Multiple Dirichlet Series

Daniel Bump

Dept. of Mathematics, Stanford University, Stanford CA 94305-2125 *email:* `bump@math.stanford.edu`

Summary. This introductory article aims to provide a roadmap to many of the interrelated papers in this volume and to a portion of the field of multiple Dirichlet series, particularly emerging new ideas. It is both a survey of the recent literature, and an introduction to the combinatorial aspects of Weyl group multiple Dirichlet series, a class of multiple Dirichlet series that are not Euler products, but which may nevertheless be reconstructed from their p-parts. These p-parts are combinatorially interesting, and may often be identified with p-adic Whittaker functions.

This survey article is intended to help orient the reader to certain topics in multiple Dirichlet series. There are several other expository articles that the reader might also want to consult, though we do not assume any familiarity with them. The article [20] which appeared in 1996 contained many of the ideas in an early, undeveloped form. The articles [9] and [26], which appeared in 2006, also survey the field from different points of view, and it is hoped that these papers will be complementary to this one. Further expository material may be found in some of the chapters of [14].

Since approximately 2003 there has been an intensive development of the subject into areas related to combinatorics, representation theory, statistical mechanics and other areas. These are scarcely touched on in [20], [9] and [26], and indeed are topics that have largely developed during the last few years. However these combinatorial developments are discussed in [14] as well as this introductory paper and other papers in this volume.

Preparation of this paper was supported in part by NSF grant DMS-1001079. We would like to thank Ben Brubaker, Solomon Friedberg and Kohji Matsumoto for helpful comments.

1 Moments of L-functions

The subject of multiple Dirichlet series originated in analytic number theory. If $\{a_n\}$ is a sequence of real or complex numbers, then a typical Tauberian theorem draws conclusions about the a_n from the behavior of the Dirichlet series $\sum_n a_n n^{-s}$. If the a_n are themselves L-functions or other Dirichlet series, this is then a multiple Dirichlet series.

One may try to study moments of L-functions this way. For example Goldfeld and Hoffstein [37] considered a pair of Dirichlet series whose coefficients are

$$Z_{\pm}(w,s) = \sum_{\pm d>0} A_d(s)|d|^{\frac{1}{2}-2w}, \tag{1}$$

where the coefficients $A_d(s)$ are essentially quadratic L-functions. More precisely, if d is squarefree

$$A_d(s) = \frac{L_2(2s-\frac{1}{2},\chi_d)}{\zeta_2(4s-1)},$$

and

$$\frac{A_{dk^2}(s)}{A_d(s)} = \sum_{\substack{d_1 d_2 d_3 = k \\ d_2,\, d_3 \text{ odd}}} \chi_d(d_3)\mu(d_3) d_2^{-4s+3/2} d_3^{-2s+1/2}$$

where μ is the Möbius function and χ_d is the quadratic character $\chi_d(c) = \left(\frac{d}{c}\right)$ in terms of the Kronecker symbol. The subscript 2 applied to the L-function and zeta function ζ means that the 2-parts have been removed.

Goldfeld and Hoffstein applied the theory of Eisenstein series of half-integral weight to obtain the meromorphic continuation and functional equations of $Z_{\pm}$. They showed that there are poles at $w = \frac{3}{4}$ and $\frac{5}{4} - s$, then used a Tauberian argument to obtain estimates for the mean values of L-functions. For example, they showed

$$\sum_{\substack{1 < \pm d < x \\ d \text{ squarefree}}} L\left(\frac{1}{2},\chi_d\right) = c_1 x \log(x) + c_2 x + O\left(x^{\frac{19}{32}+\varepsilon}\right)$$

with known constants c_1 and c_2.

Note that $Z_{\pm}$ is a double Dirichlet series (in s, w) since if we substitute the expression for the L-function $L(w,\chi_c)$ we have "essentially"

$$Z_{\pm}(s,w) = \sum_d L\left(2s-\frac{1}{2},\chi_d\right)|d|^{\frac{1}{2}-2w} = \sum_{c,d}\left(\frac{d}{c}\right)|c|^{\frac{1}{2}-2s}|d|^{\frac{1}{2}-2w}. \tag{2}$$

Equation (2) gives two heuristic expressions representing the multiple Dirichlet series with the intention of explaining as simply as possible what we expect to be true, and what form the generalizations must be. Such a heuristic form ignores a number of details, such as the fact that the coefficients are only described correctly if d is squarefree (both expressions) and that c and d are coprime (second expression). Later we will first generalize the heuristic form by attaching a multiple Dirichlet series to an arbitrary root system. The heuristic version will have predictive value, but will still ignore important details, so we will then have to consider how to make a rigorous definition.

To give an immediate heuristic generalization of (2), let us consider, with complex parameters $s_1, \cdots, s_k$ and w a multiple Dirichlet series

$$\sum_c L\left(2s_1 - \frac{1}{2}, \chi_d\right) \cdots L\left(2s_k - \frac{1}{2}, \chi_d\right) |d|^{\frac{1}{2}-2w} \tag{3}$$

for $k = 0, 1, 2, 3, \cdots$. If one could prove meromorphic continuation of this Dirichlet series to all s with $w_i = \frac{1}{2}$, the Lindelöf hypothesis in the quadratic aspect would follow from Tauberian arguments. A similar approach to the Lindelöf hypothesis in the t aspect would consider instead

$$\int_1^\infty \zeta\left(\sigma_1 \pm it\right) \cdots \zeta\left(\sigma_k \pm it\right) t^{-2w}\, dt \tag{4}$$

where for each zeta function we choose a sign $\pm$; if k is even, we may choose half of them positive and the other half negative. This is equivalent to the usual moments

$$\int_0^T \zeta\left(\sigma_1 \pm it\right) \cdots \zeta\left(\sigma_k \pm it\right)\, dt$$

which have been studied since the work in the 1920's of Hardy and Littlewood, Ingham, Titchmarsh and others. It is possible regard (4) as a multiple Dirichlet series, and indeed both (3) and (4) are treated together in Diaconu, Goldfeld and Hoffstein [32]. See [31] in this volume for a discussion of the sixth integral moment and its connection with the spectral theory of Eisenstein series on GL_3.

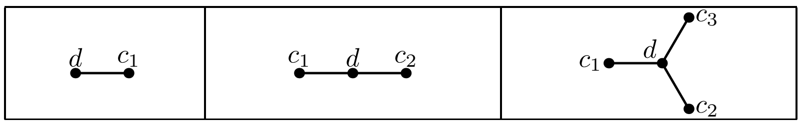
Returning to (3), there are two problems: to make a correct definition of the multiple Dirichlet series, and to determine its analytic properties. If $k = 1, 2$ or 3, these can both be solved, and the multiple Dirichlet series has global analytic continuation. In these cases, the multiple Dirichlet series was initially constructed by applying a Rankin-Selberg construction to Eisenstein series ("of half-integral weight") on the metaplectic double covers of the groups $\mathrm{Sp}(2k)$ for $k = 1, 2, 3$. No corresponding constructions could be found for $k > 3$, but since Rankin-Selberg constructions are often tricky, that did not constitute a proof that such constructions may not exist undiscovered.

In [20] a different approach was taken. If $k > 3$, then it may be possible to write down a correct definition of the multiple Dirichlet series, and indeed this has essentially been done in the very interesting special case $k = 4$. See Bucur and Diaconu [19]. Nevertheless, the approach taken in [20], which we will next explain, shows that the multiple Dirichlet series cannot have meromorphic continuation to all s_i and w if $k > 3$.

The analog for (3) of the second expression in (2) would have the form

$$\sum_{d, c_1, \cdots, c_k} \left(\frac{d}{c_1}\right) \cdots \left(\frac{d}{c_k}\right) |c_1|^{\frac{1}{2}-2s_1} \cdots |c_k|^{\frac{1}{2}-2s_k} |d|^{\frac{1}{2}-2w}. \tag{5}$$

It will be helpful to associate with this Dirichlet series a graph whose vertices are the variables $d, c_1, \cdots, c_k$. We connect two vertices if a quadratic symbol is attached to them. Our point of view (which is justified when rigorous foundations are supplied) is that due to the quadratic reciprocity law we do not have to distinguish between $\left(\frac{d}{c}\right)$ and $\left(\frac{c}{d}\right)$, Thus for heuristic purposes, the graph determines the Dirichlet series. If $k = 1, 2$ or 3, the graph looks like this:



(6)

We could clearly associate a multiple Dirichlet series with a more general graph, at least in this imprecise heuristic form. The interesting cases will be when the diagram is a Dynkin diagram.

Here we will only consider cases where the diagram is a "simply-laced" Dynkin diagram, that is, the diagram of a root system of Cartan type A, D or E. A *simply-laced* root system is one in which all roots have the same length, and these are their Cartan types. More general Dynkin diagrams are also associated with multiple Dirichlet series, and we will come to these below.

We recall that a *Coxeter group* is a group with generators $\sigma_1, \cdots, \sigma_r$, each of order two, such that the relations

$$\sigma_i^2 = 1, \qquad (\sigma_i \sigma_j)^{n(i,j)} = 1$$

give a presentation of the group, where $n(i, j)$ is the order of $\sigma_i \sigma_j$. We may associate with the Coxeter group a graph, which consists of one node for each generator σ_i, with the following conditions. If $n(i, j) = 2$, so that σ_i and σ_j commute, there is no edge connecting i and j. Otherwise, there is an edge. If $n(i, j) = 3$, it is not necessary to label the edge, but if $n(i, j) > 3$ it is labeled with $n(i, j)$. (In Dynkin diagrams it is usual to interpret these labels as double or triple bonds.) We will consider only the case where $n(i, j) = 2$ or 3. In these cases, the Coxeter group is finite if and only if the diagram is the finite union of the Dynkin diagrams of finite Weyl groups of types A_r, D_r or E_r.

As we will now explain, the group of functional equations of a multiple Dirichlet series such as (5) is expected to be the Coxeter group of its Dynkin diagram. For example, consider (5) when $k = 3$. We collect the coefficients of c_1:

$$\sum_{d, c_1, c_2, c_3} \left(\frac{d}{c_1}\right) \left(\frac{d}{c_2}\right) \left(\frac{d}{c_3}\right) |c_1|^{\frac{1}{2} - 2s_1} |c_2|^{\frac{1}{2} - 2s_3} |c_3|^{\frac{1}{2} - 2s_3} |d|^{\frac{1}{2} - 2w} =$$

$$\sum_{d, c_2, c_3} \left(\frac{d}{c_2}\right) \left(\frac{d}{c_3}\right) |c_2|^{\frac{1}{2} - 2s_2} |c_3|^{\frac{1}{2} - 2s_3} |d|^{\frac{1}{2} - 2w} \left[\sum_{c_1} \left(\frac{d}{c_1}\right) |c_1|^{\frac{1}{2} - 2s_1} \right] .$$

This equals

$$\sum_{d,c_2,c_3} \left(\frac{d}{c_2}\right)\left(\frac{d}{c_3}\right)|c_2|^{\frac{1}{2}-2s_2}|c_3|^{\frac{1}{2}-2s_3}|d|^{\frac{1}{2}-2w}L\left(2s_1-\frac{1}{2},\chi_d\right).$$

The functional equation for the Dirichlet L-function has the form

$$L\left(2s_1-\frac{1}{2},\chi_d\right)=(*)\,|d|^{1-2s_1}L\left(2(1-s_1)-\frac{1}{2},\chi_d\right)$$

where $(*)$ is a ratio of Gamma functions and powers of π. This factor is independent of d, so we see that the functional equation has the effect

$$(s_1,s_2,s_3,w)\mapsto(1-s_1,s_2,s_3,w+s_1-\tfrac{1}{2}).$$

In the general case, let the variables be $s_1,\cdots,s_r$. Thus if we are considering (5) then $r=k+1$ and $s_r=w$, but we now have in mind a general graph such as a Dynkin diagram, and r is the number of nodes. We have a functional equation which sends s_i to $1-s_i$. If $j\neq i$, then

$$s_j\longmapsto\begin{cases}s_j & \text{if } i,j \text{ are not connected by an edge;}\\ s_i+s_j-\frac{1}{2} & \text{if } i,j \text{ are connected by an edge.}\end{cases}\tag{7}$$

These functional equations generate the Coxeter group associated with the diagram, and its group of functional equations is the geometric realization of that group as a group generated by reflections.

We may now see why the Dirichlet series (3) is expected to have meromorphic continuation to all s_i and w when $k\leqslant 3$ but not in general. These are the cases where the graph is the diagram of a finite Weyl group, of types A_2, A_3 or D_4. If $k=4$, the graph is the diagram of the affine Weyl group $D_4^{(1)}$, and the corresponding Coxeter group is infinite. The meromorphic continuation in (s_1,s_2,s_3,s_4,w) cannot be to all of $\mathbb{C}^5$ since the known set of polar hyperplanes will have accumulation points.

We have now given heuristically a large family of multiple Dirichlet series, one for each simply-laced Dynkin diagram. (The simply-laced assumption may be eliminated, as we will explain later.) Only three of them, for Cartan types A_2, A_3 and D_4, are related to moments of L-functions. The case $k=3$, related to D_4, was applied in [32] to the third moment after the combinatorics needed to precisely define the Dirichlet series were established in [21].

Although only these three examples are related to quadratic moments of L-functions, others in this family have applications to analytic number theory. Chinta gave a remarkable example in [25], where the A_5 multiple Dirichlet series is used to study the distribution of central values of biquadratic L-functions. The distribution of n-th order twists of an L-function were studied by Friedberg, Hoffstein and Lieman [36], and it was shown in Brubaker and Bump [7] that these could be related to n-th order Weyl group multiple Dirichlet series of order n. (In this survey, the Dirichlet series we have considered in this section correspond to $n=2$ but we will come to general n below.)

One may also consider the Dirichlet series that are (heuristically) of the form $L(w,\pi,\chi_d)\,|d|^{-w}$ where π is an automorphic representation of GL_k. If $k=2$, then there is considerable literature of the case where $n=2$; see for example [6] and the references therein. If $k=2$ and $n=3$, there is a remarkable theory in [17]; there is a finite group of functional equations, transform the Dirichlet series into various different ones. If $k=3$ and $n=2$, then there is also a finite group of functional equations; see [21]. The papers cited in this paragraph predate the recent development of the combinatorial theory, but the combinatorics of multiple Dirichlet series involving GL_k cusp forms is under investigation by Brubaker and Friedberg.

A double Dirichlet similar to that in [36] was considered by Reznikov [53]. This is the Dirichlet series $\sum L(s,\chi^n)|n|^{-w}$, where χ is a Hecke character of infinite order for $\mathbb{Q}(i)$. Using a method of Bernstein, he proved the meromorphic continuation of this multiple Dirichlet series and determined the poles. Despite the similarity of this multiple Dirichlet series to that of [36], this series does not fit the same way in the theory of Weyl group multiple Dirichlet series.

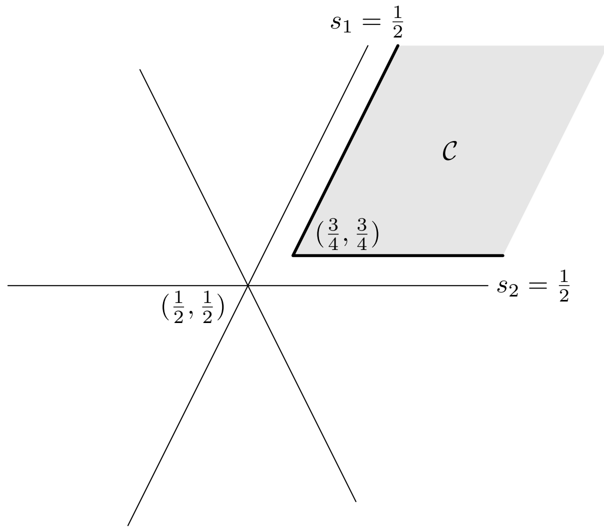
While the origins of our subject are in analytic number theory, our emphasis in this paper will not be such applications, but rather on emerging connections with areas of combinatorics, including quantum groups and mathematical physics, and the theory of Whittaker functions. As we will see, the problem of giving precise definitions of the multiple Dirichlet series, even when the general nature of the Dirichlet series is known, is a daunting combinatorial one. Early investigations, such as [25] and [21] took an *ad hoc* approach substituting computer algebra or brute force computation for real insight. This is sufficient for applications on a case-by-case basis but also unsatisfactory. In recent years, the combinatorial theory has been examined more closely, and its study may turn out to be as interesting as the original problem.

2 A method of analytic continuation

Let us consider a double Dirichlet series which might be written

$$Z^*_\Psi(s_1,s_2)=(*)Z_\Psi(s_1,s_2),\qquad Z_\Psi(s_1,s_2)=\sum_{n,m}A_\Psi(n,m)n^{-s_1}m^{-s_2}.$$

Here $(*)$ denotes some Gamma functions and powers of π. The Dirichlet series is allowed to depend on a parameter Ψ drawn from a finite-dimensional vector space Ω. It is assumed convergent in some region $\mathcal{C}$ such as the one in the following figure, which shows the region for (2). We have graphed the projection onto $\mathbb{R}^2$ obtained by taking the real parts of s_1 and s_2.



Collecting the coefficients of m^{-s_2} for each m gives a collection of Dirichlet series in one variable s_1 which have functional equations. These may be with respect to some transformation such as the following, which is a functional equation of (2).

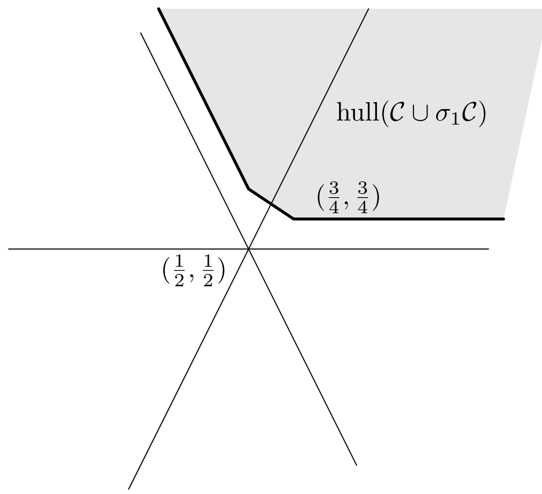
$$\sigma_1 : (s_1, s_2) \longmapsto \left(1 - s_1, s_1 + s_2 - \frac{1}{2}\right).$$

More precisely, there may be an action of σ_1 on Ω, or more properly on $\mathcal{M} \otimes \Omega$, where $\mathcal{M}$ is the field of meromorphic functions in s_1 and s_2, such that the functional equation has the form

$$Z^*_{\sigma_1 \Psi}\left(1 - s_1, s_1 + s_2 - \frac{1}{2}\right) = Z^*_{\Psi}(s_1, s_2).$$

Thus $\Psi \longmapsto \sigma_1 \Psi$ is a linear transformation of the vector space Ω which, when written out as a matrix, could involve meromorphic functions of s_1 and s_2. This is the *scattering matrix*. In some cases these meromorphic functions are holomorphic, or even just Dirichlet polynomials in a finite number of integers. For the example (2) we would take polynomials in 2^{-s_1} and 2^{-s_2}.

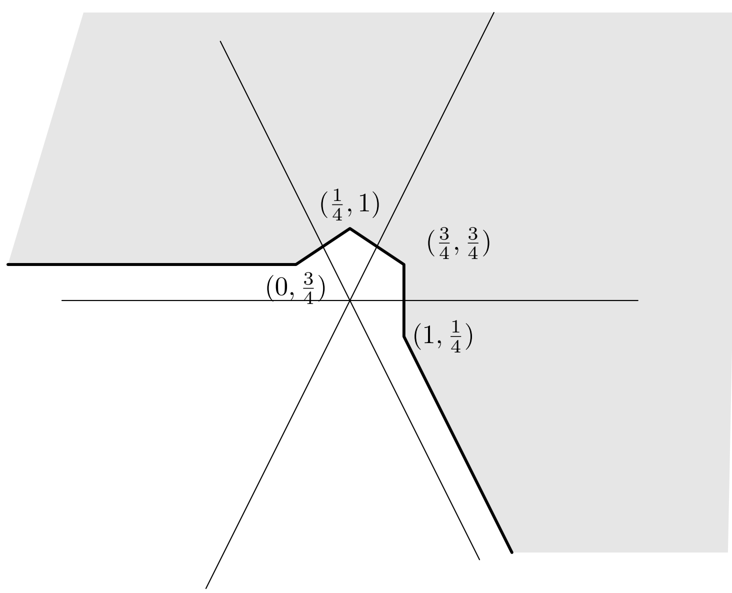
This gives the meromorphic continuation to the convex hull of $\mathcal{C} \cup \sigma_1 \mathcal{C}$.



Similarly (we assume) that collecting the coefficients of the other variable gives another functional equation, which in the example (2) is

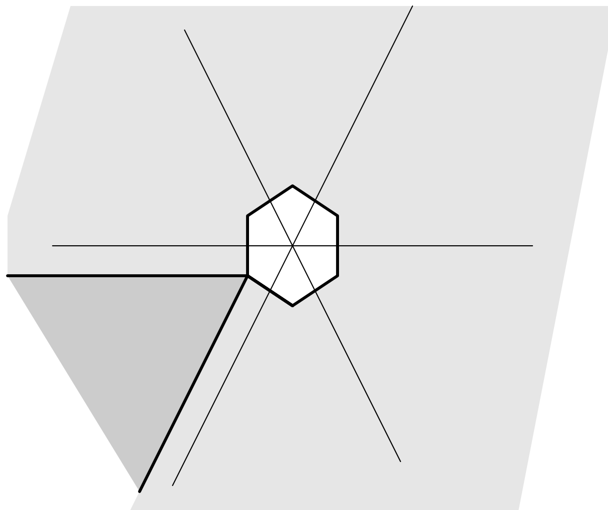
$$\sigma_2 : (s_1, s_2) \longmapsto \left(s_1 + s_2 - \frac{1}{2}, 1 - s_2\right).$$

The functional equations may be iterated, so we get analytic continuation to the the union of hull($\mathcal{C} \cup \sigma_1\mathcal{C}$) with hull($\mathcal{C} \cup \sigma_2\mathcal{C}$) and σ_1 hull($\mathcal{C} \cup \sigma_2\mathcal{C}$).



(8)

At this point, there are two ways of proceeding, one better than the other. We could continue to iterate the functional equations until we obtained meromorphic continuation to a region such as this:



There are two problems with this. One is that we have not obtained the meromorphic continuation in the area near the origin. The other is that we have obtained two different meromorphic continuations to the region $\sigma_1\sigma_2\sigma_1\mathcal{C} = \sigma_2\sigma_1\sigma_2\mathcal{C}$ that is darkly shaded. We do not know that these two meromorphic continuations agree. This agreement, the *braid relation*, should be true in a suitable sense but in fact since there is a scattering matrix involved we must be careful in formulating it. We want an action of W on Ψ extending the one already mentioned for σ_1 such that $Z^*_\Psi(s_1, s_2)$ satisfies

$$Z^*_{w(\Psi)}(w(s_1, s_2)) = Z^*_\Psi(s_1, s_2), \tag{9}$$

and the braid relation means that $\sigma_1\sigma_2\sigma_1(\Psi) = \sigma_2\sigma_1\sigma_2(\Psi)$.

A better procedure is to use a theorem in complex variables, Bochner's convexity theorem [5] to assert meromorphic continuation once one has obtained meromorphic continuation to a region such as (8) whose convex hull is $\mathbb{C}^2$. Bochner's theorem is as follows: let U be an open subset of $\mathbb{C}^r$ where $r \geqslant 2$ that is the preimage of an open subset of $\mathbb{R}^r$ under the projection map; such a set is called a *tube domain*. Then any holomorphic function on a tube domain has analytic continuation to its convex hull. In our case, we have a meromorphic function, but the polar divisor is a set of hyperplanes, and the theorem is easily extended to this case. Hence once we have meromorphic continuation to (8) we obtain meromorphicity on $\mathbb{C}^2$. The braid relation $\sigma_1\sigma_2\sigma_1(\Psi) = \sigma_2\sigma_1\sigma_2(\Psi)$ is then a consequence. See [9] and [11] for further details.

Now we come to the fundamental combinatorial question. Once one has decided *roughly* what the Dirichlet series is to look like, the exact coefficients are still not precisely defined. How can the coefficients be determined in such a way that the functional equations (9) are true for both σ_1 and σ_2? For the Dirichlet series (3), this is not too hard when $k = 1$, but when $k = 3$, the combinatorics are rather daunting. They were treated in [21] using difficult manipulations that were the only way before the combinatorial properties of Weyl group multiple Dirichlet series began to be established. Similarly in the example of Chinta [25], the method of solving the combinatorial problem was to use a computer program to find a Dirichlet series with very special combinatorial properties. There has been a great deal of progress in the basic combinatorial problem since these early papers, and this progress has implications beyond the original practical problem of giving a proper definition of a multiple Dirichlet series with a group of functional equations.

3 Kubota Dirichlet series

Let n be a positive integer: we will define some Dirichlet series related to the n-th power reciprocity law, so $n = 2$ in Section 1. Let F be a number field containing the group μ_n of n-th roots of unity. We will further assume that F contains the group μ_{2n} of $2n$-th roots of unity, that is, that -1 is an n-th power in F. The assumption that $\mu_n \subset F$ is essential; the assumption that $\mu_{2n} \subset F$ is only a matter of convenience. We will make use of the n-th power reciprocity law.

We will define a family of Dirichlet series with analytic continuation and functional equations, called *Kubota Dirichlet series*. If $n = 2$, these are the quadratic L-functions $L(2s - \frac{1}{2}, \chi_d)$. If $n = 1$, these Dirichlet series are divisor sums, actually finite Dirichlet polynomials. For general n, they are generating functions of n-th order Gauss sums.

Let S be a finite set of places of F containing all places dividing n and all archimedean ones. If v is a place of F, let F_v be the completion at v. If v is nonarchimedean, let $\mathfrak{o}_v$ be the ring of integers in F_v. Let $\mathfrak{o}_S$ be the ring of S-integers in F, that is, those elements $x \in F$ such that $x \in \mathfrak{o}_v$ for all $v \notin S$. Let $F_S = \prod_{v \in S} F_v$. We may embed $\mathfrak{o}_S$ in F_S diagonally. It is a discrete, cocompact subgroup. We may choose S so large that $\mathfrak{o}_S$ is a principal ideal domain. If $a \in F_S$ let $|a|$ denote $\prod_{v \in S} |a|_v$. It is the Jacobian of the map $x \longmapsto ax$. If $a \in \mathfrak{o}_S$ then $|a|$ is a nonnegative rational integer.

We recall the n-th order reciprocity law and n-th order Gauss sums. See Neukirch [51] for proofs. Properties of the reciprocity symbol and Gauss sums are more systematically summarized in [11].

The n-th order Hilbert symbol $(\ ,\ )_v$ is a skew-symmetric pairing of $F_v^\times \times F_v^\times$ into μ_n. Define a pairing $(\ ,\ )$ on $F_S^\times$ by

$$(\, x, y\,) = \prod_{v \in S} (x_v, y_v)_v, \qquad x, y \in F_S^\times .$$

Then the n-th power residues symbol $\left(\frac{d}{c}\right)$, defined for nonzero elements $c, d \in \mathfrak{o}$, satisfies the n-th power reciprocity law

$$\left(\frac{c}{d}\right) = (d, c)\left(\frac{d}{c}\right). \tag{10}$$

Let ψ be an additive character of F_S that is trivial on $\mathfrak{o}_S$ but no larger fractional ideal. Let

$$g(m, d) = \sum_{c \bmod d} \left(\frac{c}{d}\right) \psi\left(\frac{mc}{d}\right).$$

The sum is well defined since both factors only depend on c modulo d. It has the *twisted multiplicativity* properties:

$$g(m, dd') = \left(\frac{d}{d'}\right)\left(\frac{d'}{d}\right) g(m, d)\, g(m, d') \text{ if } \gcd(d, d') = 1,$$

$$g(cm, d) = \left(\frac{c}{d}\right)^{-1} g(m, d) \qquad \text{if } c, d \text{ are coprime.}$$

and the absolute value, for p prime in $\mathfrak{o}_S$:

$$|g(m, p)| = \sqrt{|p|} \text{ if } \gcd(m, p) = 1. \tag{11}$$

Let Ψ be a function on $F_S^\times$ such that $\Psi(\varepsilon c) = (\varepsilon, c)\Psi(c)$ when $\varepsilon \in \mathfrak{o}_S^\times (F_S^\times)^n$. The vector space of such functions is nonzero but finite-dimensional. Let

$$\mathcal{D}_\Psi(s; m) = \sum_c \Psi(c) g(m, c)\, |c|^{-2s}.$$

This has a functional equation under $s \longmapsto 1 - s$. To state it, let

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Define

$$\mathcal{D}_\Psi^*(s, m) = \boldsymbol{G}(s)^N \zeta_F(2ns - n + 1)\, \mathcal{D}_\Psi(s, m)$$

where N is the number of archimedean places (all complex) and ζ_F is the Dedekind zeta function of F. Then Kubota [46] proved a functional equation for this, as a consequence of the functional equations of Eisenstein series on the n-fold metaplectic covers of SL_2, which he developed for this purpose. In the form we need it, this is proved by the same method in Brubaker and Bump [18], and a similar result is in Patterson and Eckhardt [34].

To state these functional equations, there exists a family of Dirichlet polynomials P_η indexed by η in $F_S^\times/(F_S^\times)^n$ such that

$$\mathcal{D}_\Psi^*(s, m) = \sum_{\eta \in F_S^\times/(F_S^\times)^n} |m|^{1-2s} P_{m\eta}(s)\, \mathcal{D}_{\tilde{\Psi}_\eta}^*(1 - s, m), \tag{12}$$

where

$$\tilde{\Psi}_\eta(c) = (\eta, c)\Psi(c^{-1}\eta^{-1}).$$

The polynomial P_η is actually a polynomial in q_v^{-s} where v runs through the finite places in S, and q_v is the cardinality of the residue field. It is important for applications that P_η is independent of m.

4 A more general heuristic form

If $n = 2$, and m and c are coprime then $g(m, c)$ equals $\left(\frac{m}{c}\right)^{-1}\sqrt{|c|}$ times a factor which may be combined with Ψ and ignored for heuristic purposes. Thus $\mathcal{D}_\Psi(s; m)$ is essentially $L(2s - \frac{1}{2}, \chi_m)$. We may now give the following heuristic generalization of the Dynkin diagram multiple Dirichlet series described in Section 1. Let us start with a Dynkin diagram, which we will at first assume is simply-laced (Type A, D or E). As in Section 1, for purely heuristic purposes it is not necessary to distinguish between $\left(\frac{c}{d}\right)$ and $\left(\frac{d}{c}\right)$ since by the reciprocity law they differ by a factor (d, c) which may also be combined with Ψ, and may be ignored for heuristic purposes. Ultimately such factors must eventually be kept track of, but at the moment they are unimportant.

The nodes $i = 1, \cdots, r$ of the Dynkin diagram are in bijection with the simple roots of some root system Φ. We choose one complex parameter s_i for each i, and one "twisting parameter" m_i, which is a nonzero integer in $\mathfrak{o}_S$. The multiple Dirichlet series then has the heuristic form

$$\sum_{d_1, \cdots, d_r} \left[\prod_{i,\, j \text{ adjacent}} \left(\frac{d_i}{d_j}\right)^{-1} \right] g(m_i, d_i)|d_i|^{-2s_i}.$$

The form of the coefficient is only correct if d_i are squarefree and coprime, and even then there is a caveat, but this heuristic form is sufficient for extrapolating the expected properties of the multiple Dirichlet series. Whereas before, on expanding in powers of one of the s_i parameters, we obtained a quadratic L-function, now we obtain a Kubota Dirichlet series.

If the Dynkin diagram is not simply-laced, there are long roots and short roots. In this case there is also a heuristic form, which we will not discuss here. For each distict pair of simple roots α_i and α_j let $r(\alpha_i, \alpha_j)$ be the number of bonds connecting the nodes connecting α_i and α_j in the Dynkin diagram. Thus if θ is the angle between α_i and α_j let

$$r(\alpha_i, \alpha_j) = \begin{cases} 0 \text{ if } \alpha_i, \alpha_j \text{ are orthogonal,} \\ 1 \text{ if } \theta = \frac{2\pi}{3}, \\ 2 \text{ if } \theta = \frac{3\pi}{4}, \\ 3 \text{ if } \theta = \frac{5\pi}{6}. \end{cases}$$

Normalize the roots so short roots have length 1; thus every long root α has $||\alpha||^2 = 1, 2$ or 3, the last case occurring only with G_2. Let

$$g_\alpha(m,d) = \sum_{c \bmod d} \left(\frac{c}{d}\right)^t \psi\left(\frac{mc}{d}\right), \qquad t = \|\alpha\|^2.$$

Then the heuristic form of the multiple Dirichlet series is

$$\sum_{d_1,\cdots,d_r} \left[\prod_{i,j} \left(\frac{d_i}{d_j}\right)^{-r(\alpha_i,\alpha_j)}\right] \left[\prod_i g_{\alpha_i}(m_i,d_i)|d_i|^{-2s_i}\right]. \tag{13}$$

5 Foundations and the Combinatorial Problem

The first set of foundations for Weyl group multiple Dirichlet series were given by Fisher and Friedberg [35], and these were used in all earlier papers. Another set of foundations were explained in [9] and [11] and these have been used for the most part in subsequent papers. We recall them in this section.

Let $\boldsymbol{V}$ be the ambient vector space of Φ. Let $\langle\ ,\ \rangle$ be a W-invariant inner product on $\boldsymbol{V}$ such that the short roots have length 1. Let $B : \boldsymbol{V} \otimes \mathbb{C}^r \longrightarrow \mathbb{C}$ be the bilinear map that sends $(\alpha_i, \boldsymbol{s})$ to s_i, where $\boldsymbol{s} = (s_1, \cdots, s_r)$ to $\sum k_i s_i$ and α_i is the i-th simple root. Let $\rho^\vee$ denote the vector $(1, \cdots, 1) \in \mathbb{C}^r$. The reason for this notation is explained in [11]. The Weyl group action on $\boldsymbol{s}$, corresponding to the group of functional equations, may be expressed in terms of B: we require that

$$B\left(w\alpha, w(\boldsymbol{s}) - \frac{1}{2}\rho^\vee\right) = B\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right)$$

for $w \in W$.

We fix an ordering of simple roots of Φ, so that in order they are $\{\alpha_1, \cdots, \alpha_r\}$. Some of the formulas depend on this ordering, but in an inessential way. Following [11] let us define $\mathcal{M}$ to be the nonzero but finite dimensional space of functions on $\Psi : (F_S^\times)^r \longrightarrow \mathbb{C}$ that satisfy

$$\Psi(\varepsilon_1 C_1, \cdots, \varepsilon_r C_r) = \prod_{i=1}^r (\varepsilon_i, C_i)_S^{\|\alpha_i\|^2} \left\{\prod_{i<j} (\varepsilon_i, C_j)_S^{2\langle\alpha_i,\alpha_j\rangle}\right\} \Psi(C_1, \cdots, C_r) \tag{14}$$

when $\varepsilon_1, \cdots, \varepsilon_r \in \mathfrak{o}_S^\times (F_S^\times)^n$ and $C_i \in F_S^\times$.

We seek a function $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$ defined if the C_i and m_i are nonzero elements of $\mathfrak{o}_S$ with the following properties. There is the multiplicativity condition

$$\frac{H(C_1C_1', \cdots, C_rC_r'; m_1, \cdots, m_r)}{H(C_1, \cdots, C_r; m_1, \cdots, m_r)\, H(C_1', \cdots, C_r'; m_1, \cdots, m_r)} =$$
$$\prod_{i=1}^r \left(\frac{C_i}{C_i'}\right)^{\|\alpha_i\|^2} \left(\frac{C_i'}{C_i}\right)^{\|\alpha_i\|^2} \prod_{i<j} \left(\frac{C_i}{C_j'}\right)^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{C_i'}{C_j}\right)^{2\langle\alpha_i,\alpha_j\rangle}. \tag{15}$$

There is another multiplicativity condition which, unlike (15), *does* involve the m_i. If $\gcd(m'_1 \cdots m'_r, C_1 \cdots C_r) = 1$ we require:

$$H(C_1, \cdots, C_r; m_1 m'_1, \cdots, m_r m'_r) = \left(\frac{m'_1}{C_1}\right)^{-||\alpha_1||^2} \cdots \left(\frac{m'_r}{C_r}\right)^{-||\alpha_r||^2} H(C_1, \cdots, C_r; m_1, \cdots, m_r). \tag{16}$$

The conditions (14) and (15) together imply that if $C_1, \cdots, C_r$ is each multiplied by a unit, then the value of $\Psi H(C_1, \cdots, C_r)$ is unchanged. Since $\mathfrak{o}_S$ is a principal ideal ring, we see that $\Psi H(C_1, \cdots, C_r)$ is really a function of ideals. Let

$$Z_\Psi(s_1, \cdots, s_r; m_1, \cdots, m_r) = \sum \Psi(C_1, \cdots, C_r) H(C_1, \cdots, C_r; m_1, \cdots, m_r) |C_1|^{-2s_1} \cdots |C_r|^{-2s_r} \tag{17}$$

where the summation is over ideals (C_i). Also let

$$Z^*_\Psi(s_1, \cdots, s_r; m_1, \cdots, m_r) = \left[\prod_{\alpha \in \Phi^+} \zeta_\alpha(s) G_\alpha(s)\right] Z_\Psi(s_1, \cdots, s_r; m_1, \cdots, m_r), \tag{18}$$

where, if α is a positive root

$$\begin{aligned} \zeta_\alpha(\boldsymbol{s}) =& \zeta_F\left(1 + 2n(\alpha)\left\langle \alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee \right\rangle\right), \\ G_\alpha(\boldsymbol{s}) =& \boldsymbol{G}_{n(\alpha)}\left(\frac{1}{2} + \left\langle \alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee \right\rangle\right) \end{aligned} \tag{19}$$

with

$$n(\alpha) = \begin{cases} n \text{ if } \alpha \text{ is a short root,} \\ n \text{ if } \alpha \text{ is a long root and } \Phi \neq G_2, \text{ and } n \text{ is odd} \\ \frac{n}{2} \text{ if } \alpha \text{ is a long root and } \Phi \neq G_2, \text{ and } n \text{ is even} \\ n \text{ if } \alpha \text{ is a long root and } \Phi = G_2, \text{ and } 3 \nmid n \\ \frac{n}{3} \text{ if } \alpha \text{ is a long root and } \Phi = G_2, \text{ and } 3|n. \end{cases}$$

The Kubota Dirichlet series $\mathcal{D}_\Psi(s; m)$ is the special case if Z_Ψ where the root system is of type A_1.

We still have not fully described H, so we haven't given a proper definition of Z_Ψ. The multiplicativities (15) and (16) together imply that the function H is determined by its values on prime powers. In other words if we specify $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ for prime elements p, the function is determined.

The fundamental combinatorial problem is this: given a global field F in which -1 is an n-th power and a root system, give a correct definition of the multiple Dirichlet series extrapolating the heuristic one, such that expanding in powers of every s_i gives a sum of Kubota Dirichlet series all having the same functional equations. Naturally this must be made more precise. We will write $\boldsymbol{s} = (s_1, \cdots, s_r)$ and $\boldsymbol{m} = (m_1, \cdots, m_r)$.

Fundamental Combinatorial Problem. *Define* $H(p^{k_1},\cdots,p^{k_r};p^{l_1},\cdots,p^{l_r})$ *in such a way that for each index* i *the series* $Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ *has an expansion*

$$\sum_M \mathcal{D}_{\Psi_i}\,(s_i,M)\,P_M(\boldsymbol{s}) \tag{20}$$

for some Ψ_i*, where* P_M *is a Dirichlet polynomial, such that for each* i *we have*

$$P_M(\sigma_i\boldsymbol{s})=|M|^{1-2s_i}P_M(\boldsymbol{s}). \tag{21}$$

If this can be done, then we have a functional equation

$$Z_\Psi(\boldsymbol{s};\boldsymbol{m})=Z_{\Psi'}(\sigma_i\boldsymbol{s};\boldsymbol{m}) \tag{22}$$

for some Ψ'. Here σ_i is the simple reflection in the Weyl group action on the parameters $\boldsymbol{s}$; if the root system is simply-laced it is the action (7), or see [11] for the general case. The method of analytic continuation described in Section 2 is applicable. This yields both the meromorphic continuation and the scattering matrix, which we recall from Section 2 amounts to an action of W on Ψ such that in (22) we have $\Psi'=\sigma_i\Psi$ and more generally

$$Z^*_{w\Psi}(w\boldsymbol{s};\boldsymbol{m})=Z^*_\Psi(\boldsymbol{s};\boldsymbol{m}).$$

The normalizing factor in (18) works out as follows: the factor $\zeta_\alpha G_\alpha$ with $\alpha=\alpha_i$ is needed to normalize the Kubota Dirichlet series in (20). The other such factors are simply permuted by $\boldsymbol{s}\longmapsto\sigma_i(\boldsymbol{s})$.

Let us consider briefly how this works in the case of Type A_2. See [9] for a complete discussion and detailed proof for this case. We have noted above that specifying the coefficients $H(p^{k_1},p^{k_2};p^{l_1},p^{l_2})$ completely determines the function H. In this example, let us take $m_1=m_2=1$ so $l_1=l_2=0$ for all p. The coefficients to be described are given by the following table

Let the nonzero values of $H(p^{k_1},p^{k_2};1,1)$ be given by the following table:

		k_1		
		0	1	2
k_2	0	1	$g(1,p)$	
	1	$g(1,p)$	$g(p,p)g(1,p)$	$g(p,p^2)g(1,p)$
	2		$g(p,p^2)g(1,p)$	$g(p,p^2)g(1,p)^2$

Then, collecting terms with equal powers of $|p|^{-s_2}$ we have a decomposition (20) where the summation includes terms of the following type:

$$\mathcal{D}_{\Psi'}(s_1;1),\qquad g(1,p)|p|^{-2s_2}\mathcal{D}_{\Psi''}(s_1;p),\qquad g(1,p)g(p,p^2)|p|^{-2s_1-4s_2}\mathcal{D}_{\Psi'''}(s_1;1),$$

for suitable Ψ',Ψ'' and Ψ'''. We recognize the p-parts of these Kubota Dirichlet series from the tabulated values by collecting the terms in each column of the table.

Early papers in this subject gave *ad hoc* solutions to the combinatorial problem. Such direct verifications become fairly difficult, for example in [25] and [21].

6 p-parts

Let us define the *p-part* of Z to be the Dirichlet series

$$\sum_{k_i=0}^{\infty} H(p^{k_1},\cdots,p^{k_r};p^{l_1},\cdots,p^{l_r})|p|^{-2k_1 s_1-\ldots-2k_r s_r}. \tag{23}$$

We fix the representative p of the prime and ignore Ψ. By twisted multiplicativity, if the p-parts are known for all p, the multiple Dirichlet series is determined.

Returning to (17), let us consider the effect of the parameters $m_1,\cdots,m_r$. These are called *twisting* parameters, and the term "twisting" is supposed to evoke the usual twisting of L-functions: if $L(s,f)=\sum a_n n^{-s}$ is some L-function and χ is a Dirichlet character then $L(s,f,\chi)=\sum\chi(n)a_n n^{-s}$. The term "twisting" in the present context is both apt and in a way misleading, as we will now explain.

First suppose that $m_1,\cdots,m_r$ are coprime to $C_1,\cdots,C_r$. Then by (16) we have

$$H(C_1,\cdots,C_r;m_1,\cdots,m_r)=\\ \left(\frac{m_1}{C_1}\right)^{-||\alpha_1||^2}\cdots\left(\frac{m_r}{C_r}\right)^{-||\alpha_r||^2} H(C_1,\cdots,C_r;1,\cdots,1) \tag{24}$$

Thus these coefficients are indeed simply multiplied by an n-th order character, as the term "twisting" suggests.

On the other hand, if the m_i are not coprime to the C_i, then the effect of the m_i is much more profound. For example in $H(p^{k_1},\cdots,p^{k_r};p^{l_1},\cdots,p^{l_r})$ it is important to think of $(l_1,\cdots,l_r)$ as indexing a weight, $\sum l_i\varpi_i$, where $\varpi_1,\cdots,\varpi_r$ are the fundamental dominant weights of the root system Φ. Then we may think of the p-part (23) as being something related to the character of an irreducible representation of the associated Lie group, times a deformation of the Weyl denominator, but with the weight multiplicities replaced by sums of products of Gauss sums. In particular, varying $m_i=p^{l_i}$ affects the p-part in a profound way, no simple twisting.

With m_i general, their meaning may be explained as follows: specifying $m_1,\cdots,m_r$ is equivalent to specifying, for each p, dominant weight λ_p such that $\lambda_p=0$ for almost all p. Indeed, factor $m_i=p^{l_i}m_i'$ where $p\nmid m_i'$ and take $\lambda_p=\sum l_i\varpi_i$.

Now let the prime p and the exponents $l_1,\cdots,l_r$ be fixed, and let $\lambda=\lambda_p=\sum l_i\varpi_i$ be the corresponding dominant weight. Also let

$$\rho=\sum_{i=1}^{r}\varpi_i=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$$

be the Weyl vector. Let W be the Weyl group of Φ. If $w\in W$, let $\boldsymbol{k}(w)$ be the r-tuple of nonnegative integers $(k_1,\cdots,k_r)$ such that $\rho+\lambda-w(\rho+\lambda)=\sum k_i\alpha_i$.

The coefficients $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ in general do not admit an easy description, but if $(k_1, \cdots, k_r) = \boldsymbol{k}(w)$ for some w, then it is a product of $l(w)$ Gauss sums, where $l(w)$ is the length of w. To make this explicit, let Φ_w be the set of all positive roots α such that $w(\alpha)$ is a negative root, so $|\Phi_w| = l(w)$. Then (see [12]) we have

$$H(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{\langle \lambda+\rho, \alpha \rangle - 1}, p^{\langle \lambda+\rho, \alpha \rangle}). \qquad (25)$$

Let Supp(λ) be the *support* of $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$, that is, the set of $\boldsymbol{k} = (k_1, \cdots, k_r)$ such that $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) \neq 0$. Then by (25), $\{\boldsymbol{k}(w) | w \in W\}$ is contained in Supp(λ).

Most importantly, the $|W|$ points $\boldsymbol{k}(w)$ are the extremal values of the support. That is, Supp(λ) is contained in the convex hull of $\boldsymbol{k}(w)$. These $|W|$ extremal points are called *stable* in [11] and [12] for the following reason. If n is sufficiently large, then Supp(λ) $= \{\boldsymbol{k}(w) | w \in W\}$, and in this case, the values (25) are the *only* nonzero values of $H(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r})$. So these values are "stable" and the combinatorial problem is solved by (25).

For arbitrary n, Supp(λ) is at least contained in the convex hull of $\{\boldsymbol{k}(w) | w \in W\}$. But for interior points of this convex polytope, the description of $H(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r})$ is much more difficult. We will next look at the various approaches.

7 Multiple Dirichlet Series and Combinatorics

In this section we will introduce the modern combinatorial theory of the p-parts of Weyl group multiple Dirichlet series. There are several different methods of representing the p-parts of multiple Dirichlet series to be considered, each with its own individual combinatorial flavor. The combinatorial theory has only taken shape in the last few years. We will state things most fully in the "nonmetaplectic" case $n = 1$, leaving the reader hopefully oriented and ready to explore the general cases in the literature.

We see that correctly specifying the p-part of the function H produces a Dirichlet series Z_Ψ with global meromorphic continuation. These functions turn out to be extremely interesting. Several definitions of H emerged, and proving their equivalence proved to be nontrivial. Moreover, as the functions H were intensively studied, various clues seemed to suggest connections with the theory of quantum groups. We will discuss these points in this section.

The following main classes of definitions were found.

- Definition by the "Averaging method," sometimes known as the Chinta-Gunnells method.
- Definition as spherical p-adic Whittaker functions.
- Definition as sums over crystal bases.
- Definition as partition functions of statistical-mechanical lattice models.

The first and second definitions give uniform descriptions for all root systems and all n. The third and fourth definitions are on a case-by-case basis and have not been carried out for all n. Nevertheless they are very interesting, and it is the latter two approaches that suggest connections with quantum groups.

The equivalence of these different definitions is by no means clear or easy. However it is now mostly proved by the following scheme.

Chinta-Gunnells method ⟷ Whittaker function ⟷ Crystal description ⟷ Statistical model

The equivalence of the averaging method with the Whittaker definition was proved by Chinta and Offen [29] for Type A by generalizing the original proof of Casselman and Shalika. This was extended by McNamara [49] to arbitrary Cartan types. Two results which both assert that the Whittaker definition is equivalent to the crystal definition (in Type A) are Brubaker, Bump and Friedberg [13] and McNamara [50]. The first paper directly computes the Whittaker coefficients of Eisenstein series, and the second paper proceeds locally by partitioning the unipotent integration into cells that contribute the individual terms in the sum over the crystal. The relationship between the statistical model scheme and the crystal description must be done on a case-by-case basis, but we will discuss these below.

Yet another possibility has appeared on the horizon within the last few months.

- Approach p-adic Whittaker functions by means of Demazure-Lusztig operators and "metaplectic" generalizations of them.

The above remarks concern mainly what is, in the language of Whittaker models, the *spherical Whittaker function*. However it is useful to consider a larger class of Whittaker functions, namely the Iwahori fixed vectors in the Whittaker model. When these are considered, the Demazure-Lusztig operators and their metaplectic analogs appear.

There are other objects in mathematics that may be related to these.

- Some examples of zeta functions of prehomogeneous vector spaces seem to be specializations of Weyl group multiple Dirichlet series. These connections are under investigation by Chinta and Taniguchi.
- Jacquet conjectured that an $O(r)$ period of an automorphic form on GL_r is related to a Whittaker coefficent of the Shimura correspondent on the double cover of GL_r. Applying this to Eisenstein series, this would mean that orthogonal periods of Eisenstein series on GL_r are related to Type A_{r-1}. When $r-1=2$, this was investigated by Chinta and Offen [30].
- Zeta functions of prehomogeneous vector spaces as well as the Witten zeta functions studied by Komori, Matsumoto and Tsumura [45] in this volume are both special cases of Shintani zeta functions. It is by no means clear that the Witten zeta functions can be related to Weyl group multiple Dirichlet series but potentially there are undiscovered connections.

Let us begin with p-adic Whittaker functions. Casselman and Shalika [24] showed that the values of the spherical Whittaker function are expressible as values of the characters of irreducible representations of the L-group, times a deformation of the Weyl denominator. We begin by reviewing this important formula.

Let G be a split Chevalley group, or more generally a split reductive group defined over $\mathbb{Z}$. Let F be a nonarchimedean local field with residue field $\mathfrak{o}/\mathfrak{p} = \mathbb{F}_q$, where $\mathfrak{o}$ is the ring of integers and $\mathfrak{p}$ its maximal ideal. Let $B = TN$ be a Borel subgroup, where T is a maximal split torus, and N is the unipotent radical. The root system lives in the group $X^*(T)$ of rational characters of T and the roots so that N is the subgroup generated by the root groups of the positive roots.

We may take the algebraic groups G, T, B, N to be defined over $\mathfrak{o}$. Then $G(\mathfrak{o})$ is a special maximal compact subgroup. If w is an element of the Weyl group W, we will choose a representative for it in $G(\mathfrak{o})$, which, by abuse of notation, we will also denote as w.

Let $\hat{G}$ be the (connected) Langlands L-group. It is an algebraic group defined over $\mathbb{C}$. Then G and $\hat{G}$ contain split maximal tori T and $\hat{T}$ respectively; T we have already chosen. Then $\hat{T}(\mathbb{C})$ is isomorphic to the group of unramified characters of $T(F)$, that is, the characters that are trivial on $T(\mathfrak{o})$. If $\boldsymbol{z} \in \hat{T}(\mathbb{C})$, let $\tau_{\boldsymbol{z}}$ denote the corresponding unramified character.

Let Λ be the weight lattice of $\hat{T}$, that is, the group of rational characters. Then Λ is isomorphic to $T(F)/T(\mathfrak{o})$. The isomorphism may be chosen so that if λ is a weight and a_λ is a representative of the corresponding coset in $T(F)/T(\mathfrak{o})$, then

$$\tau_{\boldsymbol{z}}(a_\lambda) = \boldsymbol{z}^\lambda. \tag{26}$$

There are now two root systems to be considered: the root system of G relative to T, and the root system of $\hat{G}$ relative to $\hat{T}$. The latter is more important for us, so we will denote it as Φ. Thus Φ is contained in the Euclidean vector space $\mathbb{R} \otimes \Lambda$. If $\alpha \in \Phi$ then the corresponding coroot $\alpha^\vee$ is a root of G with respect to T, and we will denote by $i_\alpha : \mathrm{SL}_2 \longrightarrow G$ the corresponding Chevalley embedding.

For example, let $G = \mathrm{GL}_{r+1}$. Then $\hat{G} = \mathrm{GL}_{r+1}$. We take T and $\hat{T}$ to be the diagonal tori. We may identify the weight lattice Λ of $\hat{T}$ with $\mathbb{Z}^{r+1}$ in such a way that $\lambda = (\lambda_1, \cdots, \lambda_{r+1}) \in \mathbb{Z}^{r+1}$ corresponds to the rational character

$$\boldsymbol{z} = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_{r+1} \end{pmatrix} \longmapsto \prod_i z_i^{\lambda_i}.$$

If p is a generator of $\mathfrak{p}$ we may take

$$a_\lambda = \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_{r+1}} \end{pmatrix}.$$

Then (26) is satisfied with

$$\tau_{\boldsymbol{z}}\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{r+1}\end{pmatrix} = \prod z_i^{\mathrm{ord}_p(t_i)}.$$

Returning to the general case, let $\boldsymbol{z} \in \hat{T}(\mathbb{C})$. We may induce $\tau_{\boldsymbol{z}}$ to $G(F)$, by considering the vector space $V_{\boldsymbol{z}}$ of functions $f : G(F) \longrightarrow \mathbb{C}$ that satisfy

$$f(bg) = (\delta^{1/2}\tau_{\boldsymbol{z}})(b) f(g), \qquad b \in B(F). \tag{27}$$

The group $G(F)$ acts on $V_{\boldsymbol{z}}$ by right translation. If $\boldsymbol{z}$ is in general position, this representation is irreducible, and unchanged if $\boldsymbol{z}$ is replaced by any conjugate by an element of the Weyl group. If it is not irreducible, at least its set of irreducible constitutents are unchanged if $\boldsymbol{z}$ is conjugated.

Let $\psi : N(F) \longrightarrow \mathbb{C}$ be a character. We will assume that if α is a simple root, then the character $x \longmapsto i_\alpha \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ of F is trivial on $\mathfrak{o}$ but no larger fractional ideal. If $f \in V_{\boldsymbol{z}}$ and $g \in G(F)$, define the Whittaker function on $G(F)$ associated with f by

$$W_f(g) = \int_{N(F)} f(w_0 n g)\, \psi(n)\, dn, \tag{28}$$

where w_0 is a representative in $G(\mathfrak{o})$ of the long Weyl group element. The integral is convergent if $|\boldsymbol{z}^\alpha| < 1$ for positive roots α; for other $\boldsymbol{z}$, it may be extended by analytic continuation. The space $V_{\boldsymbol{z}}$ has a distinguished *spherical vector* f° characterized by the assumption that $f^\circ(g) = 1$ for $g \in G(\mathfrak{o})$. Let $W^\circ = W_{f^\circ}$.

Theorem 1. (Casselman-Shalika [24]) *Let $\lambda \in \Lambda$. Then*

$$\delta^{-1/2}(a_\lambda)\, W^\circ(a_\lambda) = \begin{cases} \left[\prod_{\alpha\in\Phi^+}(1 - q^{-1}\boldsymbol{z}^\alpha)\right] \chi_\lambda(\boldsymbol{z}) & \text{if } \lambda \text{ is dominant,} \\ 0 & \text{otherwise.}\end{cases} \tag{29}$$

Here with λ dominant, χ_λ is the character of the finite-dimensional irreducible representation of $\hat{G}$ having highest weight λ. Note that the product on the right-hand side is a deformation of the Weyl denominator. Thus if ρ is half the sum of the positive roots, on specializing q^{-1} to 1, the right-hand side of (29) becomes

$$\left[\prod_{\alpha\in\Phi^+}(1 - \boldsymbol{z}^\alpha)\right] \chi_\lambda(\boldsymbol{z}) = \boldsymbol{z}^\rho \sum_{w\in W} (-1)^{l(w)} \boldsymbol{z}^{w(\lambda+\rho)}, \tag{30}$$

where we have used the Weyl character formula to rewrite the specialization as a sum over the Weyl group.

We next consider how expressions such as the character χ_λ may be interpreted as the p-parts of multiple Dirichlet series.

Let q be a power of a rational prime. Let λ be a dominant weight. We consider an expression $E = \sum_\mu m(\mu)\, \boldsymbol{z}^\mu$, where the sum is over weights μ, and $m(\mu)$ is a complex number that is nonzero for only finitely many μ. More precisely, we assume $m(\mu) = 0$ unless μ is in the convex hull of the polytope spanned by the W-orbit of λ, and moreover $\lambda - \mu$ is in the root lattice, which is the lattice in Λ spanned by Φ.[1] We will also assume that m does not vanish on the W-orbit of λ, though it may vanish for roots in the interior of the polytope. We will call E a *λ-expression*. For example, χ_λ is a λ-expression, and the *numerator in the Weyl character formula*, in other words (30), is a $(\lambda+\rho)$-expression.

Given a λ-expression E, let us show how to obtain a Dirichlet polynomial, that is, a polynomial in $q^{-2s_1}, \cdots, q^{-2s_r}$ where r is the rank of $\hat{G}$. Given μ, there exist nonnegative integers $(k_1, \cdots, k_r) = (k_1(\mu), \cdots, k_r(\mu))$ such that $\sum k_i \alpha_i = \mu - w_0(\lambda)$, where w_0 is the long Weyl group element. Then we call

$$\sum_\mu m(\mu) q^{-2k_1(\mu)s_1 - \ldots - 2k_r(\mu)s_r}$$

the *Dirichlet polynomial associated with the λ-expression E*. The p-parts of the multiple Dirichlet series that we are considering are all of this type. If $n = 1$, the $(\lambda+\rho)$-expression producing the p-part is

$$\left[\prod_{\alpha\in\Phi^+} (1 - q^{-1}\boldsymbol{z}^\alpha)\right] \chi_\lambda(\boldsymbol{z}) \tag{31}$$

which, we observe, differs from (30) by the insertion of q^{-1}. Thus the p-part is a *deformation* of (30). Comparing with the Casselman-Shalika formula (29), we see that this is essentially a value of the spherical Whittaker function.

Similarly, the p-part (23) with $q = |p|$ is derived from a certain $(\lambda+\rho)$-expression, and these $(\lambda+\rho)$-expressions turn out to be values of spherical Whittaker functions on metaplectic covers of G. The integer l_i in (23) is the inner product of λ with the coroot $\alpha_i^\vee$. These $(\lambda+\rho)$-expressions might be regarded as analogs of (31) in which the integers $m(\mu)$ have been replaced by sums of products of Gauss sums. As we will explain, they are extremely interesting objects from a purely combinatorial point of view.

The averaging method of Chinta-Gunnells expresses the p-part of the multiple Dirichlet series as a ratio in which the numerator is a sum over the Weyl group, and in the case $n = 1$, it reduces to the right-hand side of (30). When $n > 1$, the Weyl group action on functions is non-obvious; the simple reflections involve Gauss sums and congruence conditions, and verifying the braid relations is not a simple matter. See Chinta and Gunnells [27] for this action,

[1] If $\hat{G}$ is semisimple then the root lattice has finite index in Λ.

and Patterson [52] for a meditation on the relationship between the method and the intertwining operators for principal series representations.

We turn next to the crystal description. Crystals arose from the representation theory of quantum groups, that is, quantized enveloping algebras. Let $\hat{\mathfrak{g}}$ be a complex Lie algebra, which for us will be the Lie algebra of $\hat{G}$. Then the quantized enveloping algebra $U_q(\hat{\mathfrak{g}})$ is a Hopf algebra that is a deformation of the usual universal enveloping algebra $U(\hat{\mathfrak{g}})$ to which it reduces when $q = 1$. The representations of $\hat{G}(\mathbb{C})$ correspond bijectively to characters of $\hat{\mathfrak{g}}$, and hence to $U(\hat{\mathfrak{g}})$; they extend naturally to representations of $U_q(\hat{\mathfrak{g}})$.

Suppose that λ is a dominant weight, which is the highest weight vector of an irreducible representation of of $\hat{G}$ and hence of $U_q(\hat{\mathfrak{g}})$. This module $U_q(\hat{\mathfrak{g}})$ has a distinguished basis, Kashiwara's global crystal basis, which is closely related to Lusztig's canonical basis. Let us denote it as $\mathcal{B}_\lambda$. Let 0 be the zero element of the module. Let $\alpha_1, \cdots, \alpha_r$ be the simple roots. If $\alpha = \alpha_i$, then $di_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $di_\alpha \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are in a certain sense approximated by maps e_i and f_i from $\mathcal{B}_\lambda$ to $\mathcal{B}_\lambda \cup \{0\}$, the *Kashiwara operators*; each such operator applied to $v \in \mathcal{B}_\lambda$ either gives 0 or another element of the basis.

Also, every element of the crystal basis lies in a well-defined weight space so there is a map $\mathrm{wt} : \mathcal{B}_\lambda \longrightarrow \Lambda$ mapping each element to its weight. We have

$$\chi_\lambda(\boldsymbol{z}) = \sum_{v \in \mathcal{B}_\lambda} \boldsymbol{z}^\lambda. \tag{32}$$

The maps e_i and f_i shift the weights: if $x, y \in \mathcal{B}_\lambda$ then $f_i(x) = y$ if and only if $e_i(y) = x$ and in this case, $\mathrm{wt}(y) = \mathrm{wt}(x) - \alpha$. When this is true, we draw an arrow $x \xrightarrow{i} y$, and the resulting directed graph, with edges labeled by indices i is the *crystal graph*.

Let us consider an example. Take $G = \mathrm{GL}_3$. Its Cartan Type is A_2. In this case, the weight lattice Λ may be identified with $\mathbb{Z}^3$. If λ is a weight, then with this identification $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ where $\boldsymbol{z}^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3}$. The weight is dominant if $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$. Assume that λ is dominant, and furthermore $\lambda_3 \geqslant 0$, so that λ is a partition. The elements of $\mathcal{B}_\lambda$ may be identified with semistandard Young tableaux with shape λ. So if $\lambda = (3, 1, 0)$ and $r = 2$, the crystal graph of $\mathcal{B}_\lambda$ is shown in Figure 1. The weight of a tableau T is $\mathrm{wt}(T) = (\mu_1, \mu_2, \mu_3)$ where μ_i is the number of entries in T equal to i.

If $w \in W$ then $\mathcal{B}_\lambda$ has a unique element $v_{w\lambda}$ of weight $w\lambda$. We will call these elements *extremal*. Consider a function f on $\mathcal{B}_\lambda$, which we assume does not vanish on the extremal elements. Then we may consider

$$E_f = \sum_{v \in \mathcal{B}_\lambda} f(v)\, \boldsymbol{z}^{\mathrm{wt}(v)}.$$

This is a λ-expression. For example if $f(v) = 1$ for all v, then E_f is the character of the irreducible representation with highest weight λ.

We call a weight *strongly dominant* if it is of the form $\lambda + \rho$ with λ dominant. A dominant weight is strongly dominant if and only if W acts freely on its orbit, or equalently, if it is in the interior of the positive Weyl chamber. Let $\lambda + \rho$ be a strongly dominant weight. We recall that the numerator (30) in the Weyl character formula, as are the p-parts of the Weyl group multiple Dirichlet series, are $(\lambda + \rho)$-expressions. So we may hope to find a function f on $\mathcal{B}_{\lambda+\rho}$ such that E_f equals this numerator or p-part. In some cases, a deformation of the Weyl character formula exists in which the numerator is a sum over $\mathcal{B}_{\lambda+\rho}$. The formula will express χ_λ as a ratio of this sum to a denominator that is a deformation of the Weyl denominator, of the form $\prod_{\alpha\in\Phi^+}(1 - t\boldsymbol{z}^\alpha)$ where t is a deformation parameter. Taking $t = 1$, only the extremal elements of $\mathcal{B}_{\lambda+\rho}$ will make a nonzero contribution, and the numerator will reduce to the numerator in the Weyl character formula. Taking $t = 0$, the only terms that contribute will be those in the image of a map $\mathcal{B}_\lambda \longrightarrow \mathcal{B}_{\lambda+\rho}$, and the sum reduces to a sum over $\mathcal{B}_\lambda$. Thus when $t = 0$, the formula reduces to (32). Most importantly for us, taking $t = q^{-1}$ and comparing with (29), we see that the sum over $\mathcal{B}_{\lambda+\rho}$ is exactly $\delta^{1/2}(a_\lambda)W^\circ(a_\lambda)$.

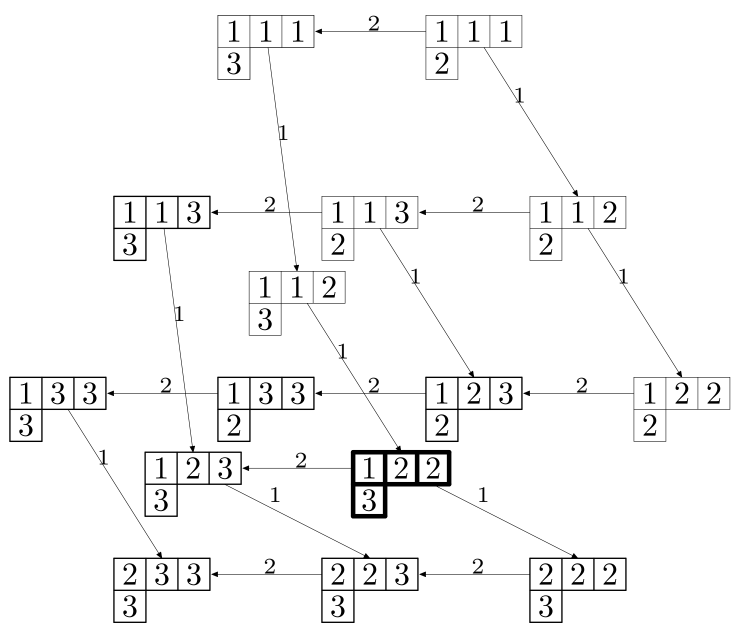


Fig. 1. The Crystal $\mathcal{B}_{(3,1,0)}$ with an element of weight $(1, 2, 1)$ highlighted.

Moreover such formulas exist for metaplectic Whittaker functions. In other words, there is often a way of summing over $\mathcal{B}_{\lambda+\rho}$ and obtaining a Whittaker function on the n-fold cover of some group. There should actually be one such formula for every reduced decomposition of the long Weyl group element into a product of simple reflections, and in some sense this is true. But in practice only certain such decompositions give clean and elegant formulas. Here is a list of cases where nice formulas are known, rigorously or conjecturally.

- Cartan Type A_r, any n. See [14] and [13]. These produce Whittaker functions on the n-fold covers of GL_{r+1}. In this case proofs are complete.
- Cartan Type B_r, n even. These would produce Whittaker functions on even covers of Sp_{2r}. For general even n, the representation is conjectural, and even in the $n = 2$ case there is still work to be done. See [10], this volume, for the case $n = 2$.
- Cartan Type C_r, n odd. See [3], this volume and Ivanov [40] for a discussion of the Yang-Baxter equation. These will produce Whittaker functions on the n-covers of Sp_{2r+1}. Only the case $n = 1$ is proved.
- Cartan Type D_r, any n. See [28], this volume. This case is still largely conjectural.

Let us explain how this works for Type A_r for arbitrary r, so $G = \mathrm{GL}_{r+1}$ (or SL_{r+1}). If $n = 1$, the formula in question is *Tokuyama's formula*. Tokuyama [54] expressed his formula as a sum over strict Gelfand-Tsetlin patterns, but it may be reformulated in terms of tableaux, or crystals. Using crystals, Tokuyama's formula may be written

$$\left[\prod_{\alpha\in\Phi^+}(1-t\boldsymbol{z}^{\alpha})\right]\chi_\lambda(\boldsymbol{z}) = \sum_{v\in\mathcal{B}_{\lambda+\rho}} G^\flat(v)\boldsymbol{z}^{\mathrm{wt}(v)-w_0\rho}, \tag{33}$$

where w_0 is the long Weyl group element, and the function $G^\flat(v)$ will be described below. Tokuyama's formula is a deformation of the Weyl character formula, which is obtained by specializing $t \longrightarrow 1$; the formula (32) is recovered by specializing $t \longrightarrow 0$. Taking $t = q^{-1}$ to be the cardinality of the residue field for a nonarchimedean local field, and combining Tokuyama's formula with the Casselman-Shalika formula (29), it gives a formula for the p-adic Whittaker function, and similar formulas give the p-part of the multiple Dirichlet series. Since this is the case we are concerned with, we will write q^{-1} instead of t for the deformation parameter, even if occasionally we want to think of it as an indeterminate.

To define $G^\flat(v)$, we first associate with v a *BZL-pattern* (for Berenstein, Zelevinsky [4] and Littelmann [47]). We choose a "long word" by which we mean a decomposition of $w_0 = \sigma_{\omega_1}\cdots\sigma_{\omega_N}$ into a product of simple reflections σ_{ω_i} $(1 \leqslant i \leqslant r)$, where N is the number of positive roots. Let b_1 be the number of times we may apply f_{ω_1} to v, that is, the largest integer such that $f_{\omega_1}^{b_1}(v) \neq 0$. Then we we let b_2 be the largest integer such that $f_{\omega_2}^{b_2}f_{\omega_1}^{b_1}(v) \neq 0$. Continuing this way, we define $b_1, \cdots, b_N$. We may characterize $\mathrm{BZL}(v) = (b_1, b_2, \cdots, b_{\omega_N})$ as the unique sequence of N nonnegative integers such that $f_{\omega_N}^{b_N}\cdots f_{\omega_1}^{b_1}(v)$ is the unique vector $v_{w_0\lambda}$ with lowest weight $w_0\lambda$.

We should be able to give a suitable definition of $G^\flat$ for any Cartan type and any long word, but in practice we only know how to give precise combinatorial definitions in certain cases. We will choose this word:

$$(\omega_1, \cdots, \omega_N) = (1, 2, 1, 3, 2, 1, \cdots)$$

corresponding to the decomposition into simple reflections:

$$w_0 = \sigma_{\omega_1} \cdots \sigma_{\omega_N} = \sigma_1 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \cdots \sigma_r \sigma_{r-1} \cdots \sigma_1,$$

where σ_i is the i-th simple reflection in W. We will write $\mathrm{BZL}(v)$ in a tabular array:

$$\mathrm{BZL}(v) = \begin{bmatrix} \ddots & & & \vdots \\ & b_4 & b_5 & b_6 \\ & & b_2 & b_3 \\ & & & b_1 \end{bmatrix} \tag{34}$$

This has the significance that each column corresponds to a single root operator f_i where $i = 1$ for the rightmost column, $i = 2$ for the next column, and so forth.

Now we will decorate the pattern by drawing boxes or circles around various b_i according to certain rules that we will now discuss. We describe the circling rule first. It may be proved that b_i satisfies the inequality $b_i \geqslant b_{i+1}$, except in the case that i is a triangular number, so that b_i is the last entry in its row; in the latter case, we only have the inequality $b_i \geqslant 0$. In either case We circle b_i if it's inequality is an equality. In other words we circle b_i if (in the first case) $b_i = b_{i+1}$ or (in the second case) $b_i = 0$.

For the boxing rule, we box b_i if $e_{\omega_{i+1}} f_{\omega_i}^{b_i} \cdots f_{\omega_1}^{b_1}(v) = 0$.

Let us consider an example. We take $(\omega_1, \omega_2, \omega_1) = (2, 1, 2)$ and v to be the element $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ in the crystal $\mathcal{B}_{(3,1,0)}$ that is highlighted in Figure 1. Then it is easy to see that

$$\mathrm{BZL}(v) = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}.$$

We decorate this as follows. Since $b_2 = b_3$, we circle b_2. Moreover $e_{\omega_1}(v) = 0$, since (referring to the crystal graph) there is no way to move in the e_1 direction. Thus b_1 is boxed and the decorated BZL pattern looks like this:

$$\mathrm{BZL}(v) = \begin{bmatrix} \textcircled{1} & 1 \\ & \boxed{1} \end{bmatrix}$$

The boxing and circling rules may seem artificial, but are actually natural, for they have the following interpretation. Kashiwara defined, in addition to the crystals $\mathcal{B}_\lambda$ corresponding to the finite-dimensional irreducible representations, a crystal $\mathcal{B}_\infty$ which is a crystal basis of the quantized universal eveloping algebra of the lower unipotent part of the Lie algebra of G. There is another crystal $\mathcal{T}_\lambda$ with precisely one element having weight λ. Then $\mathcal{B}_\infty \otimes \mathcal{T}_{\lambda+\rho}$ is a "universal" crystal with a highest weight vector having weight $\lambda+\rho$. There is then a morphism of crystals $\mathcal{B}_{\lambda+\rho} \longrightarrow \mathcal{B}_\infty \otimes \mathcal{T}_{\lambda+\rho}$. See Kashiwara [41]. This morphism may be made explicit by adopting a viewpoint similar to Littelmann [47]. Indeed, the notion of BZL patterns makes sense for $\mathcal{B}_\infty \otimes \mathcal{T}_{\lambda+\rho}$, and the set of BZL patterns is precisely the cone of patterns (34) such that

$$b_1 \geqslant 0, \qquad b_2 \geqslant b_3 \geqslant 0, \qquad b_4 \geqslant b_5 \geqslant b_6 \geqslant 0, \qquad \cdots .$$

The morphism $\mathcal{B}_{\lambda+\rho} \longrightarrow \mathcal{B}_\infty \otimes \mathcal{T}_{\lambda+\rho}$ is then characterized by the condition that corresponding elements of the two crystals have the same BZL pattern. See Figure 1 of Bump and Nakasuji [23] for a picture of this embedding.

Now the circling rule may be explained as follows: embedding $v \in \mathcal{B}_{\lambda+\rho}$ into $\mathcal{B}_\infty \otimes \mathcal{T}_{\lambda+\rho}$, an entry in the BZL pattern is circled if and only if $\mathrm{BZL}(v)$ lies on the boundary of the cone. Similarly there is a crystal $\mathcal{B}_{-\infty} \otimes \mathcal{T}_{w_0(\lambda+\rho)}$ with a unique lowest weight vector having weight $w_0(\lambda+\rho)$, and we may embed $B_{\lambda+\rho}$ into this crystal by matching up the lowest weight vectors, and an entry is boxed if and only if $\mathrm{BZL}(v)$ lies on the boundary of this opposite cone.

Returning to Tokuyama's formula, define

$$G^\flat(v) = \prod_{i=1}^{N} \left\{ \begin{array}{ll} 1 & \text{if } b_i \text{ is circled but not boxed;} \\ -q^{-1} & \text{if } b_i \text{ is boxed but not circled;} \\ 1-q^{-1} & \text{if } b_i \text{ is neither circled nor boxed;} \\ 0 & \text{if } b_i \text{ is both circled and boxed.} \end{array} \right\}$$

This was generalized to the $n \geqslant 1$ case as follows. There is, in this generality, a Whittaker function on the metaplectic group, and as in the case $n=1$, we have

$$\delta^{-1/2}(a_\lambda) W^\circ(a_\lambda) = \sum_{v \in \mathcal{B}_{\lambda+\rho}} G^\flat(v) \boldsymbol{z}^{\mathrm{wt}(v) - w_0\rho}. \tag{35}$$

Now the definition of $G^\flat$ must be slightly changed. Let a be a positive integer. Define $g(a) = q^{-a} g(p^{a-1}, p^a)$ and $h(a) = q^{-a} g(p^a, p^a)$ in terms of the Gauss sum discussed in the last section. These depend only on a modulo n. Then we have

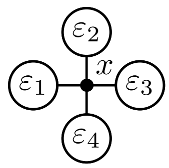
$$G^\flat(v) = \prod_{i=1}^{N} \left\{ \begin{array}{ll} 1 & \text{if } b_i \text{ is circled but not boxed;} \\ g(b_i) & \text{if } b_i \text{ is boxed but not circled;} \\ h(g_i) & \text{if } b_i \text{ is neither circled nor boxed;} \\ 0 & \text{if } b_i \text{ is both circled and boxed.} \end{array} \right\}$$

If $n=1$, this reduces to our previous definition.

Tokuyama's formula may be given another interpretation, as evaluating the partition function of a statistical mechanical system of the free-fermionic six-vertex model. For this, see [15], Chapter 19 of [14], and the paper [8] in this volume. Since more details may be found in these references, we will be brief. In a statistical mechanical system, there is given a collection of (many) *states*, and each state is assigned a *Boltzmann weight* which is a measure of how energetic the state is; more more highly energetic states are less probable.

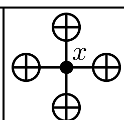
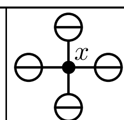
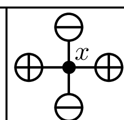
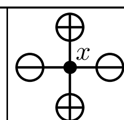
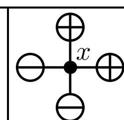
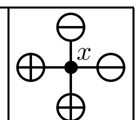
For example, the two-dimensional Ising model consists of a collection of sites, each of which may be assigned a spin $+$ or $-$. These sites might represent atoms in a ferromagnetic substance, and in the two dimensional model they lie in a plane. A state of the system consists of an assignment of spins to every site. At each site, there is a local Boltzmann weight, depending on spin at the site and at its nearest neighbors. This system was analyzed by Onsager, who found, surprisingly, that the partition function could be evaluated explicitly.

Later investigators considered models in which the spins are assigned not to the sites themselves, but to edges in a grid connecting the sites. The Boltzmann weight at the vertex depends on the configuration of spins on the four edges adjacent to the vertex. Thus if the site is x, we label the four adjacent edges with $\varepsilon_i = +$ or $-$ by the following scheme:



(36)

Of particular interest to us is the *six-vertex* or *two-dimensional ice model* which was solved by Lieb and Sutherland in the 1960's, though it is the treatment of Baxter [1] that is most important to us. There are six admissible configurations. These are given by the following table.

						
Boltzmann weight	$a_1 = a_1(x)$	a_2	b_1	b_2	c_1	c_2

The set of Boltzmann weights used may vary from site to site, so if (as in the table) the site is x we may write $a_1(x)$ to indicate this dependence.

If $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$ then the site is called *field-free*. If $a_1 a_2 + b_1 b_2 = c_1 c_2$, the site is called *free-fermionic*. The term comes from physics: in the free-fermionic case, the row transfer matrices (see [8]) are differentiated versions of the Hamiltonians of a quantum mechanical system, the XXZ model, and the quanta for this model are particles of spin 1/2, called fermions.

Hamel and King [38] and Brubaker, Bump and Friedberg [15] showed that one may exhibit a free-fermionic six-vertex model whose partition function is exactly the Tokuyama expression (33). In [8] in this volume, this is generalized to a system whose partition function is the metaplectic spherical Whittaker function (35). The explanation for this is as follows: there is a map from the set of states of the model to the $\mathcal{B}_{\lambda+\rho}$ crystal. The map is not surjective, but its image is precisely the set of $v \in \mathcal{B}_{\lambda+\rho}$ such that $G^\flat(v) \neq 0$.

Thus using this bijection of the set of states with those $v \in \mathcal{B}_{\lambda+\rho}$ such that $G^\flat(v) \neq 0$ means that Tokuyama's theorem may be formulated either as the evaluation of a sum over a crystal or as the partition function of a statistical-mechanical system. But there is a subtle and important difference between these two setups. For example, different tools are available. There is a an automorphism of the crystal graph, the *Schützenberger involution*, that takes a vertex of weight μ to one of weight $w_0\mu$, where w_0 is the long Weyl group element; this is sometimes useful in proofs. The set of states of the statistical-mechanical system has no such involution, yet another, more powerful tool becomes available: the *Yang-Baxter equation*.

To describe it, let us associate a matrix with the Boltzmann weights at a site as follows. Let V be a two-dimensional vector space with basis v_+ and v_-. (In the metaplectic case, the scheme proposed in [8] gives its dimension as $2n$.) Associate with each site an endomorphism of $V \otimes V$ as follows. With the vertices labeled as in (36), let R be the linear transformation such that the coefficient of $v_{\varepsilon_3} \otimes v_{\varepsilon_4}$ in $R(v_{\varepsilon_1} \otimes v_{\varepsilon_2})$ is the Boltzmann weight corresponding to the four spins $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$. Thus the only nonzero entries in the matrix of R with respect to the basis $v_\pm \otimes v_\pm$ are $a_1, a_2, b_1, b_2, c_1, c_2$. We will call an endomorphism R of $V \otimes V$ (or its matrix) an *R-matrix*. We then denote by R_{12}, R_{13} and R_{23} endomorphisms of $V \otimes V \otimes V$ in which R_{ij} acts on the i-th and j-th components, and the identity 1_V acts on the remaining one. For example $R_{12} = R \otimes 1_V$.

We are interested in endomorphisms R, S and T of $V \otimes V$ such that

$$R_{12}S_{13}T_{23} = T_{23}S_{13}R_{12}.$$

It is not hard to check that this is equivalent to (25) in [8]. This was called the *star-triangle relation* by Baxter, and the *Yang-Baxter equation* by others, particularly in the case where R, S, T are either all equal, or drawn from the same parametrized family. In particular let Γ be a group, and $g \longmapsto R(g)$ a map from Γ into the set of R-matrices such that

$$R_{12}(g)R_{13}(gh)R_{23}(h) = R_{23}(h)R_{13}(gh)R_{12}(g). \tag{37}$$

Then (37) is called a *parametrized Yang-Baxter equation*.

In Section 9.6 of [1], Baxter essentially found parametrized Yang-Baxter equations in the *field-free* case, where $a_1 = a_2 = a$, $b_1 = b_2 = b$ and $c_1 = c_2 = c$. Fix a complex number Δ. Then his construction gives a parametrized Yang-Baxter equation, with parameter group $\mathbb{C}^\times$, such that the image of R consists of endomorphisms of $V \otimes V$ with corresponding to such field-free R-matrices with $(a^2+b^2-c^2)/2ab = \Delta$. This construction led to the development of quantum groups. In the formulation of Drinfeld [33], this instance of the Yang-Baxter equation is related to Hopf algebra $U_q(\widehat{\mathfrak{sl}}_2)$. The parameter group indexes modules of this Hopf algebra with $\Delta = \frac{1}{2}(q+q^{-1})$, and Yang-Baxter equation is a consequence of a property (quasitriangularity) of $U_q(\widehat{\mathfrak{sl}}_2)$, or a suitable completion.

A parametrized Yang-Baxter equation with parameter group $\mathrm{SL}(2,\mathbb{C})$ was given in Korepin, Boguliubov and Izergin [44], p. 126. The parametrized R-matrices are contained within the free-fermionic six-vertex model. Scalar R-matrices may be added trivially, so the actual group is $\mathrm{SL}(2,\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})$. This nonabelian parametrized Yang-Baxter equation was rediscovered in slightly greater generality by Brubaker, Bump and Friedberg [15], who found a parametrized Yang-Baxter equation for the entire set of R-matrices in the free-fermionic six-vertex model, with parameter group $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$. It is an interesting question how to formulate this in terms of a Hopf algebra, analogous to the field-free case.

Brubaker, Bump and Friedberg [15] showed that a system may be found, with free-fermionic Boltzmann weights, whose partition function is precisely (33). This fact was generalized Bump, McNamara and Nakasuji [22], who showed that one may replace the character on the left-hand side by a factorial Schur function. Then the parametrized free-fermionic Yang-Baxter equation can be used to prove Tokuyama's formula (or its generalization to factorial Schur functions). Moreover, in [8] a different generalization is given, in which the partition function represents the metaplectic Whittaker function. In the latter case, however, no Yang-Baxter equation is known if $n > 1$.

When $n = 1$, various facts about Whittaker functions may be proved using the free-fermionic Yang-Baxter equation. One fact that may be checked is that the partition function representing (33), divided by the product on the left-hand side of the equation, is symmetric, in other words invariant under permuting the eigenvalues of $\boldsymbol{z}$. This is a step in a proof of Tokuyama's theorem. As explained in [8], this fact has a generalization to partition functions representing metaplectic Whittaker functions, and seems amenable to the Yang-Baxter equation, but no Yang-Baxter equation is known in this case.

In (35), the definition of $G^\flat$ depends on the choice of a reduced word representing the long Weyl group element. Two particular long words are considered, and the Yang-Baxter equation is used to show that both representations give the same result. If $n > 1$, this remains true, but again the Yang-Baxter equation is unavailable. Consequently different proofs, based on the Schützenberger involution of the crystal $\mathcal{B}_{\lambda+\rho}$ are given. However these arguments require extremely difficult combinatorial arguments, and it would be good to have an alternative approach based on the Yang-Baxter equation.

See [10] another application of the free-fermionic Yang-Baxter equation to metaplectic Whittaker functions, this time on the double cover of $\mathrm{Sp}(2r)$.

8 Demazure operators

Let (π, V) be a principal series representation of $G(F)$, where G is a split semisimple Lie group. The theory described above, including the Casselman-

Shalika formula (and its metaplectic generalizations) is for the spherical Whittaker function, that is, the K-fixed vector in the Whittaker model, where $K = G(\mathfrak{o})$.

Let J be the *Iwahori subgroup*, which is the inverse image of $B(\mathbb{F}_q)$ under the map $G(\mathfrak{o}) \longrightarrow G(\mathbb{F}_q)$ that is reduction mod p. We may consider more generally the space V^J of J-fixed vectors in the Whittaker model. These play an important role in the proof of the Casselman-Shalika formula which, we have seen, is a key result in the above discussion.

Until 2011, the investigations that we have been discussing in the above pages concentrated on the unique (up to scalar) K-fixed vector, rather than elements of V^J, though the Iwahori fixed vectors appeared in the work of Chinta and Offen [29] and McNamara [49] generalizing Casselman and Shalika. Still, the essence of the Casselman-Shalika proof is to finesse as much as possible in order to avoid getting involved with direct calculations of Iwahori Whittaker functions. But it turns out that there is an elegant calculus of Iwahori Whittaker functions, and this is likely to be a key to the relationship between the theory of Whittaker functions and combinatorics.

If $w \in W$, the Demazure operator ∂_w acts on the ring $\mathcal{O}(\hat{T})$ of rational functions on $\hat{T}$. To define it, first consider the case where $w = \sigma_i$ is a simple reflection. Then if f is a rational function on $\hat{T}(\mathbb{C})$,

$$\partial_{\sigma_i} f(\boldsymbol{z}) = \frac{f(\boldsymbol{z}) - \boldsymbol{z}^{-\alpha_i} f(\sigma_i \boldsymbol{z})}{1 - \boldsymbol{z}^{-\alpha_i}}.$$

The numerator is divisible by the denominator, so this is again a rational function. The definition of ∂_w is completed by the requirement that if $l(ww') = l(w) + l(w')$, where l is the length function on W, then $\partial_{ww'} = \partial_w \partial_{w'}$.

If λ is a dominant weight then $\partial_{w_0} \boldsymbol{z}^\lambda$ is the character $\chi_\lambda(\boldsymbol{z})$, and for general w we will call $\partial_w \boldsymbol{z}^\lambda$ a *Demazure character*. These first arose in the cohomology of line bundles over Schubert varieties, and they have proved to be quite important in combinatorics. As Littelmann and Kashiwara showed, they may be interpreted as operators on functions on crystals. As we will explain, Demazure operators, and the related Demazure-Lusztig operators arise naturally in the theory of Whittaker functions.

Iwahori and Matsumoto observed that V^J is naturally a module for the convolution ring of compactly supported J-biinvariant functions, and they determined the structure of this ring. Later Bernstein, Zelevinsky and Lusztig gave a different presentation of this ring. It is the (extended) affine Hecke algebra $\tilde{\mathcal{H}}_q$, and it has also turned out to be a key object in combinatorics independent of its origin in the representation theory of p-adic groups. Restricting ourselves to the semisimple case for simplicity, this algebra may be defined as follows. It contains a $|W|$-dimensional subalgebra $\mathcal{H}_q$ with generators $T_1, \cdots, T_r$ subject to the quadratic relations

$$T_i^2 = (q-1)T_i + q$$

together with the braid relations: when $i \neq j$,

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots$$

where the number of terms on either side is the order of $\sigma_i \sigma_j$, and as before σ_i is the i-th simple reflection.

The algebra $\widetilde{\mathcal{H}}_q$ is the amalgam of $\mathcal{H}_q$ with an abelian subalgebra ζ^Λ isomorphic to the weight lattice Λ. If $\lambda \in \Lambda$ let ζ^λ be the corresponding element of ζ^Λ. To complete the presentation of $\widetilde{\mathcal{H}}$ we have the relation

$$T_i \zeta^\lambda - \zeta^{\sigma_i \lambda} T_i = \zeta^\lambda T_i - T_i \zeta^{\sigma_i \lambda} = \left(\frac{v-1}{1-\zeta^{-\alpha_i}} \right) (\zeta^\lambda - \zeta^{\sigma_i \lambda}), \tag{38}$$

sometimes known as the *Bernstein relation*.

Though historically it first appeared in the representation theory of p-adic groups, the affine Hecke algebra appears in other contexts. For example, the investigation of Kazhdan and Lusztig [42], motivated by Springer's work on the representation theory of Weyl groups, used $\widetilde{\mathcal{H}}_q$ in a fundamental way, and led to applications in different areas of mathematics, such as the topology of flag varieties and the structure of Verma modules. Significantly for the present discussion, Lusztig [48] showed that $\widetilde{\mathcal{H}}_v$ (with v an indeterminate) may be realized as a ring acting on the equivariant K-theory of the flag manifold of $\hat{G}$, and Kazhdan and Lusztig [48], [43] then applied this back to the local Langlands correspondence by constructing the irreducible representations of $G(F)$ having an Iwahori fixed vector.

The equivariant K-theory of $\hat{G}$ may be described as follows. Let $\mathcal{O}(\hat{T})$ be the ring of rational functions on $\hat{T}(\mathbb{C})$. In our previous notation, it is simply the group algebra of the weight lattice Λ. If X is the flag variety of $\hat{G}$ then $K_{\hat{G}}(X) \cong \mathcal{O}(\hat{T})$. Better still, let $M = \hat{G} \times \mathrm{GL}_1$, where the GL_1 acts trivially on X. Then $K_M(X) \cong \mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ where v is a parameter.

The starting point of the investigations of Kazhdan and Lusztig is a representation of $\widetilde{\mathcal{H}}_v$ on this ring. In this representation on $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ the generators of $\mathcal{H}_v$ act by certain operators called *Demazure-Lusztig operators*, while the commutative subalgebra ζ^Λ acts by multiplication. We will call this representation of $\widetilde{\mathcal{H}}_v$ on $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ the *Lusztig representation*.

The same representation of $\widetilde{\mathcal{H}}$ appears in another way, independent of Lusztig's cohomological interpretation. There are two versions of this.

- Ion [39] observed such a representation in the space of Iwahori fixed vectors of the spherical model of an unramified principal series representation. He concluded that these matrix coefficients are expressed in terms of the non-symmetric Macdonald polynomials. His methods are based on the double affine Hecke algebra.
- Brubaker, Bump and Licata [16] found a representation equivalent to the Lusztig representation acting on Whittaker functions. Their method could also be used in the setting of [39].

After Brubaker, Bump and Licata mentioned the connection between Whittaker functions and Demazure characters, Chinta and Gunnells began looking at the metaplectic case. They found "metaplectic Demazure operators" involving Gauss sums that are related to the Chinta-Gunnells representation. Also, with A. Schilling, Brubaker, Bump and Licata looked at the possibility that the results of [16] could be reinterpreted in terms of the crystal graph, similarly to the crystal interpretation of Tokuyama's formula. This seems to be a promising line of investigation.

Let us briefly recall the results of [16]. Let $V = V_{\boldsymbol{z}}$ be as in (27). Let Ω be one of the following two linear functionals on V: it is either the Whittaker functional

$$\Omega(f) = \int_{N(F)} f(wn)\,\psi(n)\,dn,$$

where ψ is as in (28), or the spherical functional $\Omega(f) = \int_K f(k)\,dk$. If $w \in W$ let Φ_w be the element of V^J defined as follows. Every element of $G(F)$ may be written as $bw'k$ with $b \in B(F)$, $w' \in W$ and $k \in J$. Then, with $\tau_{\boldsymbol{z}}$ as in (26),

$$\Phi_w(bw'k) = \begin{cases} \delta^{1/2}\tau_{\boldsymbol{z}}(b) & \text{if } w = w', \\ 0 & \text{otherwise.} \end{cases}$$

The $|W|$ functions Φ_w are a basis of the space of J-fixed vectors in V. (The action $\pi : G(F) \to \mathrm{End}(V)$ is by right translation.) We also define

$$W_w(g) = \Omega(\pi(g)\Phi_w).$$

$$\tilde{\Phi}_w = \sum_{u \geqslant w} \Phi_u, \qquad \tilde{W}_w = \sum_{u \geqslant w} W_u$$

where $u \geqslant w$ is with respect to the Bruhat order.

Let λ be a weight; if Ω is the Whittaker functional, we require λ to be dominant. We may regard $W_w(a_\lambda)$ as an element of $\mathbb{C}[q, q^{-1}] \otimes \mathcal{O}(\hat{T})$. Then there exist operators $\mathcal{T}_i$ on $\mathbb{C}[q, q^{-1}] \otimes \mathcal{O}(\hat{T})$ such that

$$\mathcal{T}_i^2 = (q^{-1} - 1)\mathcal{T}_i + q^{-1}$$

and which also satisfy the braid relations. Therefore we obtain a representation of $\mathcal{H}_{q^{-1}}$ on $\mathbb{C}[q, q^{-1}] \otimes \mathcal{O}(\hat{T})$. It may be extended to an action of $\tilde{\mathcal{H}}_{q^{-1}}$. Now if the simple reflection σ_i is a left descent of $w \in W$, that is, $l(\sigma_i w) < l(w)$, then

$$W_{\sigma_i w}(a_\lambda) = \mathcal{T}_i W_w(a_\lambda).$$

(See [16]). The operators $\mathfrak{T}_i$ are slightly different in the two cases (Ω the Whittaker or Spherical functional.) In both cases they are essentially the Demazure-Lusztig operators. For definiteness, we will describe them when Ω is the Whittaker functional. If f is a function on $\hat{T}(\mathbb{C})$, define

$$\partial_i' f(\boldsymbol{z}) = \frac{f(\boldsymbol{z}) - \boldsymbol{z}^{\alpha_i} f(\sigma_i \boldsymbol{z})}{1 - \boldsymbol{z}^{\alpha_i}} = \frac{f(\sigma_i \boldsymbol{z}) - \boldsymbol{z}^{-\alpha_i} f(\boldsymbol{z})}{1 - \boldsymbol{z}^{-\alpha_i}}.$$

This is the usual Demazure operator conjugated by the map $\boldsymbol{z} \mapsto -\boldsymbol{z}$. Then the operators $\mathfrak{T}_i$ are given by

$$\mathfrak{D}'_i = (1 - q^{-1}\boldsymbol{z}^{\alpha})\partial'_{\alpha}, \qquad \mathfrak{T}'_i = \mathfrak{D}'_i - 1.$$

The Whittaker functions W_w can thus be obtained from W_{w_0} by applying the $\mathfrak{T}_i$. Moreover W_{w_0} has a particularly simple form:

$$W_{w_0}(a_\lambda) = \begin{cases} \delta^{1/2}(a_\lambda)\boldsymbol{z}^{w_0\lambda} & \text{if } \lambda \text{ is dominant,} \\ 0 & \text{otherwise.} \end{cases}$$

In conclusion, the Lusztig representation arises naturally in the theory of Whittaker functions or, in Ion's setup, K, J-biinvariant matrix coefficients. It gives a calculus whereby the Whittaker functions may be computed recursively from the simplest one W_{w_0}.

It is also important to consider $\tilde{W}_w$. For example, $\tilde{W}_1$ is the spherical Whittaker function which we have discussed at length in the previous sections. In the theory of multiple Dirichlet series it might be useful to substitute $\tilde{W}_w$ for the p-part at a finite number of places. In the study of the $\tilde{W}_w$ the remarkable combinatorics of the Bruhat order begins to play an important role. See [16] for further information. An important issue is to extend the theory of the previous sections to the theory of the $\tilde{W}_w$, and to carry out this unified theory in the metaplectic context.

References

1. R. Baxter. *Exactly solved models in statistical mechanics.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982.
2. J. Beineke, B. Brubaker, and S. Frechette. Weyl group multiple Dirichlet series of type C. *Pacific J. Math.*, To Appear.
3. J. Beineke, B. Brubaker, and S. Frechette. A crystal description for symplectic multiple Dirichlet series, in this volume.
4. A. Berenstein and A. Zelevinsky. Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics. *Duke Math. J.*, 82(3):473–502, 1996.
5. S. Bochner. A theorem on analytic continuation of functions in several variables. *Ann. of Math. (2)*, 39(1):14–19, 1938.
6. B. Brubaker, A. Bucur, G. Chinta, S. Frechette and J. Hoffstein. Nonvanishing twists of GL(2) automorphic L-functions. *Int. Math. Res. Not.*, 78:4211–4239, 2004.
7. B. Brubaker and D. Bump. Residues of Weyl group multiple Dirichlet series associated to $\widetilde{\mathrm{GL}}_{n+1}$. In *Multiple Dirichlet series, automorphic forms, and analytic number theory*, volume 75 of *Proc. Sympos. Pure Math.*, pages 115–134. Amer. Math. Soc., Providence, RI, 2006.
8. B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and G. Gunnells. Metaplectic ice, in this volume.

9. B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series. I. In *Multiple Dirichlet series, automorphic forms, and analytic number theory*, volume 75 of *Proc. Sympos. Pure Math.*, pages 91–114. Amer. Math. Soc., Providence, RI, 2006.
10. B. Brubaker, D. Bump, G. Chinta, and G. Gunnells. Metaplectic Whittaker functions and crystals of type B, in this volume.
11. B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple dirichlet series. ii. the stable case. *Invent. Math.*, 165(2):325–355, 2006.
12. B. Brubaker, D. Bump, and S. Friedberg. Twisted Weyl group multiple Dirichlet series: the stable case. In *Eisenstein series and applications*, volume 258 of *Progr. Math.*, pages 1–26. Birkhäuser Boston, Boston, MA, 2008.
13. B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series, Eisenstein series and crystal bases. *Ann. of Math. (2)*, 173(2):1081–1120, 2011.
14. B. Brubaker, D. Bump, and S. Friedberg. *Weyl group multiple Dirichlet series: type A combinatorial theory*, volume 175 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2011.
15. B. Brubaker, D. Bump, and S. Friedberg. Schur polynomials and the Yang-Baxter equation. *Comm. Math. Phys.*, 308(2):281-301, 2011.
16. B. Brubaker, D. Bump, and A. Licata. Whittaker functions and Demazure operators. *Preprint*, 2011.
17. B. Brubaker, S. Friedberg, and J. Hoffstein. Cubic twists of GL(2) automorphic L-functions. *Invent. Math.*, 160(1):31–58, 2005.
18. Ben Brubaker and Daniel Bump. On Kubota's Dirichlet series. *J. Reine Angew. Math.*, 598:159–184, 2006.
19. Alina Bucur and Adrian Diaconu. Moments of quadratic Dirichlet L-functions over rational function fields. *Mosc. Math. J.*, 10(3):485–517, 661, 2010.
20. D. Bump, S. Friedberg, and J. Hoffstein. On some applications of automorphic forms to number theory. *Bull. Amer. Math. Soc. (N.S.)*, 33(2):157–175, 1996.
21. D. Bump, S. Friedberg, and J. Hoffstein. Sums of twisted GL(3) automorphic L-functions. In *Contributions to automorphic forms, geometry, and number theory*, pages 131–162. Johns Hopkins Univ. Press, Baltimore, MD, 2004.
22. D. Bump, P. McNamara and M. Nakasuji. Factorial Schur Functions and the Yang-Baxter Equation, Preprint, 2011.
23. D. Bump and M. Nakasuji. Integration on p-adic groups and crystal bases. *Proc. Amer. Math. Soc.*, 138(5):1595–1605, 2010.
24. W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.
25. G. Chinta. Mean values of biquadratic zeta functions. *Invent. Math.*, 160(1):145–163, 2005.
26. G. Chinta, S. Friedberg, and J. Hoffstein. Multiple Dirichlet series and automorphic forms. In *Multiple Dirichlet series, automorphic forms, and analytic number theory*, volume 75 of *Proc. Sympos. Pure Math.*, pages 3–41. Amer. Math. Soc., Providence, RI, 2006.
27. G. Chinta and P. Gunnells. Constructing Weyl group multiple Dirichlet series. *J. Amer. Math. Soc.*, 23:189–215, 2010.
28. G. Chinta and P. Gunnells. Littelmann patterns and Weyl group multiple Dirichlet series of type D, in this volume.
29. G. Chinta and O. Offen. A metaplectic Casselman-Shalika formula for GL_r, *Amer. J. Math.*, to appear.

30. G. Chinta and O. Offen. Orthgonal period of a GL(3,Z) Eisenstein series. In Offen Krötz and Sayan, editors, *Representation Theory, Complex Analysis and Integral Geometry I.* Birkhäuser, to appear.
31. A. Diaconu, P. Garrett, and D. Goldfeld. Natural boundaries and integral moments of L-functions, in this volume.
32. C. A. Diaconu, D. Goldfeld, and J. Hoffstein. Multiple Dirichlet series and moments of zeta and L-functions. *Compositio Math.*, 139(3):297–360, 2003.
33. V. G. Drinfeld. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 798–820, Providence, RI, 1987. Amer. Math. Soc.
34. C. Eckhardt and S. J. Patterson. On the Fourier coefficients of biquadratic theta series. *Proc. London Math. Soc. (3)*, 64(2):225–264, 1992.
35. B. Fisher and S. Friedberg. Sums of twisted GL(2) L-functions over function fields. *Duke Math. J.*, 117(3):543–570, 2003.
36. S. Friedberg, J. Hoffstein, and D. Lieman. Double Dirichlet series and the n-th order twists of Hecke L-series. *Math. Ann.*, 327(2):315–338, 2003.
37. D. Goldfeld and J. Hoffstein. Eisenstein series of 1/2-integral weight and the mean value of real Dirichlet L-series. *Invent. Math.*, 80(2):185–208, 1985.
38. A. M. Hamel and R. C. King. Bijective proofs of shifted tableau and alternating sign matrix identities. *J. Algebraic Combin.*, 25(4):417–458, 2007.
39. B. Ion. Nonsymmetric Macdonald polynomials and matrix coefficients for unramified principal series. *Adv. Math*, 201(1):36–62, 2006.
40. D. Ivanov. Symplectic ice, in this volume, 2011.
41. M. Kashiwara. On crystal bases. In *Representations of groups (Banff, AB, 1994)*, volume 16 of *CMS Conf. Proc.*, pages 155–197. Amer. Math. Soc., Providence, RI, 1995.
42. D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.
43. D. Kazhdan and G. Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Invent. Math.*, 87(2):153–215, 1987.
44. V. Korepin, N. Boguliubov and A. Izergin *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1993.
45. Y. Komori, K. Matsumoto, and H. Tsumura. On Witten multiple zeta-functions associated with semisimple lie algebras iii. in this volume.
46. T. Kubota. *On automorphic functions and the reciprocity law in a number field.* Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 2. Kinokuniya Book-Store Co. Ltd., Tokyo, 1969.
47. P. Littelmann. Cones, crystals, and patterns. *Transform. Groups*, 3(2):145–179, 1998.
48. G. Lusztig. Equivariant K-theory and representations of Hecke algebras. *Proc. Amer. Math. Soc.*, 94(2):337–342, 1985.
49. P. J. McNamara. The metaplectic Casselman-Shalika formula. *Preprint*, 2011. `http://arxiv.org/abs/1103.4653.`
50. P. J. McNamara. Metaplectic Whittaker functions and crystal bases. *Duke Math. J.*, 156(1):29–31, 2011.
51. J. Neukirch. *Class field theory*, volume 280 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1986.
52. S. J. Patterson Excerpt from an unwritten letter, in this volume.

53. A. Reznikov. A double Dirichlet series for Hecke L-functions of a CM field. *Preprint*, 2010.
54. T. Tokuyama. A generating function of strict Gelfand patterns and some formulas on characters of general linear groups. *J. Math. Soc. Japan*, 40(4):671–685, 1988.