Moments of the Riemann Zeta Function and Eisenstein Series II

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Abstract. The fourth moment of the Riemann zeta function and the second moment of the L-function of a Maass cusp form are studied using a construction of Epstein, Hafner and Sarnak.

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In [BB2], we argue that the $2k$-th moments of the Riemann zeta function should be approached using Eisenstein series on $GL(2k)$. Yet it may also be possible to study the $2k$-th moments of ζ using Eisenstein series on $GL(k)$. In this paper we consider an automorphic approach to the fourth moment of ζ, and also to the second moment of the L-function of a Maass cusp form. Our estimates are not new, but the approach (based on a construction of Epstein, Hafner and Sarnak [EHS]) may be of interest.

Using an Eisenstein series on $GL(2)$, we showed in [BB2] how it is possible to obtain fairly precise asymptotics for the second moment of ζ. In the present paper we will show that it is also possible to obtain a sharp estimate for the *fourth* moment from this same $GL(2)$ Eisenstein series. We will show that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt = O(T\log^4(T)).$$

This bound was first obtained by Hardy and Littlewood [HL]. Although it is sharp, it is not very precise, for Ingham [I] proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt = \frac{1}{2\pi^2} T\log^4(T) + O(T\log^3(T)).$$

Before we consider the case of an Eisenstein series, we consider the second moment of the L-function of a Maass cusp form. We obtain a sharp bound in this easier case using this same method. Again, more precise estimates are already in the literature. For the analogous problem about holomorphic modular forms, the second moment was considered by Good [G1], [G2] and [G3]. Using an approximate functional equation in the first two papers, he was able to show that

$$\int_0^T |L(f, \tfrac{k}{2}+it)|^2\, dt = cT\, \log(T) + dT + O((T\log(T))^{2/3})$$

for constants c and d, and deduced that

$$L(f, \tfrac{k}{2}+it) = O(|t|^{1/3}\log^{5/6}|t|). \tag{1}$$

The paper [G3] gives an alternative approach to the second moment of a $GL(2)$ cusp form based on a variant of the Rankin-Selberg method. The second moments of Maass forms have been estimated by Kuznetsov [K] and Jutila [J]. The analog of (1) was obtained by Meurman [Meu].

The method of the present paper is adapted from Epstein, Hafner and Sarnak [EHS], who used the same construction to study zeros on the half line for L-functions of Maass forms. It is also similar to that of Sarnak [S], who considers the fourth moments of Grössencharacter L-functions over the field $\mathbb{Q}(i)$.

There are a couple of points where (because we are considering a different problem) we must modify the approach taken in [EHS]. First, there it is assumed that the eigenvalue of the Laplacian on their Maass cusp form is not $\frac{1}{4}$. Since we are interested in the fourth moment of zeta we must take precisely the Eisenstein series with this eigenvalue. Because of this, the hypergeometric function which appears is not $F_{2,1}$ but a solution with a logarithmic singularity. Second, we find that a single value of their parameter θ is not sufficient. We find that two distinct values of θ are needed to prevent a phase cancellation between two contributions in the "denominator" Φ.

Although better bounds for the moments can be obtained by different methods, we think this method is interesting because it is a *direct* approach based on an integral representation of the L-function, rather than one based on the Kuznetsov trace formula or approximate functional equations. We would characterize the approaches of Sarnak [S] and Good [G3] as direct in this sense and therefore having good potential for generalization to higher moments. Both these methods apply to the fourth moment. The method of Good [G3] is under development by Diaconu and Goldfeld.

In a statement which might apply to any automorphic attack on the moment problem, Sarnak [S] commented "It seems likely that if these methods were pushed through for $GL(k,\mathbb{Z})$ one would obtain sharp mean value estimates for $|\zeta(\frac{1}{2}+it)|^{2k}$ ...." Although this has not yet been done, this hope still seems possible.

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1. The case of a Maass cusp form. Let

$$f(z) = \sum_{n \neq 0} a_n \sqrt{y}\, K_{i\nu}(2\pi |n| y)\, \cos(2\pi n x)$$

be an even Maass cusp form for $SL(2,\mathbb{Z})$. (Thus $a_n = a_{-n}$.) Here $\frac{1}{4} + \nu^2$ is the eigenvalue of the Laplacian, and since (as Selberg proved) the eigenvalue $\frac{1}{4} + \nu^2$ of an $SL(2,\mathbb{Z})$ cusp form is bounded below by $\frac{1}{4}$, ν is real and nonzero. Let $L(f,s) = \sum_{n=1}^{\infty} a_n\, n^{-s}$ be the L-function of f.

Theorem 1. *We have*

$$\int_0^T |L(f, \tfrac{1}{2} + it)|^2\, dt = O(T\, \log(T)). \tag{2}$$

We will prove this using a method similar to that of Sarnak [S]. Epstein, Hafner and Sarnak [EHS] apply the same technique to a different problem. They consider

$$\int_0^{\infty} f(re^{i\theta})\, r^{it}\, \frac{dr}{r}, \qquad 0 \leq \theta < \frac{\pi}{2}. \tag{3}$$

Using Gradshteyn and Ryzhik [GR] 6.699.4, (3) equals

$$\Phi(t,\theta,\nu)\, L(f, \tfrac{1}{2} + it)$$

where

$$\Phi(t,\theta,\nu) = \tfrac{1}{4}\, \pi^{-it-1/2}\, \Gamma\left(\frac{i\nu}{2} + \frac{it}{2} + \frac{1}{4}\right) \Gamma\left(-\frac{i\nu}{2} + \frac{it}{2} + \frac{1}{4}\right) \times$$
$$\sin(\theta)^{-it} F_{2,1}\left(\frac{i\nu}{2} + \frac{it}{2} + \frac{1}{4}, -\frac{i\nu}{2} + \frac{it}{2} + \frac{1}{4}; \frac{1}{2}; -\cot^2(\theta)\right)$$

By Parseval's identity,

$$\int_0^{\infty} |f(re^{i\theta})|^2\, \frac{dr}{r} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi(t,\theta,\nu)\, L(f, \tfrac{1}{2} + it)|^2\, dt. \tag{4}$$

Lemma 2. *There exists a constant C depending only on f such that*

$$\int_0^{\infty} |f(re^{i\theta})|^2\, \frac{dr}{r} \leq C \log \csc(\theta). \tag{5}$$

Proof. Using $f(z) = f(-1/z)$ and the fact that f is even, the integrand is invariant under $r \to r^{-1}$. Thus we must bound the integral from 1 to ∞, which we split into two parts. Since f is bounded

$$\int_1^{\csc(\theta)} |f(re^{i\theta})|^2 \frac{dr}{r} = O(\log \csc(\theta)).$$

On the other hand we estimate

$$\int_{\csc(\theta)}^{\infty} |f(re^{i\theta})|^2 \frac{dr}{r}$$

by replacing $r \to \csc(\theta)\, y$. It becomes

$$\int_1^{\infty} |f(iy + y \cot\, \theta)|^2 \frac{dy}{y} = O(1).$$

This proves (5). $\square$

In [EHS] it is shown that there exist constants M, c and ϵ such that if

$$(1-\epsilon)M \leq t\theta \leq M, \tag{6}$$

then

$$\left| F_{2,1}\left(\frac{s+it}{2}, \frac{1-s+it}{2}; \frac{1}{2}; -\cot^2(\theta)\right)\right| > ce^{\frac{\pi|t|}{2}}. \tag{7}$$

Using Stirling's formula, it follows that in this range

$$|\Phi(t,\theta,\nu)| \gg t^{-1/2}. \tag{8}$$

We choose $\theta = M/T$. The estimate (8) is then valid for $(1-\epsilon)T < t < T$. Then

$$\int_{(1-\epsilon)T}^{T} |L(f, \tfrac{1}{2} + it)|^2 \, dt \ll \frac{T}{2\pi} \int_{(1-\epsilon)T}^{T} |\Phi(t,\theta,\nu)\, L(f, \tfrac{1}{2} + it)|^2 \, dt. \tag{9}$$

Using (4) and (5), and noting that $\log \csc(\theta)$ is of the order $\log(T)$, we get

$$\int_{(1-\epsilon)T}^{T} |L(f, \tfrac{1}{2} + it)|^2 \, dt = O(T \, \log(T)).$$

We can break the interval from 0 to $\log(T)$ into intervals of length $-\log(1-\epsilon)$, and for $\log(t)$ in each of these intervals we have a similar estimate. The number of intervals is of the order $\log(T)$ giving us

$$\int_0^T |L(f, \tfrac{1}{2} + it)|^2 \, dt = O(T \log^2(T)).$$

We've missed (2) by a factor of $\log(T)$.

We will explain the reason for this failure before we consider how to remedy it. Figure 1 shows the graph of $|\Phi(60, \theta, 10)|$ as a function of θ. For general t and ν, the general nature of Φ will resemble this example provided $t > \nu$.

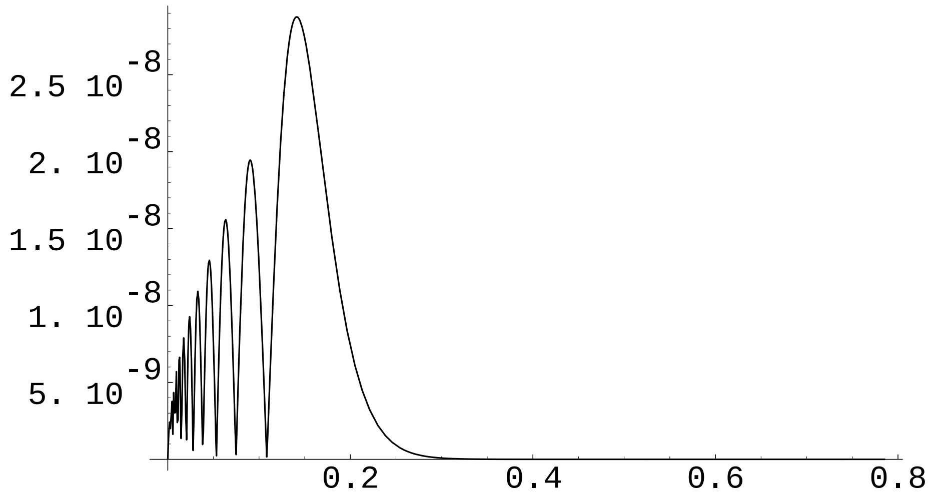

Figure 1. $|\Phi(60, \theta, 10)|$

The maximum is attained at $\theta = 0.141255$, far to the right of the value $2/(60e) = 0.01226$ taken by Epstein, Hafner and Sarnak [EHS]. If one wanted to prove (2) with the best implied constant, it would be better to use θ at the rightmost peak.

However this is not why we missed (2) by a power of $\log(T)$. To explain this point, Figures 2 and 3 show the graphs of $|\Phi|$ as a function of t with θ fixed.

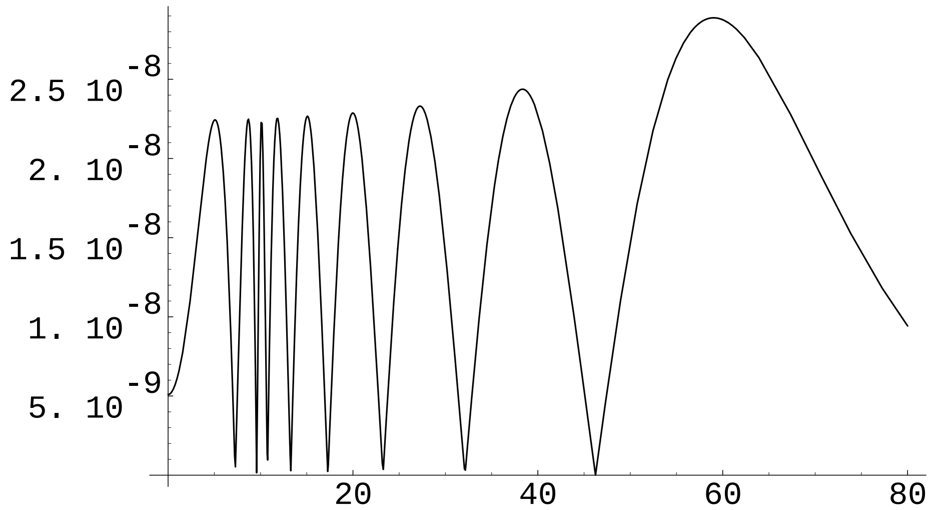

Figure 2. $|\Phi(t, 0.141225, 10)|$

Taking $\theta = 0.01226$ we get a similar picture (Figure 3).

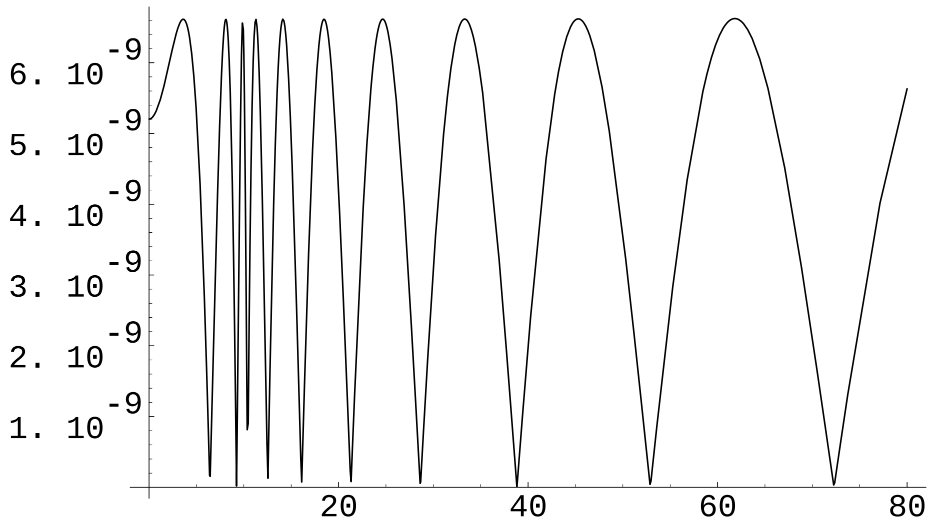

Figure 3. $|\Phi(t, 0.01226, 10)|$

Epstein, Hafner and Sarnak only proved the estimate (7) subject to (6) because with θ fixed, t must be kept within the width of a single oscillation of $\Phi(t, \theta, \nu)$. We would like to integrate in (9) from 0 to T, but the lower bound (8) for Φ is not true in this range, as a glance at Figure 2 or 3 will show. This is the reason we missed (2) by a factor of $\log(T)$.

In [EHS], the estimate (7) is proved by using the identity

$$(10)\quad \Phi(t,\theta,\nu) = \tfrac{1}{4}\,\pi^{-it-1/2}\,\Gamma\left(\frac{i\nu}{2}+\frac{it}{2}+\frac{1}{4}\right)\Gamma\left(-\frac{i\nu}{2}+\frac{it}{2}+\frac{1}{4}\right)\times$$
$$\sin(\theta)^{-it}F_{2,1}\left(\frac{i\nu}{2}+\frac{it}{2}+\frac{1}{4},-\frac{i\nu}{2}+\frac{it}{2}+\frac{1}{4};\frac{1}{2};-\cot^2(\theta)\right)=$$
$$\tfrac{1}{4}\,\pi^{-it}\Bigg[\sin(\theta)^{i\nu+\frac{1}{2}}\,\frac{\Gamma(-i\nu)\,\Gamma\left(\frac{i\nu+it+\frac{1}{2}}{2}\right)}{\Gamma\left(\frac{-i\nu-it+\frac{1}{2}}{2}\right)}F_{2,1}\left(\frac{i\nu+it+\frac{1}{2}}{2},\frac{i\nu-it+\frac{1}{2}}{2},i\nu+1,\sin^2(\theta)\right)+$$
$$\sin(\theta)^{-i\nu+\frac{1}{2}}\,\frac{\Gamma(i\nu)\,\Gamma\left(\frac{-i\nu+it+\frac{1}{2}}{2}\right)}{\Gamma\left(\frac{i\nu-it+\frac{1}{2}}{2}\right)}F_{2,1}\left(\frac{-i\nu+it+\frac{1}{2}}{2},\frac{-i\nu-it+\frac{1}{2}}{2},-i\nu+1,\sin^2(\theta)\right)\Bigg].$$

The reason that Epstein, Hafner and Sarnak used a value of θ well to the left of the rightmost peak was to stay within the range where the two hypergeometric functions on the right side of (10) can be estimated by their power series expansion.

Having explained why we failed to obtain $O(T\log(T))$, we may now explain how to remedy it. The oscillations in $\Phi(t,\theta,\nu)$ as a function of t come about because the two terms on the right side are themselves oscillating, and they go in and out of phase. We may obtain a satisfactory lower bound for Φ by taking two different values of θ differing by a suitable phase shift.

Lemma 3. *Assume that $0<\nu$ is real. There exist constants M_1, M_2, B and c depending only on ν such that if $\theta_1 = M_1/T$ and $\theta_2 = M_2/T$ then for $B<t<T$*

$$\max\left(|\Phi(t,\theta_1,\nu)|,|\Phi(t,\theta_2,\nu)|\right) > c\,T^{-1/2}.$$

Remark. This is also true when $\nu=0$, but must be proved separately. See Lemma 5.

Proof. As in [EHS], the estimate $F_{2,1}(a,b;c,z)=1+O(|abz/c|)$, valid when

$$|z|\max_{l\geq 0}\left|\frac{(a+l)(b+l)}{(c+l)(1+l)}\right| < \frac{1}{2}$$

is easily proved using the series definition of $F_{2,1}$. (Their (4.5) omits the $1+l$ in the denominator by mistake.) Thus as in their Proposition 4.2, if we choose M_1 and M_2 around $2/e$, we can use (10) and approximate $F_{2,1}$ by its constant term. If $t>B=\max(1,2\nu)$, then we may use Stirling's formula to estimate the Gamma functions in (10). Let $\Gamma(-i\nu)=g\,e^{iw}$ with $g>0$ and w real, and let

$$\alpha = (\nu+t)\log\left(\frac{\nu+t}{2}\right)-\nu-t-\frac{\pi}{4}-t\log(\pi)+w,$$

$$\beta = (-\nu + t)\log\left(\frac{-\nu+t}{2}\right) + \nu - t - \frac{\pi}{4} - t\log(\pi) - w.$$

We see that $\Phi(t, \theta, \nu)$ is about

$$\tfrac{1}{4}\, g\, \sqrt{\sin(\theta)}\, \left\{ e^{i[\nu \log(\sin(\theta)) + \alpha]} + e^{i[-\nu \log(\sin(\theta)) + \beta]} \right\}.$$

It is elementary that if ϕ_1, ϕ_2 are real and $\phi_1 - \phi_2 = \frac{\pi}{2}$, then for any real α and β, one of

$$e^{i(\phi_1+\alpha)} + e^{i(\phi_1+\beta)}, \qquad e^{i(\phi_2+\alpha)} + e^{i(\phi_2+\beta)}$$

is at least $\sqrt{2}$. Therefore if θ_1 and θ_2 are chosen so that $\log\sin(\theta_1) - \log\sin(\theta_2) = \frac{\pi}{2\nu}$ then one of $|\Phi(t, \theta_1, \nu)|$ or $|\Phi(t, \theta_2, \nu)|$ is at least $2^{-3/2}\, g\, \sqrt{\sin(\theta)}$. The constant g is nonzero and depends only on ν. Note that we are taking θ_i proportional to $1/T$, so $\sqrt{\sin(\theta_i)}$ is of the order $T^{-1/2}$. $\square$

Proof of Theorem 1. We choose M_1 and M_2 as in Lemma 3, and θ_1, θ_2 accordingly. Using (4) and Lemma 2,

$$\frac{1}{2\pi} \int_0^T |\Phi(t, \theta_i, \nu)\, L(f, \tfrac{1}{2} + it)|^2 \, dt \ll \log\csc(\theta_i) = O(\log(T)).$$

Adding this relation for θ_1 and θ_2, the left hand side is bounded below by

$$\frac{c^2}{2\pi T} \int_0^T |L(f, \tfrac{1}{2} + it)|^2 \, dt$$

and the theorem follows. $\blacksquare$

2. Eisenstein series. In this section we will prove by the same method of Sarnak [S] the following theorem of Hardy and Littlewood [HL].

Theorem 4. *We have*

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt = O(T \log^4(T)). \tag{11}$$

The automorphic form will be the Eisenstein series

$$E^*(z, s) = \pi^{-s}\,\Gamma(s)\,\zeta(2s)\,E(z, s), \qquad E(z, s) = \tfrac{1}{2} \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$

with $s = \frac{1}{2}$. Since $\nu = 0$ Lemma 3 does not apply.

Lemma 5. *There exist constants M and c such that if $\theta = M/T$ then for T sufficiently large and $0 < t < T$ we have*

$$|\Phi(t, \theta, 0)| > cT^{-1/2}. \tag{12}$$

Proof. The hypergeometric function $F_{2,1}(a, a; \frac{1}{2}; z)$ which appears on the left side of (10) when $\nu = 0$ has a logarithmic singularity at ∞. It is analytic in the complement of $\{z \in \mathbb{R} \,|\, z > 1\}$ and in this region it can be expressed in terms of two linearly independent solutions, one of which, namely $z^{-a}F_{2,1}(a, a+\frac{1}{2}; 1; 1/z)$, is analytic at ∞, and the other has a logarithmic singularity there. We use the theorem on p. 123 of Mehlenbacher [Meh] and the precise formula (3) on p. 55 of Rainville [Rain] for the relevant logarithmic solution of the hypergeometric differential equation, which is uniquely characterized by the vanishing of h_0 in (11) on p. 122 of [Meh]. After some simplification we get

$$\begin{aligned} \Phi(t, \theta, 0) &= \tfrac{1}{4}\pi^{-it-1/2}\,\Gamma\left(\frac{it}{2} + \frac{1}{4}\right)^2 \sin(\theta)^{-it}\, F_{2,1}\left(\frac{1}{4} + \frac{it}{2}, \frac{1}{4} + \frac{it}{2}; \frac{1}{2}; -\cot^2(\theta)\right) = \\ &- \tfrac{1}{4}\pi^{-it}\,\frac{\Gamma\left(\frac{it}{2} + \frac{1}{4}\right)}{\Gamma\left(\frac{it}{2} - \frac{1}{4}\right)} \sin(\theta)^{-it}\, \tan(\theta)^{it+1/2} \times \\ &\left[\left(\psi\left(\frac{1}{4} + \frac{it}{2}\right) + \psi\left(\frac{1}{4} - \frac{it}{2}\right) + 2\gamma + \log\,\tan^2(\theta)\right) F_{2,1}\left(\frac{1}{4} + \frac{it}{2}, \frac{3}{4} + \frac{it}{2}; 1; -\tan^2(\theta)\right)\right. \\ &\left. + S(t, \theta)\right], \end{aligned} \tag{13}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, $\gamma = -\psi(1)$ is Euler's constant and $S(t, \theta)$ is

$$\sum_{n=1}^{\infty} \frac{(\frac{1}{4} + \frac{it}{2})_n(\frac{3}{4} + \frac{it}{2})_n(-\tan^2(\theta))^n}{n!^2} \left\{H\left(\frac{1}{4} + \frac{it}{2}, n\right) + H\left(\frac{3}{4} + \frac{it}{2}, n\right) - 2H(1, n)\right\},$$

with

$$H(a, n) = \sum_{k=1}^{n} \frac{1}{a + k - 1}.$$

Here the factor in braces, involving H, is about $\log((1+1/4n)^2 + (t/2n)^2)$. As in Lemma 3, we may approximate both the $F_{2,1}$ series and $S(t, \theta)$ by their first terms uniformly in $(\frac{1}{4} + \frac{it}{2})(\frac{3}{4} + \frac{it}{2})\tan^2(\theta)$ provided

$$\max_{l \geq 0} \left| \frac{\left(\frac{1}{4} + \frac{it}{2} + l\right)\left(\frac{3}{4} + \frac{it}{2} + l\right)}{(1 + l)^2} \tan^2(\theta) \right| < \frac{1}{2}.$$

This will certainly be satisfied if M is sufficiently small. Using Stirling's formula

$$\psi(z) = \log(z) - \frac{1}{2z} + \frac{1}{12z^2} + O(z^{-3})$$

the factor

$$\left|\psi\left(\frac{1}{4} + \frac{it}{2}\right) + \psi\left(\frac{1}{4} - \frac{it}{2}\right) + 2\gamma + \log\,\tan^2(\theta)\right| \tag{14}$$

is monotone decreasing and may be bounded below by its value at the right endpoint; this is about $2|\log(2) - \gamma - \log(M)|$. On the other hand, we may approximate $S(t,\theta)$ by its first term, which is monotone increasing in absolute value and hence bounded above by its value when $t = T$, which is about $M^2/4$. If we choose M small enough, the lower bound for (14) will be larger in absolute value than the upper bound for $S(t,\theta)$. The controlling factor in (13) is thus $\tan(\theta)^{it+1/2}$ accounting for the $T^{-1/2}$ in (12). $\square$

The constant term $E_0^*(z,s) = \int_0^1 E^*(z+u,s)\,du$ equals

$$\pi^{-s}\Gamma(s)\,\zeta(2s)\,y^s + \pi^{s-1}\Gamma(1-s)\,\zeta(2-2s)\,y^{1-s}$$

unless $s = \frac{1}{2}$. In that case, with $z = x + iy$,

$$E_0^*(z,\tfrac{1}{2}) = \sqrt{y}\,\big(\gamma - \log(4\pi) + \log(y)\big).$$

Subtracting the constant term gives a function

$$E^*(z,s) - E_0^*(z,s) = \sum_{n\neq 0} a_n(y,s)\,e^{2\pi i n x} \tag{15}$$

of rapid decay when $y \to \infty$. Here when $n \neq 0$,

$$a_n(y,s) = 2\,\tau_{s-1/2}(|n|)\,\sqrt{y}\,K_{s-1/2}(2\pi|n|y),$$

where

$$\tau_{s-1/2}(n) = \tau_{1/2-s}(n) = \sum_{\substack{n=d_1 d_2\\ 0<d_i}} \left|\frac{d_1}{d_2}\right|^{s-1/2}.$$

In particular, we will denote by $\tau(n) = \tau_0(n)$ the number of divisors of n.

Zagier [Z] defined a "renormalized" Mellin transform

$$\mathbf{RN}\int_0^\infty E^*(r\,e^{i\theta})\,r^w\,\frac{dr}{r}.$$

If $\mathrm{re}(w)$ is sufficiently large, this is just the Mellin transform of $E^* - E_0^*$, but one can also give a definition which is valid for all w.

A version of Zagier's definition may be found in Section 1 of Beineke and Bump [BB1]. (The difference between Zagier's truncation and that of [BB1] is that Zagier truncates the interval $(0,\infty)$ at ∞, while in [BB1], the interval is truncated at both ends.) In the case we are interested in, this definition is, for $-\infty < U' < U < \infty$,

$$\begin{aligned}
(16)\quad \mathbf{RN}\int_0^\infty E^*(re^{i\theta},\tfrac12)\,r^w\,\frac{dr}{r} &= \int_{U'<\log\, r<U} E^*(re^{i\theta},\tfrac12)\,r^w\,\frac{dr}{r} \\
&+\int_{U<\log\, r}[E^*(re^{i\theta},\tfrac12) - r^{1/2}\,\sin(\theta)^{1/2}(\gamma-\log(4\pi)+\log(r)+\log\,\sin(\theta))]\,r^w\,\frac{dr}{r} \\
&+\int_{\log\, r<U'}[E^*(re^{i\theta},\tfrac12) - r^{-1/2}\,\sin(\theta)^{1/2}(\gamma-\log(4\pi)-\log(r)+\log\,\sin(\theta))]\,r^w\,\frac{dr}{r} \\
&-\frac{2\sqrt{\sin(\theta)}\,e^{U(w+1/2)}}{(1+2w)^2}\big(-2+(1+2w)(\gamma+U-\log(4\pi)+\log\,\sin(\theta))\big) \\
&-\frac{2\sqrt{\sin(\theta)}\,e^{U'(w-1/2)}}{(1-2w)^2}\big(-2+(1-2w)(\gamma-U'-\log(4\pi)+\log\,\sin(\theta))\big).
\end{aligned}$$

Differentiating with respect to U and U' one sees that this definition is independent of the choice of U and U', and it is equivalent to Zagier's definition.

Lemma 6. *We have*

$$(17)\qquad \mathbf{RN}\int_0^\infty E^*(re^{i\theta},\tfrac12+i\nu)\,r^{it}\,\frac{dr}{r} = 4\,\Phi(t,\theta,\nu)\,\zeta(\tfrac12+it+i\nu)\,\zeta(\tfrac12+it-i\nu).$$

Proof. With ν fixed, we may prove this first when it is a large real number. In this case, we may let $U, U' \to -\infty$ and

$$\mathbf{RN}\int_0^\infty E^*(re^{i\theta},\tfrac12+i\nu)\,r^{it}\,\frac{dr}{r} = \int_0^\infty (E^*-E_0^*)(re^{i\theta},\tfrac12+i\nu)\,r^{it}\,\frac{dr}{r}.$$

As in the cusp form case, we apply Gradshteyn and Ryzhik [GR] 6.699.4 to (15) to obtain

$$4\,\Phi(t,\theta,\nu)\sum_{r=1}^\infty \tau_{i\nu}(r)\,r^{-it-1/2}.$$

With our assumption that it is a large real number, this series is convergent and represents $\zeta(\frac12+it+i\nu)\,\zeta(\frac12+it-i\nu)$ and we obtain (17) in this case. The general case follows by analytic continuation in t. $\square$

Now we take $U = U' = 0$ in (16) and apply Lemma 6. Observing that $E^*(re^{i\theta}, \frac{1}{2})$ is invariant under $r \to r^{-1}$, we may invert the integral when $r < 1$ to obtain

$$(18) \quad 4\,\Phi(t,\theta,0)\,\zeta(\tfrac{1}{2}+it)^2 = \int_1^\infty [E^*(re^{i\theta},\tfrac{1}{2}) - E_0^*(re^{i\theta},\tfrac{1}{2})]\,(r^{it}+r^{-it})\,\frac{dr}{r} + 4\frac{\sqrt{\sin(\theta)}}{1+4t^2}\left[2\left(\frac{1-4t^2}{1+4t^2}\right) - \gamma + \log(4\pi) - \log\sin(\theta)\right].$$

Ramanujan [R] proved that

$$(19) \qquad \sum_{n=0}^{\infty}\frac{\tau(n)^2}{n^s} = \prod_p\left[\sum_{k=0}^{\infty}\frac{(k+1)^2}{p^{ks}}\right] = \prod_p \frac{1-p^{2s}}{(1-p^{-s})^4} = \frac{\zeta(s)^4}{\zeta(2s)}$$

and that

$$(20) \qquad \sum_{n\le x}\tau(n)^2 \sim \frac{1}{\pi^2}\,x\,\log^3(x).$$

Wilson [W] published proofs of (19) and (20). It is worth mentioning that the estimate (20) follows from (19) by a Tauberian theorem of Stark, stated in Diaconu, Goldfeld and Hoffstein [DGH], Section 2.3. Similarly $\sum_{n=0}^{\infty}\tau(n)\,n^{-s} = \zeta(s)^2$ so one obtains the estimate

$$(21) \qquad \sum_{n\le x}\tau(n) \sim x\,\log(x)$$

of Dirichlet [D].

Proposition 7. *Suppose that* $0 \le \theta < \frac{\pi}{2}$. *As* $\theta \to 0$ *we have*

$$\int_1^\infty |E^*(re^{i\theta},\tfrac{1}{2}) - E_0^*(re^{i\theta},\tfrac{1}{2})|^2\,\frac{dr}{r} = O\big(\log^4\,\csc(\theta)\big).$$

Compare Titchmarsh [T1], section 5.2, and Sarnak [S], Lemma 3.

Proof. We substitute

$$E^*(re^{i\theta},\tfrac{1}{2}) - E_0^*(re^{i\theta},\tfrac{1}{2}) = 2\sum_{n\neq 0}\tau(|n|)\,\sqrt{r\,\sin(\theta)}\,K_0(2\pi|n|r\sin(\theta))\,e^{2\pi i n r\cos(\theta)}.$$

Multiplying this by its conjugate, we must bound

$$(22) \quad \sum_{m,n\neq 0}\tau(|n|)\,\tau(|m|)\,\sin(\theta)\int_1^\infty K_0(2\pi|n|r\,\sin(\theta))K_0(2\pi|m|r\,\sin(\theta))\,e^{2\pi i(n-m)\,\cos(\theta)}\,dr.$$

We consider the diagonal terms $n = m$ first.

Lemma 8. *We have*

$$\sum_{n>0} \tau(|n|)^2 \sin(\theta) \int_1^\infty K_0(2\pi n r \sin(\theta))^2\, dr = O(\log^4 \csc(\theta)).$$

Proof. Using summation by parts, this is

$$- \sin(\theta) \int_1^\infty \left[\sum_{n\le x} \tau(|n|)^2\right] \left[\frac{d}{dx} \int_1^\infty K_0(2\pi x r \sin(\theta))^2\, dr\right] dx =$$

$$- \sin(\theta) \int_1^\infty \left[\sum_{n\le x} \tau(|n|)^2\right] \left[\frac{d}{dx}\frac{1}{x} \int_x^\infty K_0(2\pi r \sin(\theta))^2\, dr\right] dx =$$

$$\sin(\theta) \int_1^\infty \left[\sum_{n\le x} \tau(|n|)^2\right] \left[\frac{1}{x^2} \int_x^\infty K_0(2\pi r \sin(\theta))^2\, dr + \frac{1}{x} K_0(2\pi x\ \sin(\theta))^2\right] dx.$$

Replacing r by $r/\sin(\theta)$ and using (20), the first contribution becomes

$$\int_1^\infty \left[\sum_{n\le x} \tau(|n|)^2\right] \frac{1}{x^2} \int_{x\ \sin(\theta)}^\infty K_0(2\pi r)^2\, dr\, dx \ll \int_1^\infty \frac{\log^3(x)}{x} \int_{x\ \sin(\theta)}^\infty K_0(2\pi r)^2\, dr\, dx.$$

Replacing x by $x/\sin(\theta)$ shows that this is dominated by

$$\int_{\sin(\theta)}^\infty \frac{(\log(x) - \log\ \sin(\theta))^3}{x} \int_x^\infty K_0(2\pi r)^2\, dr\, dx \ll \int_{\sin(\theta)}^1 \frac{(\log(x) - \log\ \sin(\theta))^3}{x}\, dx + O(1).$$

This contribution is thus $O(\log^4\ \csc(\theta))$. On the other hand the second contribution is dominated by

$$\sin(\theta) \int_1^\infty \log^3(x) K_0(2\pi x\ \sin(\theta))^2\, dx = \int_{\sin(\theta)}^\infty (\log(x) - \log\ \sin(\theta))^3\, K_0(2\pi x)^2\, dx$$

This term is $O(\log^3 \csc(\theta))$. □

Next we turn to the off-diagonal terms.

Lemma 9. *Let f be nonnegative and monotone decreasing on $[1, \infty)$ such that*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0.$$

Let $0 \ne a \in \mathbb{R}$. Then

$$\left| \int_1^\infty f(x)\, e^{iax}\, dx \right| \le \frac{2\, f(1)}{|a|}.$$

(Compare Titchmarsh [T2], Lemma 4.3 on p. 61.)

Proof. Integrating by parts,

$$\int_1^\infty f(x)\, e^{iax}\, dx = \left[\frac{e^{iax}}{ia}\, f(x)\right]_1^\infty - \int_1^\infty \frac{e^{iax}}{ia}\, f'(x)\, dx.$$

Both pieces are $\leq f(1)/|a|$ in absolute value. For the first piece this is obvious. The second piece is bounded by

$$\frac{1}{|a|}\int_1^\infty |f'(x)|\, dx = \frac{1}{|a|}\int_1^\infty -f'(x)\, dx = \frac{f(1)}{|a|}.$$

This proves the Lemma. $\square$

Lemma 10. *We have*

$$\sum_{n\neq m} \frac{\tau(|n|)\,\tau(|m|)}{|m-n|} \sin(\theta) K_0(2\pi|n|\sin(\theta))\, K_0(2\pi|m|\sin(\theta)) = O(\log^4 \csc(\theta)).$$

Proof. We separate the integral into two parts. First consider

$$\sin(\theta) \sum_{0<|n-m|<\csc(\theta)} \frac{\tau(|n|)\,\tau(|m|)}{|m-n|} K_0(2\pi|n|\sin(\theta))\, K_0(2\pi|m|\sin(\theta)). \tag{23}$$

This may be estimated using an inequality of Hilbert (Hardy, Littlewood and Pólya [HLP], p. 212.) We will give a direct method of bounding (23). Let $\mathbb{T}$ be the unit circle, and $H = L^2(\mathbb{T})$. Let $c(n)$ be a sequence satisfying $c(-n) = \overline{c(n)}$, such that the function $f(x) = \sum_{-\infty}^{\infty} c(n)\, e^{2\pi inx}$ is in $L^\infty(\mathbb{T})$. The Toeplitz operator $T : H \to H$ consisting of multiplication by f is self-adjoint. It is clear that T is a bounded operator and the l^∞ norm of f is the operator norm of T. In particular, if $g(x) = \sum d(n)\, e^{2\pi inx}$, we have

$$\langle Tg, g\rangle \leq ||f||_\infty \,\langle g, g\rangle\,.$$

This means that

$$\sum c(n-m)\, d(n)\, \overline{d(m)} \leq ||f||_\infty \sum |d(n)|^2.$$

We can apply this with

$$f(x) = \sum_{0<|n|<\csc(\theta)} \frac{1}{|n|} e^{2\pi inx} = 2\sum_{n=1}^{\lfloor \csc(\theta)\rfloor} \frac{\cos(2\pi nx)}{n}.$$

Clearly f takes its supremum at 0, so the constant $||f||_\infty$ is about $2\log\,\csc(\theta)$. Thus (23) is dominated by

$$\sin(\theta)\,\log\,\csc(\theta)\sum_{n\neq 0}\tau(|n|)^2\,K_0(2\pi|n|\,\sin(\theta))^2 \ll$$

$$\sin(\theta)\,\log\,\csc(\theta)\sum_{n\neq 0}\tau(|n|)^2\,e^{-4\pi|n|\,\sin(\theta)}.$$

This may be estimated by summation by parts. It equals

$$-\sin(\theta)\,\log\,\csc(\theta)\int_1^\infty\left[\sum_{n\leq x}\tau(|n|)^2\right]\frac{d}{dx}e^{-4\pi x\,\sin(\theta)}\,dx$$

$$\ll \sin^2(\theta)\,\log\,\csc(\theta)\int_1^\infty x\,\log^3(x)e^{-4\pi x\,\sin(\theta)}\,dx,$$

where we have used (20). To estimate this integral, replace $x\to x/\sin(\theta)$, then split the integral into the intervals $(\sin(\theta),1)$ and $(1,\infty)$. On the first interval, bound $e^{-4\pi x\,\sin(\theta)}$ by 1, then perform the integration. The second integral is also easily bounded. One obtains $O(\log^4\csc(\theta))$ for these terms.

What remains is

$$\sin(\theta)\sum_{\csc(\theta)\leq|n-m|}\frac{\tau(|n|)\,\tau(|m|)}{|m-n|}K_0(2\pi|n|\sin(\theta))\,K_0(2\pi|m|\sin(\theta))$$

$$\ll \sin^2(\theta)\sum_{n,m\neq 0}\tau(|n|)\,\tau(|m|)K_0(2\pi|n|\sin(\theta))\,K_0(2\pi|m|\sin(\theta))$$

$$=\left[\sin(\theta)\sum_{n\neq 0}\tau(|n|)K_0(2\pi|n|\sin(\theta))\right]^2.$$

A similar calculation using (21) gives

$$\sin(\theta)\sum_{n\neq 0}\tau(|n|)K_0(2\pi|n|\sin(\theta)) = O\big(\log\,\csc(\theta)\big).$$

Therefore this second contribution is $O(\log^2\csc(\theta))$. $\square$

Combining Lemma 9 and Lemma 10 gives the bound for the off-diagonal terms in (22), and the Theorem is proved. $\blacksquare$

Proof of Theorem 4. We now proceed as in the cuspidal case. We take $\theta = M/T$ with M as in Lemma 5. Using Parseval's identity, (18) and the triangle inequality for the L^2 norm,

$$\int_{-\infty}^{\infty} |\Phi(t,\theta,0)|^2 |\zeta(\tfrac{1}{2}+it)|^4 \, dt$$

is bounded by the quantity estimated in Proposition 7, plus

$$\sqrt{\int_{-\infty}^{\infty} \left| 4\frac{\sqrt{\sin(\theta)}}{1+4t^2} \left[2\left(\frac{1-4t^2}{1+4t^2}\right) - \gamma + \log(4\pi) - \log \sin(\theta) \right] \right|^2 dt} = O(T^{-1/2} \log(T)).$$

For $t \in [0,T]$ we have the lower bound in Lemma 5, and the Theorem follows. ■

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