Paul Garrett, 7/15

Overview of

- Integral resp of sums/integrals of moments
- Spectral analysis of kernel (Poincaré)

for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ and more. We will omit (see Adrian's talks) the application to convexity breaking.

What we get: (expanded form of) $\int_{Z_{\mathbb{A}}\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A})} \mathrm{Pé}(g)|f(g)|^2 dg$ where Pé is a Poincaré series. We may replace $|f(g)|^2$ by a pair of cusp forms. This is to be expressed as

$$\sum_{\text{cuspidal } F \text{ on } \mathrm{GL}_{r-1}} \int_{-\infty}^{\infty} \left| L^{\text{fin}}\left(\frac{1}{2}+it, f\otimes F\right)\right|^2 \text{“}t^{-w}\text{”}\, dt + \text{more degenerate ...}$$

Here the part written t^{-w} depends on the archimedean components f_∞ and F_∞ and is hence in quotation marks. It is really a function derived from the kernel and is only asymptotically t^{-w}. Following Good, Diaconu and Goldfeld we can obtain actual moments from this expression using a Tauberian argument.

For ease of exposition, assume that f has trivial central character.

1. Orthonormal basis for $L^2_{\text{cusp}}(Z_{\mathbb{A}}\mathrm{GL}_{r-1}(k)\backslash \mathrm{GL}_{r-1}(\mathbb{A}))$ – without loss of generality (spherical case for simplicity), rep'n generated by F, p_F is irreducible (unitary) and (therefore! Type I-ness!) $\pi_F = \otimes_v \pi_{F,v}$.

Meaning of $L^{\text{fin}}(s, f\otimes F)$ – this is not $L^{\text{fin}}(s, \pi_f \times \pi_F)$. Jacquet-Piatetski-Shapiro-Shalika $L^{\text{fin}}(s, \pi_f\times\pi_F)$ is $L^{\text{fin}}(s, f\otimes F)$ times a polynomial in $q_v^{\pm s}$ where v runs through a finite set of bad places.

How do we get $L(s, f\otimes F)$? Assume f is spherical at v_0 but F is not spherical at v_0. Fail: if you take the spherical vector in π_{f,v_0} then you cannot get $\neq 0$. Succeed: (Jacquet-PS-S – $\exists$ good choices) take vector in π_{f,v_0} non-spherical at v_0. Watch out: there is a global normalization issue ($\sim$ Hoffstein-Lockhart, Bernstein-Reznikov, Sarnak). We cannot make both $|f|_2 = 1$ and $a_f(1) = 1$, and you want to choose one of these. It is better to take $a_f(1) = 1$, so $|f|_2$ will appear somewhere in the formula.

(Discussion. Hoffstein-Lockhart is proved on GL_2. Higher symmetric power information, functoriality, ineffective constant ...)c

Go back: with “live” auxiliary s', which is the equivalent of the parameter v in Diaconu's lecture, what do you really get?

Really have ($\mathrm{re}(s') \gg 0$ — must meromorphically continue to $s' = 0$)

$$\sum_{\text{cuspidal } F \text{ on } \mathrm{GL}_{r-1}} L\left(s' + \frac{1}{2} + it, f\otimes F\right)\overline{L\left(\frac{1}{2}+it, f\otimes F\right)}\text{“}t^{-w}\text{”}\, dt + \dots$$

=spectral side. Part of the spectral decomposition of Pé is on the other side, and this has not been discussed.

We now define Pé. It is

$$\text{Pé}(g) = \sum_{Z_k M_k \mathrm{GL}_r(k)} \phi(\gamma g), \qquad M = \left(\begin{array}{|c|c|} \hline \mathrm{GL}_{r-1} & 0 \\ \hline 0 & * \\ \hline \end{array}\right).$$

Spherical everywhere locally for $|f|^2$, $\phi = \otimes\, \phi_v$. At nonarchimedean places

$$\phi_v(g_v) = \begin{cases} \left|\frac{\det(A)}{d^{r-1}}\right|^{s'} & g = m \cdot k, m = \left(\begin{array}{|c|c|} \hline A & 0 \\ \hline 0 & d \\ \hline \end{array}\right), k \in K_{v,\max} \\ 0 & \text{not of this form} \end{cases},$$

At $v|\infty$ ($\sim$ Good, DG – type choice – two? ...) But we can set up things without *committing*

$$\phi_v(mzg) = \left|\frac{\det(A)}{d^{r-1}}\right|^{s'} \phi_v(g).$$

Have NOT committed

$$\phi_v \text{ a} \begin{array}{|c|c|} \hline 1_{r-1} & * \\ \hline & 1 \\ \hline \end{array} \quad \longleftarrow \mathbb{R}^{r-1}, \mathbb{C}^{r-1} - \text{ if spherical have } O(r-1) \text{ or } U(r-1) \text{ invariance.}$$

Discussion. In complex cae try

$$\phi_v \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \left(\frac{1}{1 + \|x\|^2}\right)^{s'/2} = (!) \;\; \phi_{\ldots}\left(w_0 \begin{pmatrix} 1_{r-1} & x \\ & 1 \end{pmatrix}\right) \quad — I^G_{P^{r-1,1}}.$$

Spectral decomposition of Pé (considerable good:tractable surprise)

$$\mathrm{re}(s') \gg \qquad \text{try: first pole if any at 1.}$$

$$\begin{aligned} \textit{Pé} \; = & \\ (*)_\infty E^{r-1,1}(s'+1) & \\ + \sum_{\text{cuspidal } F \text{ on } \mathrm{GL}_2, \text{ spherical}} (\text{arch}) L\left(\frac{1}{2} + s', F\right) \cdot E^{r-2,2}\left(\frac{s'}{2} + 1 \otimes F\right) & \\ + \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} E^{r-2,1,1}(\ldots s' \ldots) & \\ + (!) \text{ no further terms.} & \end{aligned}$$

where the Eisenstein series is induced from E. See note on "half-degenerate" Eisenstein series from Paul Garrett's web page.

! Wiith pretty good choice of archimedean data as $s' \longrightarrow 0$ ...

Why does Pé terminate so early? *Poisson summation*

$$\textit{Pé}(g) = \sum_{Z_k M_k \backslash G_k} \phi(\gamma g) = \sum_{\gamma \in P_k^{r-1,1} \backslash G_k} \; \sum_{\beta \in U_k} \phi(\beta \gamma g),$$

where

$$\beta \in U_k = \begin{pmatrix} I_{r-1} & * \\ & 1 \end{pmatrix}.$$

We do Poisson summation on the β smmation.

$$\sum_{\gamma \in P_k^{r-1,1}\backslash G_k} \left(\sum_{\psi \in (U_{\mathbb{A}}/U_k)^*} \hat{\phi}_{\gamma g}(\phi) \right), \qquad \phi_g(u) = \phi(ug)$$

(deceptively, might think $E^{r-1,1}$ ... ?! x)

- $\psi = 1$ term really is an $E^{r-1,1}$ (unsurprisingly —)
- $\psi \neq 1$ terms: M_k transitive on $\psi \neq 1$: fix $\psi(u) = \psi_0(u_{r-1})$ ($u_{r-1} = \text{bottom}$).

$$\text{isotropy subgroup in } M = \begin{array}{|c|c|c|} \hline * & * & \\ \hline & 1 & \bullet \\ \hline & & 1 \\ \hline \end{array} \begin{array}{l} \longleftarrow \text{trivial char} \\ \longleftarrow \psi \\ \\ \end{array}$$

$$\sum_{\xi \in k^\times} \hat{\phi}_g(\psi^\xi) \ \text{ has isotropy subgroup } \quad \begin{array}{|c|c|c|} \hline * & * & * \\ \hline & * & \\ \hline & & 1 \\ \hline \end{array} \begin{array}{l} \longleftarrow + \text{invt under } \mathbb{A}-\text{points of } r-2 \times \text{part of } 2\times 2 \\ \longleftarrow \text{get} \\ \\ \end{array}$$

Seen to be

$E^{r-2,2}$(det on $r-2 \otimes \mathrm{GL}_2$-automorphised subsum on lower right GL_2).

Goldfeld asks why the Poincaré series is orthogonal to the GL_2 cusp forms – now we can sort of see it.

$$\sum_{\psi \neq 1} = \sum_{\begin{array}{|c|c|c|} \hline * & * & * \\ \hline & * & * \\ \hline & & 1 \\ \hline \end{array} \backslash G_k} \left(\sum_{\xi \in k^*} \hat{\phi}_{\gamma g}(\psi^\xi) \right).$$

It is illuminating to use the GL_2 business. Use sphericalness:

$$g = \begin{pmatrix} A & * \\ & D \end{pmatrix} \longleftarrow 2 \times 2$$

$$= \sum_{\gamma \in P^{r-2,2}\backslash G_k} \left[\sum_{\alpha \in P^{1,1}\backslash \mathrm{GL}_2} \left(\sum_{\xi \in k^*} \hat{\phi}_{\alpha\gamma g}(\psi^\xi) \right) \right].$$

At $g = \begin{pmatrix} A & * \\ & D \end{pmatrix}$ the inner sum is Pé on GL_2 – has reasonable spherical decomposition from $E^{r-2,2}$ from each fragment.

(Break)

(Before proof of LHS ...)

The other terms of $\int$ Pé $|f|^2 =$

$$\sum_{\text{cuspidal } F \text{ on } \mathrm{GL}_{r-1}} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes F)|^2 \text{“}t^{-w}\text{”}\, dt$$

$$+ \sum_{\substack{r_1+\ldots+r_l=r-1 \\ \text{cusp forms } F_i \text{ on } \mathrm{GL}_{r_i}}} \prod_{j=1}^{l} \left| L\left(\frac{1}{2}+it\pm it_1\pm\cdots\pm it_l, f\otimes F_j\right)\right|^2$$

The most degenerate continuous part is

$$\int \cdots \int \prod \left| L\left(\frac{1}{2}+it\pm it_1\pm\cdots\pm it_{r-2}, f\right)\right|^2 \text{“}t^{-w}\text{”}\, dt\, dt_1 \cdots dt_{r-1}.$$

For example, if $F = F_1 = \cdots = F_l$,

$$\left| L\left(\frac{1}{2}+it, f\otimes F\right)\right|^{2^{\frac{r-1}{\text{size}}}}.$$

One can try to take a truncated Eisenstein series $\Lambda^T E$ ...

How to compute LHS

Proof: obvious unwinding + spectral decomposition/expansion (less obviously) on GL_{r-1} to get $\int_{\text{vertical}}$ of Euler products. $f \longrightarrow$ its Whittaker function, which is a product of local data. Now in the old days (20-th century), we would expect purely a product of local data. But now (21-st century) we look harder and sometimes get an integral or sum of Euler products. If something is an Euler product, it is interesting. One of the technical advantages of working on the adele group is that you can see in advance that the decomposition is coming.

So, what happens?

$$\int \text{Pé}(g)|f(g)|^2 = (\text{usual unwinding}) \; =$$

$$\int_{Z_k M_k\backslash G_{\mathbb{A}}} \phi(g) \sum_{\gamma\in\Theta_k\backslash M_k} W(\gamma g)\overline{f(g)}\, dg, \qquad \Theta = \left(\begin{array}{c|c|c} * & * & \\ \hline & 1 & \\ \hline & & 1 \end{array}\right).$$

Now we use a truly great trick. We get

$$\int_{Z_k\Theta_k\backslash G_{\mathbb{A}}} \phi(g)W(g)\overline{f(g)}dg.$$

This depends on the left M_k-invariance of ϕ. The domain of integration does not decompose locally, nor does f. But you make a Fourier expansion

$$\int_{Z_k\Theta_k\backslash G_{\mathbb{A}}} \phi(g)W(g)\sum_{\gamma\in\Theta_k\backslash M_k}\overline{W(\gamma g)}dg.$$

(This will be **recanted** below. It's right but there is another way to proceed.) For simplicity, consider the spherical case (Iwasawa).

$$\mathrm{GL}_1\colon \mathbb{A}^\times/k^\times = \mathbb{A}_1^\times/k^\times \text{ (compact)} \times (0,\infty) \longleftarrow \text{gives vertical integral.}$$

$$\mathrm{GL}_r(k)\backslash\mathrm{GL}_r(\mathbb{A}) = \mathrm{GL}_r(k)^1\backslash\mathrm{GL}_r(\mathbb{A})^1 \times (0,\infty)$$

Naming

$$U = \begin{array}{|c|c|}\hline I_{r-1} & * \\ \hline & 1 \\ \hline\end{array}, \qquad M = \begin{array}{|c|c|}\hline *_{r-1} & \\ \hline & 1 \\ \hline\end{array}, \qquad N = \begin{array}{|c|c|c|}\hline I_{r-2} & * & \\ \hline & 1 & \\ \hline & & 1 \\ \hline\end{array}, \qquad H = \begin{array}{|c|c|c|}\hline * & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline\end{array}.$$

Unfortunate notational change: $H \longleftrightarrow M$. Apart from the last column in U ... and moreover the definition of N is **corrected below**.

Recant the second Fourier-Whittaker expansion – it's better to produce another way.

$$= \int_{N_k\backslash M_{\mathbb{A}}\times U_{\mathbb{A}}} \phi(mu)W(mu)\overline{f(mu)}dmdu. \tag{1}$$

The last column is not yet resolved. How to proceed? There are three possibilities

0th. $f=$ its own spherical expansion on GL_r

1st. (wrong) $f=$ Fourier-Whittaker

2nd (right) Restrict f to GL_{r-1} (a multiplicity one subgroup) and do a spectral expansion.

Spectral expansion on GL_{r-1}, in somewhat symbolic notation.

$$F = \left\{\int \text{ or } \sum\right\}\langle F,\eta\rangle\cdot\eta\, d\eta, \qquad F(1) = \left\{\int \text{ or } \sum\right\}\langle F,\eta\rangle\cdot\eta(1)\, d\eta.$$

Expand $\overline{f(mu)}$ in m.

$$\overline{f(mu)} = \int_{(\eta)}\eta(1)\int_{M_k\backslash M_{\mathbb{A}}}\overline{\eta(m')}\, f(m'mu)dm'd\eta.$$

So (1) equals

$$\int_{(\eta)}\eta(1)\int_{N_k\backslash M_{\mathbb{A}}\times U_{\mathbb{A}}}\phi(mu)W(mu)\int_{M_k\backslash M_{\mathbb{A}}}\overline{\eta(m')}\, f(m'mu)dm'd\eta.$$

Lied before: N_k is really the unipotent radical of GL_{r-1} embedded in GL_r:

$$\begin{pmatrix} 1 & * & \cdots & * & 0 \\ & 1 & & \vdots & \vdots \\ & & \ddots & * & 0 \\ & & & & 1 \end{pmatrix}$$

Use Whittaker expansion

$$f(g) = \sum_{N_k \backslash M_k} W(\gamma g).$$

$$= \int_{(\eta)} \eta(1) \int_{N_k \backslash M_{\mathbb{A}} \times U_{\mathbb{A}}} \phi(mu) W(mu) \int_{M_k \backslash M_{\mathbb{A}}} \overline{\eta(m')} \overline{W(m' mu)} dm' dm\, du d\eta.$$

More $\int_{(\eta)}$ inside, replace m' by $m' m^{-1}$ (right translate η by m).

$$= \int_{(\eta)} \eta(1) \int_{N_k \backslash M_{\mathbb{A}} \times U_{\mathbb{A}}} \phi(mu) \eta(m) W(mu) \int_{N_k \backslash M_{\mathbb{A}}} \overline{\eta(m')} \overline{W(m' u)} dm dm\, du d\eta.$$

Everything is fine but pesky $U_{\mathbb{A}}$. Not η but all other stuff $= \otimes_v$ and pesky $U_{\mathbb{A}}$ is not inside η's.

This is where the archimedean stuff is a complication.

Lemma 1. *For $v < \infty$. For m, m' such that $W_v(m), W_v(m')$*

$$\int_{U_v} \phi_v(m) \psi(m u m^{-1}) \psi(m' u m'^{-1}) du = \int_{U_v \cap K_v} 1$$

which is (boring/good) independent of m, m'.

Proof. $\phi(u) = 0 \Longrightarrow u \in U_v \cap K_v$ (or close)

$$\begin{aligned} & \psi(m u m^{-1}) W(m) = W(m u m^{-1} \cdot m) \\ = \ & W(mu) = W(m) 1 \end{aligned}$$

Do same for m'. □

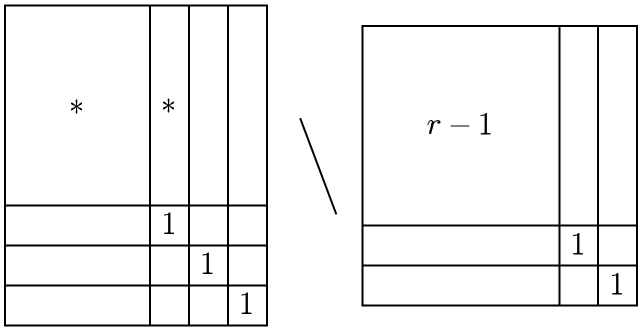
Ach! radically different at ∞. $\mathcal{K}_\infty(m_\infty, m'_\infty, \phi_\infty) = \int$ at infinity., what is left of the pesky unipotent integration at the archimedean places.

$$\int \mathrm{Pé}|f|^2 = \int_{(\eta)} \int_{N_k M_{\mathbb{A}}} \int_{N_k M_{\mathbb{A}}} \mathcal{K}_\infty(m, m') \phi_{\mathrm{fin}}(m) W(m) \eta(m) \overline{W(m')}\, \overline{\eta(m')} dm\, dm'\, d\eta.$$

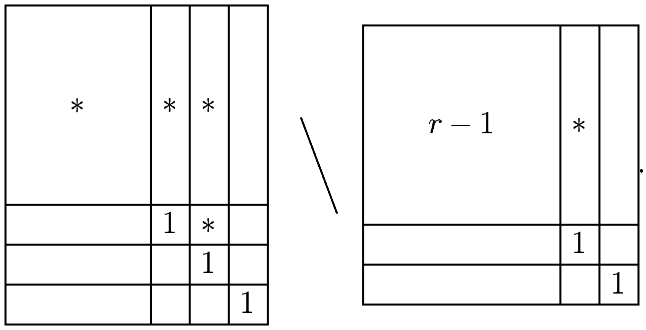
Then Fourier-Whittaker expansion of η (twice)

$$\eta(m) = \sum_{*} W_\eta(\gamma m)$$

We may as well include extra stuff that doesn't do anything in the summation (choice of coset representatives). The whole action is in the upper left block.



this coset decomposition can be identified with



We can then unwind ...

$$\int_{N_k\backslash M_{\mathbb{A}}} F(m)dm = \int_{N_{\mathbb{A}}\backslash M_{\mathbb{A}}}\int_{N_k\backslash N_{\mathbb{A}}} F(nm)dn\,dm.$$

(W_f left $N_{\mathbb{A}}$, ψ equivariant, left $N_{\mathbb{A}} \subset M'_{\mathbb{A}}$-invariant dendependent of ϕ) ($\mathcal{K}_\infty$ only depends on last column!)

$$\int_{N_k\backslash M_{\mathbb{A}}} \phi(m)\eta(m)W_f(m)dm = \int_{N_{\mathbb{A}}\backslash M_{\mathbb{A}}}\int_{N_k\backslash N_{\mathbb{A}}} \phi(nm)\eta(nm)W_f(nm)dn\,dm$$

where $\phi(m)$ is basically $|\det(M)|^{s'}$ if

$$m = \begin{array}{|c|c|}\hline A & \\ \hline & 1 \\ \hline\end{array}$$

$$\begin{aligned}\int_{N_{\mathbb{A}}\backslash M_{\mathbb{A}}} \phi(m)W_f(m)\left(\int_{N_k\backslash N_{\mathbb{A}}} \psi(n)\eta(nm)dn\right)dm &= \\ \int_{N_{\mathbb{A}}\backslash M_{\mathbb{A}}} \phi(m)W_f(m)W_\eta(m)dm &= (\text{arch?})L_{\text{fin}}\left(s'+s+\frac{1}{2}, f\otimes F\right).\end{aligned}$$

We are writing $\eta = |\det|^s \otimes F$.

Question: why does the residual spectrum not contribute? Answer: no Whittaker models.

Whole

$$= \sum_F \int_{\mathrm{re}(s)=\frac{1}{2}} (\mathrm{arch}) \times L^{\mathrm{fin}}(s'+s, f \otimes F)\overline{L^{\mathrm{fin}}(s, f \otimes F)}\, ds.$$