Orbital Integrals
and the Satake Isomorphism

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Orbital integrals are basic distributions on the group which are invariant under conjugation. They were introduced in the study of Lie groups by Harish-Chandra, who saw them as a tool to prove the Plancherel formula. Eventually, he introduced them into the representation theory of p-adic groups in [5], and applied them to the study of the character of a representation, among other things, again with the Plancherel formula in mind. Orbital integrals also arise in the trace formula.

The Satake Isomorphism [10], on the other hand, describes the structure of the spherical Hecke algebra. We will explain the relationship between these two, restricting ourselves to the special case of $\mathrm{GL}(n, F)$, where F is a nonarchimedean local field. We recommend Cartier [3], Part IV for further information, because the proofs in that paper there are quite short and to the point.

In these notes, F will be a nonarchimedean local field, $\mathfrak{o}$ its ring of integers. Let ϖ be a prime element of $\mathfrak{o}$, $\mathfrak{p} = (\varpi)$ the maximal ideal, and $q = |\mathfrak{o}/\mathfrak{p}|$ the cardinality of the residue field, so $q^{-1} = |\varpi|$.

1 Generalities about representations

We recall without proof some facts about Hecke algebras and smooth representations. Proofs and a more expanded discussion may be found in Bump [2], Section 4.2. Later we will introduce the spherical Hecke algebra and principal series representations, which for GL(2) are studied in some detail in Bump [2], Sections 4.5 and 4.6.

Let G be a totally disconnected locally compact group. This means that G has a basis of neighborhoods of the identity consisting of open, compact subgroups. For example, if $G = \mathrm{GL}(n, F)$, where F is a nonarchimedean local field, then we may see as follows that G satsifies our requirements. Let $\mathfrak{o}$ be the ring of integers, $\mathfrak{p}$ the prime ideal of $\mathfrak{o}$, and $K(\mathfrak{p}^n)$ the group of elements of $K = \mathrm{GL}(n, \mathfrak{o})$ which are congruent to the identity modulo $\mathfrak{p}^n$. The groups $K(\mathfrak{p}^n)$ are compact and open, and form a basis of neighborhoods of the identity. Hence this is a totally disconnected locally compact group.

We assume for simplicity that G is unimodular,and we denote by $\int \quad dg$ a Haar integral on G.

A *representation* (π, V) is a homomorphism from G into the group of invertible endomorphisms of a vector space V, which will often be infinite dimensional. The representation is called *smooth* the stabilizer of every $v \in V$ is an open subgroup of G. It is called *admissible* if it is smooth, and if for every open compact subgroup K of G, the space V^K of K-fixed vectors is finite dimensional.

A function f on G is called *smooth* if it is locally constant. The *Hecke algebra* $\mathcal{H}$ consists of all compactly supported locally constant complex-valued functions on G. It is a ring under convolution, without unit if G is noncompact. If $\phi \in \mathcal{H}$, and if (π, V) is any smooth representation, then we define an endomorphism $\pi(\phi)$ of V by

$$\pi(\phi)\, v = \int_G \phi(g)\, \pi(g)\, v\, dg. \tag{1}$$

Note that $\phi(g)\, \pi(g)\, v$ only takes on a finite number of values since $\phi(g)$ is locally constant and compactly supported, and since the stabilizer of v is open. Consequently the integral (1) is essentially a finite sum. It is easy to see that

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \circ \pi(\phi_2),$$

so $\phi \to \pi(\phi)$ is a representation of the ring $\mathcal{H}$.

If K is a compact, open subgroup of G, let

$$\epsilon_K = \frac{1}{\mathrm{vol}(K)}\, \mathbf{1}_K,$$

where $\mathbf{1}_K$ is the characteristic function of K. This is an idempotent in $\mathcal{H}$. Let $\mathcal{H}_K$ be the ring of K-bi-invariant functions in $\mathcal{H}$. Clearly a necessary and sufficient condition for K-bi-invariance is that $\epsilon_K * \phi * \epsilon_K = \phi$, so $\mathcal{H}_K = \epsilon_K *$

$\mathcal{H} * \epsilon_K$. Moreover, for every $\phi \in \mathcal{H}$, there exists a K sufficiently small that $\phi * \epsilon_K = \epsilon_K * \phi = \phi$ and so $\mathcal{H} = \bigcup_K \mathcal{H}_K$. If (π, V) is an admissible representation, then clearly $\pi(\epsilon_K)$ is a projection of V onto the finite-dimensional space of K-fixed vectors. Consequently if $\phi \in \mathcal{H}_K$, then the image of $\pi(\phi)$ is contained in V^K, and in particular we have a representation of $\mathcal{H}_K$ on V^K. Note that in contrast with $\mathcal{H}$, the ring $\mathcal{H}_K$ has a unit, namely ϵ_K.

A smooth representation is called *irreducible* if it has no proper nontrivial invariant subspaces. A module over a ring is called *simple* if it has no proper nontrivial submodules.

Proposition 1 *Let (π, V) be an admissible representation of G. Then π is irreducible if and only if V^K is either zero or a simple $\mathcal{H}_K$ module for every open and compact subgroup K of G.*

Proof See Proposition 4.2.3 of Bump [2]. □

Let $\hat{v} : V \to \mathbb{C}$ be a linear functional. We say that $\hat{v}$ is *smooth* if there exists an open subgroup K of G such that

$$\hat{v}\left(\pi(g)\, v\right) = \hat{v}(v)$$

for all $g \in K$, and all $v \in V$. Let $\hat{V}$ be the space of all smooth linear functionals. We will use the notation

$$\langle v, \hat{v} \rangle = \hat{v}(v)$$

for the pairing of V with $\hat{V}$. We have an action $\hat{\pi}$ of G on $\hat{V}$ defined by

$$\langle v, \hat{\pi}(g)\, \hat{v} \rangle = \left\langle \pi(g^{-1})\, v, \hat{v} \right\rangle.$$

Since $\hat{v}$ is smooth, its stabilizer is open, and so the representation $(\hat{V}, \hat{\pi})$ is smooth. It is called the *contragredient representation.*

Proposition 2 *If (π, V) is admissible the contragredient representation to V is also admissible, and $\hat{\hat{V}} \cong V$.*

Proof Fix a compact open subgroup K of G. Let $\hat{K}$ be the set of all isomorphism classes of finite-dimensional irreducible continuous representations of K. Let $\rho \in \hat{K}$. By abuse of notation, let us denote by (ρ, V_ρ) one particular representative of the isomorphism class ρ of representations. The kernel of

ρ is open as a consequence of the "no small subgroups" argument (Bump [2], Exercise 3.1.1). Thus ρ is essentially a finite-dimensional irreducible representation of the finite group $K/\ker(\rho)$. Now returning to the admissible representation (π, V), define the *ρ-isotypic part* $V(\rho)$ to be the direct sum of all subspaces of V which are isomorphic as K-modules to V_ρ. Since π is smooth, every vector in V is fixed by some subgroup of finite index in K, and hence

$$V = \bigoplus_{\rho \in \hat{K}} V(\rho).$$

The assumption of irreducibility means each $V(\rho)$ is *finite-dimensional*. Now a linear functional is smooth if and only if it is nonzero on only finitely many $V(\rho)$, and consequently we may identify $\hat{V}$ with

$$\bigoplus_{\rho \in \hat{K}} V(\rho)^*,$$

where $V(\rho)^*$ is the usual dual space. Now since $V(\rho)$ is finite-dimensional, so is $V(\rho)^*$ and so $\hat{V}$ is admissible; and $V(\rho)^{**} \cong V(\rho)$, so $\hat{\hat{V}} \cong V$. See Proposition 4.2.2 of Bump [2] for further details about these arguments. $\square$

If $G = \mathrm{GL}(n, F)$, where F is a nonarchimedean local field, and if (π, V) is an irreducible admissible representation, then the contragredient representation has a specific realization which is sometimes useful. Indeed, $g \mapsto \pi({}^t g^{-1})$ defines a representation of G on the same space V, and it is a theorem that this representation is equivalent to the contragredient. This is Theorem 4.2.3 of [2], which is stated there for $\mathrm{GL}(n)$, but only proved for $\mathrm{GL}(2)$. A proof of the general case may be found in Bernstein and Zelevinsky [1].

If R is an algebra over a field k, and M is an R-module which is finite-dimensional as a k-vector space, then a *matrix coefficient* of M is a function on R of the form $r \mapsto T(r.m)$ where $m \in M$, and T is a fixed k-linear functional on M.

Proposition 3 *Let (π_1, V_1) and (π_2, V_2) be two irreducible admissible representations of G. Let K be an open compact subgroup such that V_1^K and V_2^K are nonzero, and isomorphic as $\mathcal{H}_K$-modules. Then π_1 and π_2 are isomorphic.*

This is a strengthening of Proposition 4.2.7 of Bump [2]. The proof consists of a minor modification of the proof of Proposition 4.6.3 of [2].

Proof According to Proposition 4.2.7 of [2], π_1 and π_2 will be isomorphic if we can confirm that $V_1^L \cong V_2^L$ for every open and compact subgroup L, and it is clearly sufficient to consider L which are contained in K. Note that V_1^L and V_2^L are simple $\mathcal{H}_L$-modules, and to show that they are isomorphic, it is sufficient by Proposition 4.6.3 of [2] to show that they have a matrix coefficient in common. By Proposition 4.2.5 of Bump, the fact that $V_1^K \neq 0$ implies that $\hat{V}_1^K \neq 0$, and moreover the pairing of $V \times \hat{V} \to \mathbb{C}$ induces a $\mathcal{H}_K$-equivariant dual pairing $V_1^K \times \hat{V}_1^K \to \mathbb{C}$. As a consequence $\hat{V}_1^K \cong \hat{V}_2^K$ as $\mathcal{H}_K$-modules. Let us fix nonzero K-fixed vectors $v_1 \in V_1^K$ and $\hat{v}_1 \in \hat{V}_1^K$ such that $\langle v_1, \hat{v}_1 \rangle \neq 0$. If v_2 and $\hat{v}_2$ are the images of v_1 and $\hat{v}_1$ under the $\mathcal{H}_K$-isomorphisms $V_1^K \cong V_2^K$ and $\hat{V}_1 \cong \hat{V}_2$, then for $\phi \in \mathcal{H}_K$ we have

$$\langle \pi(\phi)\, v_1, \hat{v}_1 \rangle = \langle \pi(\phi)\, v_2, \hat{v}_2 \rangle \,. \tag{2}$$

We will prove that this identity remains true for all $\phi \in \mathcal{H}$; in particular it is true when $\phi \in \mathcal{H}_L$, showing that indeed V_1^L and V_2^L have a matrix coefficient in common. We make use of the projection $P : \phi \longmapsto \epsilon_K * \phi * \epsilon_K$ from $\mathcal{H} \to \mathcal{H}_K$. Since $P \circ P = P$, $\mathcal{H}$ splits into the direct sum of the kernel and image of P, and it is sufficient to verify (2) for these separately. The image of P is $\mathcal{H}_K$, and if $\phi \in \mathcal{H}_K$, then (2) is already known. If ϕ is in the kernel, then we will see that both sides of (2) vanish. We have $\pi(\epsilon_K)\, v_1 = v_1$ and $\hat{\pi}(\epsilon_K)\, \hat{v}_1 = \hat{v}_1$. Thus if $P(\phi) = 0$ we have

$$\langle \pi(\phi)\, v_1, \hat{v}_1 \rangle = \langle \pi(\phi)\, \pi(\epsilon_K)\, v_1, \hat{\pi}(\epsilon_K)\, \hat{v}_1 \rangle = \langle \pi(\epsilon_K)\, \pi(\phi)\, \pi(\epsilon_K)\, v_1, v_1 \rangle = \\ \langle \pi\left(P(\phi)\right)\, v_1, \hat{v}_1 \rangle = 0,$$

and similarly for the right-hand side. □

This means that to classify the irreducible admissible representations of G having a vector fixed by a particular compact subgroup K, it is sufficient to classify the simple finite-dimensional representations of the ring $\mathcal{H}_K$.

2 The Spherical Hecke algebra for $\mathrm{GL}(n)$

Now let F be a nonarchimedean local field, let $\mathfrak{o}$, $\mathfrak{p}$, ϖ and q be as before. Let $G = \mathrm{GL}(n, F)$ and let $K^\circ = \mathrm{GL}(n, \mathfrak{o})$. We will normalize the left Haar integral $\int_G \; dg$ so that K° has volume one. Let $\mathcal{H}^\circ = \mathcal{H}_{K^\circ}$. In this section, we will study this *spherical Hecke algebra* and its representations.

We now describe a method of distinguishing the double cosets $K^\circ\backslash G/K^\circ$. If $g \in G$, let $\gcd(g)$ be the fractional ideal of $\mathfrak{o}$ generated by the entries in g. Evidently $\gcd(g)$ is invariant under both left and right multiplication by K°. We may refine this invariant of the double coset $K^\circ g K^\circ$ as follows: if $1 \leqslant k \leqslant n$, let

$$\wedge^k : GL(n,F) \longrightarrow GL\left(\binom{n}{k}, F\right)$$

be the k-th exterior power representation; the matrix entries $\wedge^k g$ are the $k \times k$ minors of g. Then $\gcd\left(\wedge^k g\right)$ is the fractional ideal generated by these minors. Clearly $\wedge^k(K^\circ) \subset GL\left(\binom{n}{k}, \mathfrak{o}\right)$, so $\gcd\left(\wedge^k g\right)$ is also invariant under left and right multiplication by K°.

Proposition 4 (The Elementary Divisor Theorem) *Let R be a principal ideal domain, let M be a free R-module of rank n, and let N be a submodule of M which is also free of rank n. Then there exists an R-basis $\xi_1, \cdots, \xi_n$ of M and nonzero elements $D_1, \cdots, D_n$ of R such that each D_{i+1} divides D_i $(i = 1, \cdots, n-1)$ and $D_1\xi_1, \cdots, D_n\xi_n$ is a R-basis of N.*

Proof See Theorem III.7.8 of Lang [6]. $\square$

Proposition 5 (The p-adic Cartan Decomposition) *Every double coset in $K^\circ\backslash G/K^\circ$ has a unique representative of the form*

$$\begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix}, \qquad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n. \tag{3}$$

Proof We show first that if $g \in G$, then there exist elements $\kappa_1, \kappa_2 \in K^\circ$ such that $\kappa_1 g \kappa_2$ is diagonal. If we know this for elements $g \in \mathrm{Mat}_n(\mathfrak{o})$, then we can deduce it for all g, since we can multiply g by a scalar to put it in $\mathrm{Mat}_n(\mathfrak{o})$. Hence there is no loss of generality in assuming $g \in \mathrm{Mat}_n(\mathfrak{o})$. We apply the Elementary Divisor Theorem with $R = \mathfrak{o}$, $M = \mathfrak{o}^n$, and N the submodule generated by the columns of g. Let $\xi_1, \cdots, \xi_n$ and D_i be such that $\xi_1, \cdots, \xi_n$ generate $\mathfrak{o}^n$ and $D_1\xi_1, \cdots, D_n\xi_n$ generate the same $\mathfrak{o}$-module as the columns of g, there exists $\kappa_2 \in K^\circ$ such that

$$(D_1\xi_1, \cdots, D_n\xi_n) = g\kappa_2.$$

We may rewrite this

$$(\xi_1, \cdots, \xi_n) \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{pmatrix} = g\kappa_2,$$

and the first matrix on the left is an element of K°, so we have shown that

$$\begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{pmatrix} \in K^\circ g K^\circ.$$

We may clearly adjust the D_i's by units, proving that every coset has a representative of the form (3).

It remains to be shown the matrices (3) lie in distinct double cosets. Indeed, the invariants $\gcd\left(\wedge^k g\right)$ determine $\lambda_1, \cdots, \lambda_n$, since clearly if g equals the matrix (3) we have $\gcd(g) = \lambda_n$, $\gcd\left(\wedge^2 g\right) = \lambda_{n-1}\lambda_n$, and so forth. $\square$

Theorem 1 *(i) The spherical Hecke algebra $\mathcal{H}^\circ$ is commutative.*
(ii) If (π, V) is an irreducible admissible representation of $\mathrm{GL}(n, F)$*, then the space V^{K° of K°-fixed vectors in V is at most one-dimensional.*

The following "involution" method of proof is due to Gelfand. The second assertion is sometimes expressed by the statement that K° is a *Gelfand subgroup* of $\mathrm{GL}(n, F)$.

Proof Define a map $\iota : \mathcal{H}^\circ \to \mathcal{H}^\circ$ by $(\iota f)(g) = f({}^t g)$. Since K° is stable under transposition, this is a well-defined transformation of $\mathcal{H}^\circ$, and because transposition is an anti-automorphism of G, it is easy to see that ι is an anti-automorphism of $\mathcal{H}^\circ$:

$$\iota(f_1 * f_2) = \iota(f_2) * \iota(f_1).$$

On the other hand, ι is just the identity map, since every double coset has a representative which is diagonal, hence stable under transposition, by Proposition 5. We see that the identity map is an anti-automorphism of $\mathcal{H}^\circ$. This means that $\mathcal{H}^\circ$ is commutative.

For the second statement, V^{K°, if nonzero, is a finite-dimensional simple module over the commutative $\mathbb{C}$-algebra $\mathcal{H}^\circ$. It is therefore one-dimensional. $\square$

We call an irreducible admissible representation (π, V) *spherical* if it has a K°-fixed vector. The commutative Hecke algebra $\mathcal{H}^\circ$ is called the *spherical Hecke algebra.* We recall the partial order on partitions, in which $\mu \preccurlyeq \nu$ means that

$$\mu_1 + \ldots + \mu_r \leqslant \nu_1 + \ldots + \nu_r$$

for each r. Now let us study the product of two double cosets.

Proposition 6 *Suppose that λ and μ are partitions of k and l, respectively, of length $\leqslant n$. for each $1 \leqslant r \leqslant n$. We will denote by $\lambda + \mu$ the partition $\{\lambda_1 + \mu_1, \cdots, \lambda_n + \mu_n\}$ of $k + l$. Let*

$$g = \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix}, \qquad h = \begin{pmatrix} \varpi^{\mu_1} & & \\ & \ddots & \\ & & \varpi^{\mu_n} \end{pmatrix},$$

where $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \geqslant \cdots \geqslant \mu_n$. Suppose that $(K^\circ g K^\circ)(K^\circ h K^\circ)$ contains a double coset

$$K^\circ \begin{pmatrix} \varpi^{\nu_1} & & \\ & \ddots & \\ & & \varpi^{\nu_n} \end{pmatrix} K^\circ, \qquad \nu_1, \cdots, \nu_n \tag{4}$$

Then ν is a partition of $k + l$, and $\nu \preccurlyeq \lambda + \mu$.

Proof Since g and $h \in \mathrm{Mat}_n(\mathfrak{o})$, we have $K^\circ g K^\circ h K^\circ \subset \mathrm{Mat}_n(\mathfrak{o})$ and so the ν_i are nonnegative integers. Comparing determinants, ν is a partition of $k + l$. To prove that $\nu \preccurlyeq \lambda + \mu$, it is sufficient to check the inequalities

$$\nu_n \geqslant \lambda_n + \mu_n, \tag{5}$$

$$\nu_{n-1} + \nu_n \geqslant \lambda_{n-1} + \mu_{n-1} + \lambda_n + \mu_n, \tag{6}$$

etc. since subtracting these inequalities from the equality

$$\nu_1 + \ldots + \nu_n = k + l = \lambda_1 + \mu_1 \cdots + \lambda_n + \mu_n$$

will give $\nu \leqslant \lambda + \mu$. The matrix entries of an element of $K^\circ g K^\circ$ have greatest common divisor equal to $\mathfrak{p}^{\lambda_n}$, and the matrix entries of an element of $K^\circ h K^\circ$ have greatest common divisor $\mathfrak{p}^{\mu_n}$; it is evident that the matrix entries of an element of $(K^\circ g K^\circ)(K^\circ h K^\circ)$ lie in the ideal $\mathfrak{p}^{\lambda_n+\mu_n}$, and therefore we have (5). Repeating this argument with $\wedge^2 g$ and $\wedge^2 h$ replacing g and h gives (6), and so forth. $\square$

For $1 \leqslant r \leqslant n$, let T_r be $q^{-r(n-r)/2}$ times the characteristic function of the double coset

$$K^\circ \tau_r K^\circ, \qquad \tau_r = \begin{pmatrix} \varpi I_r & \\ & I_{n-r} \end{pmatrix}. \tag{7}$$

Of these "Hecke operators" the last one, T_n is invertible, having as its inverse the characteristic function of the double coset $K^\circ \tau_r^{-1} K^\circ$. Also, if $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ is a sequence of integers satisfying $\lambda_1 \geqslant \ldots \geqslant \lambda_n$, let T_λ equal

$$q^{\frac{1}{2}((1-n)\lambda_1+(3-n)\lambda_2+\ldots+(n-1)\lambda_n)}$$

times the characteristic function of the double coset (3). By the p-adic Cartan decomposition, these form a $\mathbb{Z}$-basis of $\mathcal{H}^\circ$.

Proposition 7 (Tamagawa [11], Satake [10]) *The ring $\mathcal{H}^\circ$ is a polynomial ring over in $T_1, \cdots, T_n$ and T_n^{-1}:*

$$\mathcal{H}^\circ \cong \mathbb{C}[T_1, \cdots, T_{n-1}, T_n, T_n^{-1}].$$

This structure theorem is a special case of the *Satake isomorphism* describing the structure of the spherical Hecke algebra of a reductive p-adic group.

Proof Let us show that T_λ lies in the $\mathbb{C}$-algebra generated by $T_1, \cdots, T_n, T_n^{-1}$, where $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ is a sequence of integers satisfying $\lambda_1 \geqslant \ldots \geqslant \lambda_n$. Clearly $T_n^{-r} * T_\lambda = T_{\lambda'}$, where

$$\lambda' = \{\lambda_1 - r, \cdots, \lambda_n - r\},$$

and so we may assume that $\lambda_n = 0$. Then the $\lambda_i \geqslant 0$, and λ is a partition, so (3) lies in $\mathrm{Mat}_n(\mathfrak{o})$. With this assumption, we will prove that T_λ is a polynomial in $T_1, \cdots, T_n$. (T_n^{-1} is not needed if the $\lambda_i \geqslant 0$.)

If the λ_i are all equal zero, then $T_\lambda = 1$ and there is nothing to prove, so assume that $\lambda_1 > 0$. Let $1 \leqslant k \leqslant n-1$ be the largest integer such that $\lambda_k \neq 0$, and let

$$\mu = \{\lambda_1 - 1, \cdots, \lambda_k - 1, 0, \cdots, 0\}.$$

By induction, T_μ lies in the $\mathbb{C}$-algebra generated by $T_1, \cdots, T_n$. We ask which double cosets occur in the support of $T_k * T_\mu$. Evidently the double coset of (3) occurs, and every other double coset is of the form (4) with ν a partition of $|\lambda|$, which strictly preceeds λ in the partial order. By induction, the characteristic of each such double coset is a polynomial in $T_1, \cdots, T_n$. We see that $T_k * T_\mu$ lies in the $\mathbb{C}$-algebra generated by $T_1, \cdots, T_n$, and it differs from a nonzero multiple of T_λ by a sum of elements T_ν which lie in this ring; hence T_λ is a polynomial in $T_1, \cdots, T_n$.

We must also show that the T_i are algebraically independent. We note that T_i is the characteristic function of a set supported on the matrices of determinant equal to ϖ^i times a unit, so we may grade the ring $\mathcal{H}^\circ$ by degree, T_i having degree i. Given relation of algebraic dependence, we may clearly separate out the part which is homogeneous of given degree and obtain a homogeneous relation

$$\sum_{|\lambda|=k} a(\lambda)\, T_1^{\lambda_1-\lambda_2} \cdots T_{n-1}^{\lambda_{n-1}-\lambda_n} T_n^{\lambda_n} = 0. \tag{8}$$

The point is that the homogeneous degree of the monomial

$$T_1^{\lambda_1-\lambda_2} \cdots T_{n-1}^{\lambda_{n-1}-\lambda_n} T_n^{\lambda_n} \tag{9}$$

is

$$(\lambda_1 - \lambda_2) + 2(\lambda_2 - \lambda_3) + \cdots + n\lambda_n = \lambda_1 + \ldots + \lambda_n = |\lambda|.$$

Now let us expand this out in terms of the T_λ, which are a $\mathbb{Z}$-basis of $\mathcal{H}^\circ$. It is a consequence of Proposition 6 that when (9) is expanded out, T_λ will occur, together with terms of the form T_ν, where ν runs through partitions of k strictly preceeding λ in the partial order. Thus if λ is minimal in the partial ordering subject to the condition that $a(\lambda) \neq 0$, it is clear that the coefficient of T_λ in the expansion of (8) is nonzero. Thus (8) does not vanish. This contradiction shows that the T_i are algebraically independent, and we have proved that $\mathcal{H}^\circ$ is a polynomial ring $\mathbb{C}[T_1, \cdots, T_{n-1}, T_n, T_n^{-1}]$. □

Proposition 8 (Iwasawa decomposition) *Let $B(F)$ be the Borel subgroup of upper triangular matrices in $G = \mathrm{GL}(n, F)$, and let $K^\circ = \mathrm{GL}(n, \mathfrak{o})$. Then $G = B(F)\, K^\circ$.*

Proof See Bump [2], Proposition 4.5.2. □

Now let us construct representations of $G = \mathrm{GL}(n, F)$ which have a K°-fixed vector. These are the *spherical principal series* representations.

We recall that a *quasicharacter* of a locally compact group is a continuous homomorphism into $\mathbb{C}^\times$; a quasicharacter of $F^\times$ is called *nonramified* if it is trivial on $\mathfrak{o}^\times$.

The *modular quasicharacter* a topological group H is the quasicharacter $\delta_H : H \to \mathbb{C}$ such that if $d_L h$ and $d_R h$ denote left and right Haar measures, respectively, on H, then $d_R h = \delta_H(h)\, d_L h$. The the modular quasicharacter of $B(F)$ is

$$\delta \begin{pmatrix} y_1 & * & \cdots & * \\ & y_2 & & * \\ & & \ddots & \vdots \\ & & & y_n \end{pmatrix} = |y_1|^{n-1}|y_2|^{n-3}\cdots|y_n|^{1-n}.$$

Fix quasicharacters $\chi_1, \cdots, \chi_n$ of $F^\times$. Then we have a quasicharacter $\chi : B(F) \longrightarrow \mathbb{C}^\times$ given by

$$\chi \begin{pmatrix} y_1 & * & \cdots & * \\ & y_2 & & * \\ & & \ddots & \vdots \\ & & & y_n \end{pmatrix} = \chi_1(y_1)\cdots\chi_n(y_n).$$

Let $V = V(\chi_1, \cdots, \chi_n)$ be the space of locally constant functions $f : G \to \mathbb{C}$ such that

$$f(bg) = \chi(b)\, \delta^{1/2}(b)\, f(g). \tag{10}$$

We define an action $\pi = \pi(\chi_1, \cdots, \chi_n)$ of G on V by right translation:

$$(\pi(h)f)\,(g) = f(gh).$$

It is easily verified that $\pi(h)f \in V$ with this definition, and since the functions f are locally constant, the stabilizer of any particular f is open, so this is a smooth representation. We may see that it is admissible as follows: If K is any open subgroup, we want to show that V^K is finite. This is a subspace of $V^{K\cap K^\circ}$, so without loss of generality $K \subset K^\circ$. The index of K in K° is finite, and if $x_1, \cdots, x_N$ are a complete set of coset representatives, so $K^\circ = \bigcup x_i K$, then we claim that $f \in V^K$ is completely determined by the values $f(x_i)$. Indeed, since f is right K-invariant, knowledge of these values determines f on K° and by (10) and the Iwasawa decomposition,f is therefore completely known. We see that V^K is finite-dimensional, so this representation is admissible.

Proposition 9 *Assume that the quasicharacters χ_i are nonramified. The space of K°-fixed vectors in the representation $\pi(\chi_1, \cdots, \chi_n)$ is one-dimensional.*

Proof We will show that the space of K°-fixed vectors is spanned by the function f° defined by

$$f^\circ(bk) = \chi(b)\, \delta^{1/2}(b) \qquad , \qquad b \in B(F),\ k \in K^\circ. \tag{11}$$

It is called the *standard spherical vector*. It is a consequence of the Iwasawa decomposition that every element of G can be written as bk as in (11), so the definition (11) makes sense provided the right-hand side is well-defined, independent of the decomposition of g as bk. This is true on our assumption that $\chi_1, \cdots, \chi_n$ are nonramified, since if $bk = b'k'$ where $b,\ b' \in B(F)$ and $k,\ k' \in K^\circ$, then $b^{-1}b' \in B(F) \cap K^\circ$ is upper triangular with units on the diagonal, and so $\chi(b^{-1}b') = \delta(b^{-1}b') = 1$. Thus f° is well defined. It is clear from the Iwasawa decomposition that a K°-fixed vector is a constant multiple of this f°, so V^{K° is exactly one-dimensional. □

Proposition 10 *Assume that the quasicharacters χ_i are nonramified, and let f° be the K°-fixed vector (11). Then*

$$T_r f^\circ = e_r(t_1, \cdots, t_n)\, f^\circ, \tag{12}$$

where e_r is the r-th elementary symmetric polynomial, and $t_i = \chi_i(\varpi)$.

Proof It is clear that $T_r * f \in V^{K^\circ}$ for any f since T_r is K°-bi-invariant, and so $T_r * f^\circ = cf^\circ$ for some constant c. Evidently $c = (T_r * f^\circ)(1)$, so we must show that

$$(T_r * f^\circ)(1) = e_r(t_1, \cdots, t_n).$$

Since $K^\circ \tau_r K^\circ$ is the continuous image of $K^\circ \times K^\circ$ under the map $(k_1, k_2) \mapsto k_1 \tau_r k_2$, it is compact; and since K° is open, there are a finite number of right cosets in $K^\circ \tau_r K^\circ / K^\circ$. Let Λ be a complete set of coset representatives for these, so that

$$K^\circ \tau_r K^\circ = \bigcup_{\beta \in \Lambda} \beta K^\circ, \tag{13}$$

Since $K^\circ \tau_r K^\circ \subset \mathrm{Mat}_n(\mathfrak{o})$, the matrix entries in β are all integers. It follows from the Iwasawa decomposition that we may chose the representatives β to be upper triangular, and we have the freedom to change them by an element of $K^\circ \cap B(F)$ on the right. We may then arrange that the diagonal entries of β are all powers of π. In order to lie in the same double coset as τ_r, it is necessary that $\gcd(\wedge^k \tau_r)$ and $\gcd(\wedge^k \beta)$ agree. The implications of this are as follows: of the diagonal entries of β, there are exactly r ϖ's and exactly $n-r$ 1's. Moreover, if S is the set (of cardinality r) of $1 \leqslant i \leqslant n$ such that $\beta_{ii} = \varpi$, and if i, j are distinct elements of S, then β_{ij} must lie in $\mathfrak{p}$. If these conditions are not satisfied, then β will not lie in the same double coset as τ_r. Given that we have the freedom to change β on the right by an upper triangular unipotent element of K, we may change each β_{ij} $(i < j)$ by any multiple of β_{ii}. If $i, j \in S$ then $\mathfrak{p} | \beta_{ij}$ while $\beta_{ii} = \varpi$, and so we may assume that $\beta_{ij} = 0$. On the other hand if $i \notin S$, then $\beta_{ii} \in \mathfrak{o}^\times$ while $\beta_{ij} \in \mathfrak{o}$, and so again we may assume that $\beta_{ij} = 0$. On the other hand, if $i \in S$ and $j \notin S$, then β_{ij} may be any element of $\mathfrak{o}$ modulo $\mathfrak{p}$.

With this in mind, let us fix $S = \{\lambda_1, \cdots, \lambda_r\}$, where $\lambda_1 < \lambda_2 < \cdots < \lambda_r$. We ask how many β there are in the decomposition (13) whose diagonal entries equal to ϖ are β_{ii} with $i \in S$. We have just shown that if $i < j$ with $i \in S$ and $j \notin S$, then β_{ij} can be chosen to be an arbitrary element of $\mathfrak{o}/\mathfrak{p}$; and all other entries above the diagonal can be assumed to be zero. If $i = \lambda_1$, there are $n - r - \lambda_1 + 1$ values of j such that $j > i$ and $j \notin S$, so there are $n - r - \lambda_1 + 1$ elements to be chosen inthe λ_1 row, and similarly, there are $n - r - \lambda_2 + 2$ entries to be chosen in the λ_2 row, and so forth; the total number of elements to be chosen is

$$\sum_{i=1}^{r} (n - r - \lambda_i + i) = r\left(n - r - \frac{1}{2}\right) - \sum_{i=1}^{r} \lambda_i,$$

and so the total number of β for this choice of S is

$$q^{r\left(n-r-\frac{1}{2}\right) - \sum_{i=1}^{r} \lambda_i}.$$

For such a β, we have

$$f^\circ(\beta) = \delta^{1/2}(\beta)\, \chi(\beta) = q^{-(n+1)\frac{r}{2} + \sum \lambda_i}\, t_{\lambda_1} \cdots t_{\lambda_r}.$$

Recalling that T_r is $q^{-r(n-r)/2}$ times the characteristic function of the double coset $K^\circ \tau_r K^\circ$, we see that $(T_r f^\circ)(1)$ equals

$$\begin{aligned} & q^{-\frac{r(n-r)}{2}} \sum_{\lambda_1 < \cdots < \lambda_r} q^{r\left(n - \frac{r-1}{2}\right) - \sum_{i=1}^{r} \lambda_i}\; q^{-\frac{(n+1)r}{2} + \sum \lambda_i}\, t_{\lambda_1} \cdots t_{\lambda_r} \\ & = e_r(t_1, \cdots, t_r), \end{aligned}$$

as required. $\square$

3 Orbital Integrals

In this section we will drop the $\circ$ and denote by K and $\mathcal{H}$ what were denoted K° and $\mathcal{H}^\circ$ in the previous section. Let A be the diagonal subgroup of $G = \mathrm{GL}(n)$. Let B be the standard Borel subgroup and N its unipotent radical. To simplify things we will normalize Haar measures so that K and $A(\mathfrak{o})$ have volume 1, although there are good reasons *not* to insist such a normalization.

Let $\mathfrak{n}$ be the Lie algebra of $N(F)$, which is the F-vector space of upper triangular nilpotent matrices. If $\gamma \in A(F)$, let $\mathrm{Ad}_{\mathfrak{n}}(\gamma)$ be the linear transformation of $\mathfrak{n}$ induced by conjugation by γ. We will denote by $\Delta : A(F) \longrightarrow \mathbb{C}$ the function

$$\Delta(\gamma) = |\det(1 - \mathrm{Ad}_{\mathfrak{n}}(\gamma))|\,.$$

It is nonzero if γ is regular (i.e. has distinct eigenvalues).

Also, let $D(\gamma) = \delta(\gamma)^{-1/2}\Delta(\gamma)$, where $\delta : B(F) \longrightarrow \mathbb{C}$ is the modular quasicharacter. For example, if $n = 2$, then

$$\mathrm{Ad}_{\mathfrak{n}}\begin{pmatrix} a & \\ & b \end{pmatrix} : \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & ab^{-1}x \\ 0 & 0 \end{pmatrix},$$

so

$$\Delta(\gamma) = |1 - ab^{-1}|, \qquad D(\gamma) = \frac{|a-b|}{\sqrt{|ab|}}.$$

More generally if

$$\gamma = \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{pmatrix}, \tag{14}$$

then

$$\Delta(\gamma) = \prod_{i<j} |1 - \gamma_i\gamma_j^{-1}|, \qquad D(\gamma) = \prod_{i<j} \frac{|\gamma_i - \gamma_j|}{|\det(\gamma)|^{(n-1)/2}}\,.$$

We note that

Proposition 11 *Suppose that γ is regular. Then the Jacobian of the map*

$$n \longrightarrow n\gamma n^{-1}\gamma^{-1} \tag{15}$$

of $N(F)$ to itself is $\Delta(\gamma)$. More precisely, if f is any integrable function on $N(F)$, then

$$\int_{N(F)} f(n)\, dn = \Delta(\gamma) \int_{N(F)} f(n^{-1}\gamma n\gamma^{-1})\ dn.$$

Proof We make use of the following fact. If f is any function on F^N, and if $x_k = a_k y_k + \mu_k(y_1, \cdots, y_{k-1})$ for some functions μ_k for each $1 \leqslant k \leqslant N$ then

$$\prod |a_k| \int_{F^N} f(x_1, \cdots, x_N)\, dy_1 \cdots dy_N = \int_{F^N} f(y_1, \cdots, h_N)\, dy_1 \cdots dy_N.$$

The reason is that writing the integral on the left-hand side as an iterated integral, making the y_N integration the inner one, we may dispense with the translation by $\mu_N(y_1, \cdots, y_{N-1})$ by a change of variables shifting y_N. Then one may eliminate μ_{N-1} in the same way, and eventually one obtains

$$\prod |a_k| \int_{F^N} f(a_1 y_1, \cdots, a_N y_N) dy_1 \cdots dy_N,$$

and the statement follows.

To apply this, let γ be as in (14), and let $n = (n_{ij})$. In the map (15), each coordinate n_{ij} $(i < j)$ is mapped to $(\gamma_i \gamma_j^{-1} - 1)$ times itself, plus terms depending on $n_{i'j'}$ with $j' - i'$ strictly less than $j - i$. Thus

$$\prod_{i<j} |1 - \gamma_i \gamma_j^{-1}| \int_{N(F)} f(n^{-1}\gamma n \gamma^{-1})\, dn = \int_{N(F)} f(n)\, dn,$$

as required. □

Now let $\varphi \in \mathcal{H}$, and define the *orbital integral*

$$F_\varphi(\gamma) = D(\gamma) \int_{A(F)\backslash G(F)} \varphi(g^{-1}\gamma g)\, dg.$$

We are normalizing the Haar measures so that both $K = G(\mathfrak{o})$ and $A(\mathfrak{o})$ have volume 1.

Let db be left Haar measure on B. Then, if the Haar measures are chosen compatibly, for any integrable function φ on $G(F)$, we have

$$\int_{G(F)} \varphi(g)\, dg = \int_{B(F)} \int_K \varphi(bk)\, dk\, db. \tag{16}$$

Proposition 12 *We have*

$$F_\varphi(\gamma) = \delta(\gamma)^{-1/2} \int_{N(F)} \varphi(n\gamma)\, dn. \tag{17}$$

Proof Write

$$\int_{A(F)\backslash G(F)} \varphi(g^{-1}\gamma g)\, dg.$$

Using (16), we may write this

$$\int_{A(F)\backslash B(F)} \int_K \varphi(k^{-1}b^{-1}\gamma bk)\, dk\, db = \int_{N(F)} \varphi(n^{-1}\gamma n)\, dn,$$

where we have dropped the k integration since φ is K-bi-invariant, and identified the coset space $A(F)\backslash B(F)$ with $N(F)$. Therefore

$$F_\varphi(\gamma) = D(\gamma) \int_{N(F)} \varphi(n^{-1}\gamma n)\, dn = \Delta(\gamma)\delta(\gamma)^{-1/2} \int_{N(F)} \varphi(n^{-1}\gamma n\gamma^{-1} \cdot \gamma)\, dn.$$

The result now follows from Proposition 11. □

The formula (17) shows that $F_\varphi(\gamma)$, originally defined for φ regular can be extended to a continuous function of all γ. Moreover, it shows that $F_\varphi : A(F) \longrightarrow \mathbb{C}$ is constant on the cosets in $A(F)/A(\mathfrak{o}) = \Lambda$, and that it is zero off a compact subset of $A(F)$. Thus we may regard it as an element of the group algebra $\mathbb{C}[\Lambda]$, in which the coefficient of $\lambda \in \Lambda$, representated by $\gamma \in A(F)$, is $F_\varphi(\gamma)$. The multiplication in $\mathbb{C}[\Lambda]$ coincides with convolution of functions on $A(F)$, so we denote it as $*$.

Theorem 2 *The orbital integral* $\varphi \longmapsto F_\varphi$ *is a ring homomorphism from* $\mathcal{H}$ *to* $\mathbb{C}[\Lambda]$.

Proof Let φ_1 and φ_2 be elements of $\mathcal{H}$. Then for $\gamma \in A(F)$,

$$F_{\varphi_1 * \varphi_2}(\gamma) = \delta(\gamma)^{-1/2} \int_{N(F)} \int_{G(F)} \varphi_1(g)\varphi_2(g^{-1}n\gamma)\, dg =$$

$$\delta(\gamma)^{-1/2} \int_{N(F)} \int_K \int_{B(F)} \varphi_1(bk)\varphi_2(k^{-1}b^{-1}n\gamma)\, db\, dk\, dn =$$

$$\delta(\gamma)^{-1/2} \int_{N(F)} \int_{B(F)} \varphi_1(b)\varphi_2(b^{-1}n\gamma)\, db\ dn$$

We dropped the k integration since the integrand on the second line is independent of $k \in K$. This equals

$$\delta(\gamma)^{-1/2} \int_{A(F)} \int_{N(F)} \int_{N(F)} \delta(a)^{-1} \varphi_1(ua) \varphi_2(a^{-1}u^{-1}n\gamma) \, du \, dn \, da =$$
$$\delta(\gamma)^{-1/2} \int_{A(F)} \int_{N(F)} \int_{N(F)} \delta(a)^{-1} \varphi_1(ua) \varphi_2(a^{-1}n\gamma) \, du \, dn \, da,$$

where we made the variable change $n \longrightarrow un$. Noting that $A(F)$ normalizes $N(F)$, and making the variable change $n \longrightarrow ana^{-1}$, which produces a Jacobian factor of $\delta(a)$, this equals

$$\delta(\gamma)^{-1/2} \int_{A(F)} \int_{N(F)} \int_{N(F)} \varphi_1(ua) \varphi_2(na^{-1}\gamma) \, du \, dn \, da =$$
$$\int_{A(F)} F_{\varphi_1}(a) F_{\varphi_2}(a^{-1}\gamma) \, da = (F_{\varphi_1} * F_{\varphi_2})(\gamma).$$

$\square$

Proposition 13 *Let $1 \leqslant r \leqslant n$, and let T_r be $q^{-r(n-r)/2}$ times the characteristic function of the double coset of (7), as in the previous section, and let*

$$\gamma = \begin{pmatrix} \varpi^{k_1} & & \\ & \ddots & \\ & & \varpi^{k_n} \end{pmatrix}.$$

(We do not assume $k_1 \geqslant \cdots \geqslant k_n$). Then

$$F_{T_r}(\gamma) = \begin{cases} 1 & \text{if exactly } r \text{ of the } k_i = 1, \text{ the rest } 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let us ask when

$$n\gamma \in K \tau_r K \tag{18}$$

with $n = (n_{ij}) \in N(F)$ and τ_r as in (7). As explained in the previous section, a necessary and sufficient condition for two matrices $g, h \in \mathrm{GL}(n, F)$ to lie in the same K-double coset is that $\gcd(\wedge^k g) = \gcd(\wedge^k h)$ for all k. Thus we must have

$$\gcd \wedge^k (n\gamma) = \gcd(\wedge^k \tau_r).$$

First we show that all $k_i \leqslant 1$ and exactly r of them are nonzero. Indeed, the determinant of γ must equal the determinant of τ_r, so $\sum k_i = r$, and if any $k_i > 1$, then gcd $(n\gamma) \geqslant k_i$, which implies that each $k_i = 0$ or 1. Evidently there must be exactly r 1's among the k_i. Otherwise $F_{T_r}(\gamma) = 0$.

If we assume this, then a necessary and sufficient condition for (18) is that when $i < j$ we have

$$n_{ij} \in \begin{cases} \mathfrak{p}^{-1} & \text{if } k_i = 0, k_j = 1, \\ \mathfrak{o} & \text{otherwise}. \end{cases}$$

This means that the principal minor formed with the i and j rows and columns of $n\gamma$ is of the form

$$\begin{pmatrix} \varpi^{k_i} & x_{ij} \\ & \varpi^{k_j} \end{pmatrix}, \qquad x_{ij} = n_{ij}\varpi^{k_j},$$

where either one of k_i and k_j is 0 and $x \in \mathfrak{o}$, or both k_i and k_j are 1 and $x \in \mathfrak{p}$. From this it is easy to see that left and right multiplication by elements of $N(\mathfrak{o})$ is sufficient to replace all x_{ij} by zero, showing that $n\gamma$ and γ are in the same double coset, and of course γ is in the same double coset as τ_r. Conversely, we claim that if any $x \notin \mathfrak{p}$, or if $k_i = k_j = 1$ and $x \notin \mathfrak{p}$, then $n\gamma$ is not in the same double coset as τ_r. If $x \notin \mathfrak{p}$, then $n\gamma \notin \mathrm{Mat}_n(\mathfrak{o})$, so this is clear. If $k_i = k_j = 1$ and $x \notin \mathfrak{p}$ then $\wedge^{n-r+1}(n\gamma)$ will have one matrix entry which is a unit, namely the $(n-r+1)\times(n-r+1)$ minor formed with the i-th row and the l-th row for each l such that $\gamma_l = 1$, and with the j-th column and the l-th column for each l such that $\gamma_l = 1$. Thus $\gcd(\wedge^{n-r+1}(n\gamma)) = 1$ while $\gcd(\wedge^{n-r+1}\tau_r) = \mathfrak{p}$, showing that these elements are in distinct double cosets.

We see that the volume of the set of $n \in N(F)$ such that $T_r(n\gamma) \neq 0$ is q^N, where N is the number of pairs i, j with $i < j$ such $k_i = 0$ and $k_j = 1$. It is easy to check that $\delta(\gamma)^{-1/2}q^N = q^{r(n-r)/2}$. Since the value of $T_r(n\gamma) = T_r(\tau_r) = q^{-r(n-r)/2}$, we see that $F_{T_r}(\gamma) = 1$. $\square$

Theorem 3 *Let χ_i be nonramified quasicharacters of $F^\times$, and let $t_i = \chi_i(\varpi)$. Let f° be the standard spherical vector in the space of the representation $\pi = \pi(\chi_1, \cdots, \chi_n)$ of* $\mathrm{GL}(n, F)$. *Let $\varphi \in \mathcal{H}$. Then*

$$\pi(\varphi)f^\circ = \varphi^\vee(t_1, \cdots, t_n),$$

where $\varphi^\vee$ *is a symmetric polynomial. If*

$$\gamma = \begin{pmatrix} \varpi^{k_1} & & \\ & \ddots & \\ & & \varpi^{k_n} \end{pmatrix},$$

then the coefficient of $x_1^{k_1} \cdots x_n^{k_2}$ *in* $p_\varphi(x_1, \cdots, x_n)$ *is* $F_\varphi(\gamma)$*.*

Proof In view of Theorem 2 and the fact that the T_r generate $\mathcal{H}$, it is enough to check this when $\varphi = T_r$. In this case, $\varphi^\vee$ is the r-th elementary symmetric polynomial by (12). The statement follows now from Proposition 13. $\square$

4 Spherical functions

We continue with the notations of the previous section.

The representations $\pi(\chi_1, \cdots, \chi_n)$ are irreducible for "most" χ_i. More precisely, it may be shown that unless $\chi_i \chi_j^{-1}(y) = |y|$ or $|y|^{-1}$ for some $1 \leqslant i < j \leqslant n$ (and all y), the representation $\pi(\chi_1, \cdots, \chi_n)$ is irreducible, and in this case, the isomorphism class of $\pi(\chi_1, \cdots, \chi_n)$ is unchanged if the χ_i are permuted. In particular, it is irreducible if all χ_i are unitary.

In any case, the representation $\pi(\chi_1, \cdots, \chi_n)$ has a composition series of finite length, and the composition factors are unchanged if the χ_i are permuted. These facts are proved in Bump [2] when $n = 2$ (see especially Theorem 4.5.1). A composition factor of a representation having a composition series of finite length is sometimes called a *subquotient* since (as is easily seen) it is a quotient of a subrepresentation.

Lemma 1 *Let* $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ *be a short exact sequence of* $\mathrm{GL}(n, F)$*-modules. Then the sequence* $0 \longrightarrow U^K \longrightarrow V^K \longrightarrow W^K \longrightarrow 0$ *is exact.*

Proof The exactness of $0 \longrightarrow U^K \longrightarrow V^K \longrightarrow W^K$ is a purely algebraic fact which is easily checked. To prove the surjectivity of $V^K \longrightarrow W^K$, let $w \in W^K$ and let $v \in V$ have image w. Then $\int_K k \cdot v \, dk$ is in V^K, and its image is also W. $\square$

Since $\pi(\chi_1, \cdots, \chi_n)$ contains a unique spherical vector, it follows that it has a unique spherical composition factor.

Theorem 4 *Let (π, V) be a spherical representation of* $\mathrm{GL}(n, F)$*. Then π is a composition factor of $\pi(\chi_1, \cdots, \chi_n)$ for some unramified quasicharacters χ_i.*

Proof By Theorem 1, the space V^K of K-fixed vectors is exactly one-dimensional. Let $\xi^\circ \in V^K$ be a nonzero element. Then if $\varphi \in \mathcal{H}$, the spherical Hecke algebra, $\pi(\phi)\xi^\circ = \chi_\pi(\phi)\xi^\circ$ for some complex number $\chi_\pi(\phi)$ and $\phi \longmapsto \chi_\pi(\phi)$ is a character. Recall that $\mathcal{H}$ is a polynomial ring in $T_1, T_2, \cdots, T_n, T_n^{-1}$. If $\lambda_i = \chi_\pi(T_i)$, then $\lambda_n \neq 0$ since T_n is a unit in $\mathcal{H}$. We may therefore find nonzero complex numbers $t_1, \cdots, t_n$ such that $\lambda_i = e_i(t_1, \cdots, t_n)$. Let χ_i be the unramified characters such that $\chi_i(\varpi) = t_i$. Then by Proposition 10, the spherical constituent of $\pi(\chi_1, \cdots, \chi_n)$ produces the same character of $\mathcal{H}$ as χ_π. But the isomorphism class of π is determined by χ_π by Proposition 3. $\square$

Proposition 14 *The contragredient of $\pi(\chi_1, \cdots, \chi_n)$ is $\pi(\chi_1^{-1}, \cdots, \chi_n^{-1})$. The invariant bilinear pairing of these two spaces is*

$$(f, \hat{f}) \longmapsto \int_K f(k)\, \hat{f}(k)\, dk.$$

Proof Let $\mathrm{Ind}_{B(F)}^{G(F)}(\delta)$ be the space of functions $F : G(F) \longrightarrow \mathbb{C}$ such that $F(bg) = \delta(b)\, F(g)$ for $b \in B(F)$. If $\varphi \in C_c(G(F))$ then

$$(P\varphi)(g) = \int_{B(F)} \varphi(bg)\, db$$

defines an element of $B(F)$ because right Haar measure on $B(F)$ is $d_R b = \delta(b)\, db$, so

$$(P\varphi)(\beta g) = \int_B \varphi(b\beta g)\, db = \int_B \varphi(b\beta g)\, \delta(b)^{-1}\, d_R b = \delta(\beta)(P\varphi)(g),$$

where the last step uses the variable change $b \longrightarrow b\beta^{-1}$. The map $P : C_c(G(F)) \longrightarrow \mathrm{Ind}_{B(F)}^{G(F)}(\delta)$ is surjective, since using (16) it is easy to check that eny $f \in \mathrm{Ind}_{B(F)}^{G(F)}(\delta)$ satisfies if $g \in K$,

$$f = P\varphi, \qquad \varphi(g) = \begin{cases} f(g) & \text{if } g \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $I : \mathrm{Ind}_{B(F)}^{G(F)}(\delta) \longrightarrow \mathbb{C}$ be the linear functional defined by

$$I(f) = \int_K f(k)\, dk.$$

By (16), we have

$$I(P\varphi) = \int_{G(F)} \varphi(g)\, dg,$$

and it follows from this formula that the functional I is invariant under right translation.

Now if $f \in V(\chi_1, \cdots, \chi_n)$ and $\hat{f} \in V(\chi_1^{-1}, \cdots, \chi_n^{-1})$, then $f\hat{f} \in \mathrm{Ind}_{B(F)}^{G(F)}(\delta)$ and $(f, \hat{f}) \longmapsto I(f\hat{f})$ is an invariant bilinear form. This is the pairing (14). It is nondegenerate since if $0 \neq f \in V(\chi_1, \cdots, \chi_n)$ we can take $\hat{f}$ whose values on K are the conjugate of the values of f, in which cashe $I(f\hat{f}) \neq 0$. This dual pairing shows that we may identify $\pi(\chi_1^{-1}, \cdots, \chi_n^{-1})$ with the contragredient of $\pi(\chi_1, \cdots, \chi_n)$. $\square$

If (π, V) is a smooth representation and $(\hat{\pi}, \hat{V})$ is its contragredient, let $\langle\ ,\ \rangle$ denote the dual pairing between V and $\hat{V}$. If $v \in V$ and $\hat{v} \in \hat{V}$, the function

$$g \longmapsto \langle \pi(g)v, \hat{v} \rangle$$

is called a *matrix coefficient* of the representation. In particular, let (π, V) be a spherical representation, $(\hat{\pi}, \hat{V})$ its contragredient, and let v°, $\hat{v}^\circ$ be spherical vectors in V and $\hat{V}$. The function

$$\omega(g) = \langle \pi(g)v^\circ, \hat{v}^\circ \rangle$$

is the unique K-bi-invariant matrix coefficient. It is called the *spherical function*. We will see in the next Proposition that $\omega(1) \neq 0$, and we will choose the normalization such that $\omega(1) = 1$.

Proposition 15 *Suppose that the spherical representation π is a subquotient of $\pi(\chi_1, \cdots, \chi_n)$, and let $f^\circ \in V(\chi_1, \cdots, \chi_n)$ be the standard spherical vector. The spherical function may be normalized so that*

$$\omega(g) = \int_K f^\circ(kg)\, dk.$$

Proof Since $\pi(\chi_1, \cdots, \chi_n)$ has a unique spherical subquotient, as does its contragredient, it has a unique K-bi-invariant matrix coefficient, which is also a matrix coefficient of π, that is, the spherical function. By Proposition 14, the K-bi-invariant matrix coefficient of $\pi(\chi_1, \cdots, \chi_n)$ is

$$\omega(g) = \int_K (\pi(g) f^\circ)(k)\, \hat{f}^\circ(k)\, dk.$$

Since $\pi(g) f^\circ(k) = f^\circ(kg)$ and $\hat{f}^\circ(k) = 1$, the statement follows. $\square$

Proposition 16 *Let π be a spherical representation of $\mathrm{GL}(n, F)$, and let ω be the spherical function of π. Then for $\phi \in \mathcal{H}$, we have*

$$\chi_\pi(g) = \int_G \omega(g)\, \phi(g)\, dg.$$

Proof We use the formula (15) for ω to write

$$\int_G \omega(g)\, \phi(g)\, dg = \int_G \int_K f^\circ(kg)\ \phi(g)\, dk\, dg =$$
$$\int_K \int_G f^\circ(kg)\ \phi(g)\, dg\, dk\,.$$

Makin the variable change $g \longmapsto k^{-1} g$ and recalling that ϕ is K-bi-invariant, this equals

$$\int_G f^\circ(g)\ \phi(g)\, dg = \pi(\phi) f^\circ(1).$$

Since $\pi(\phi) f^\circ = \chi_\pi(\phi)\, f^\circ$, $\chi_\pi(\phi) = (\pi(\phi) f^\circ)(1)$, and the statement is proved. $\square$

If $\lambda \in \Lambda = A(F)/A(\mathfrak{o})$, let φ_λ be the characteristic function of the double coset $K\lambda K$.

Proposition 17 *Let $\pi = \pi(\chi_1, \cdots, \chi_n)$, and let $t = (t_1, \cdots, t_n)$ with $t_i = \chi_i(\varpi)$. Let ω be the spherical function of ϖ, and let $\lambda \in \Lambda$. Then*

$$\varphi_\lambda^\vee(t) = \mathrm{vol}(K\lambda K)\, \omega(\lambda).$$

Proof This is immediate from Proposition 16 and the definition of $\varphi^\vee$. $\square$

We next recall the *Macdonald formula* for the spherical function on $\mathrm{GL}(n,F)$. This was originally proved in Macdonald [8] for semisimple algebraic groups over a local field. We recommend Macdonald [9] or Casselman [4] for proofs. Proofs for GL(2) may be found in Bump [2] and Langlands [7] (Lemma 5.1).

Theorem 5 (Macdonald) *Let $\pi = \pi(\chi_1, \cdots, \chi_n)$, and let $t = (t_1, \cdots, t_n)$ with $t_i = \chi_i(\varpi)$. Let ω be the spherical function of ϖ. Suppose that $k_1 \geqslant \cdots \geqslant k_n$ and*

$$g = \begin{pmatrix} \varpi^{k_1} & & \\ & \ddots & \\ & & \varpi^{k_n} \end{pmatrix}.$$

Then

$$\omega(g) = \left[\prod_{i=1}^{n-1} (1 + q^{-1} + \ldots + q^{-i})^{-1}\right] \delta(g)^{1/2} \sum_{w \in W} w \left(\prod t_i^{k_i} \prod_{i<j} \frac{1 - q^{-1} t_i t_j^{-1}}{1 - t_i t_j^{-1}} \right),$$

where $\delta = \delta_B$ and $w = S_n$ acting on the t_i by permuting them.

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