up of open problems. Though they prove the continuation and functional equation of their series, they do not give an explicit formula for each coefficieint. I should point out that they also offer a characterization of the coefficients but do not prove that there is a unique function that satisfies their characterization. I believe that their series corresponds to the first (i.e. $(1, \cdots, 1)$) Whittaker coefficient.

4 Open Problems

Here are some open problems, in roughly increasing order of expected difficulty.

1. Reconcile the Chinta-Gunnells and Brubaker-Bump-Friedberg-Hoffstein descriptions when $\Phi = A_r$ and $n = 2$. For example, one could find an explicit formula for the Chinta-Gunnells coefficients, and show that it is the same as the BBFH one. Or one could establish the uniqueness mentioned above and then check that the series formed from the BBFH coefficients for fixed prime p satisfies the desired properties.

2. Extend the Chinta-Gunnells description to $n > 2$. And extend their work to the case of the WMDs corresponding to general Whittaker coefficients, the so-called twisted case.

3. Extend the BBFH description to root systems other than $\Phi = A_r$. See Ben's write-up for a discussion of possible strategies related to this problem.

4. Use the BBFH description to prove the meromorphic continuation and functional equations in general.

5. Extend the group-theoretic description of the coefficients given for the stable case in [4, 6] to the non-stable case. Use this to prove the continuation and f.e.s. Then link this new, not-yet-existent, description, to the BBFH one and to the Chinta-Gunnells one.

6. Prove the conjectured link between WMDs and the Whittaker coefficients of metaplectic Eisenstein series in general. One strategy for doing this would be to analyze the geometry of the associated flag variety in an effort to determine how geometric components of the flag contributed individually to the resulting exponential sums written using Plücker coordinates. The hope is that one would see a natural way in which to parametrize these contributions using GT patterns.

References

[1] S. Bochner. A theorem on analytic continuation of functions in several variables. *Ann. of Math. (2)*, 39(1):14–19, 1938.

[2] B. Brubaker and D. Bump. On Kubota's Dirichlet series. *J. Reine Angew. Math.*, to appear.

[3] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series I. In *Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, Proceedings of the 2005 Bretton Woods Workshop*, Proceedings of Symposia in Pure Mathematics. The American Mathematical Society, to appear.

[4] B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series II. The stable case. *Invent. Math.*, to appear.

[5] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series III: Twisted unstable A_r. *Annals of Math.*, to appear.

[6] B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series IV. The stable twisted case. preprint.

[7] G. Chinta. Mean values of biquadratic zeta functions. *Invent. Math.*, 160(1):145–163, 2005.

[8] G. Chinta and P. Gunnells. Weyl group multiple Dirichlet series constructed from quadratic characters. preprint (2006).

[9] L. Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[10] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[11] T. Kubota. Topological covering of SL(2) over a local field. *J. Math. Soc. Japan*, 19:114–121, 1967.

[12] T. Kubota. *On automorphic functions and the reciprocity law in a number field.* Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 2. Kinokuniya Book-Store Co. Ltd., Tokyo, 1969.

[13] H. Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.