Adrian Diaconu, 7/14

This is a continuation from yesterday's talk.

We will specialize the section in the Poincare series. Let $v, w \in \mathbb{C}$,

$$\phi_\nu\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{cases} (x^2+1)^{-w/2} & \nu \text{ real} \\ (|x|^2+1)^{-w} & \nu \text{ complex.} \end{cases}$$

Here in the complex case $|\,|$ is the "usual" absolute value so $|\,|_\nu = |\,|^2$

Let

$$\chi_{0,\nu}(m) = |y|_\nu^v, \qquad m = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$= \mathrm{Pe}(v,w) = \mathrm{Pe}(g) = \sum_{\gamma \in M_k \backslash G_k} \phi(\gamma g).$$

Theorem. *The Poincare series* Pe *has meromorphic continuation to* $\mathbb{C}^2$. *For* $v = 0$, $\mathrm{Pe}(0, w)$ *has its first pole at* $w = 1$, *of order* $r_1 + r_2 + 1$ ($r_1, r_2 =$ *number of real and complex places, respectively).*

Spectral decomposition

$$L^2(Z_A G_k \backslash G_A, \omega) = L^2_{\text{cusp}}(Z_A G_k \backslash G_A, \omega) \oplus L^2_{\text{cusp}}(Z_A G_k \backslash G_A, \omega)^\perp.$$

The Poincare series is not L^2, but after subtracting an Eisenstein series $E(g)$, $\mathrm{Pe} - E$ square integrable. We define $\mathrm{Pe}^* = \mathrm{Pe} - E$. The residual spectrum does not contribute anything to the Poincare series. Now if f is a cusp form,

$$\int_{Z_A G_k \backslash G_A} \mathrm{Pe}(g) \overline{f(g)}\, dg$$

is an Euler product. (The integral is convergent even though Pe is not L^2.) At finite primes ν the corresponding local factor $L_\nu(\chi_{0,\nu} |\,|^{1/2}, f) = L_\nu(v + \frac{1}{2}, f)$. If $\nu | \infty$ the local factor is a ratio of Gamma functions, which has exponential decay in the local parameters of f, so when we sum over cusp forms we have absolute convergence. This give analytic continuation when we do the spectral expansion of the Eisenstein series. (This is Good's argument.)

Good's method

At $v = 0$, let

$$I(0, w) = I(\chi_0) = \int \mathrm{Pe}(g) |f(g)|^2\, dt.$$

We have

$$I(0,w)=I(\chi_0)=\sum_{\chi\in\hat{C}_{0,S}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^2K_\infty\left(\frac{1}{2}+it,w,\chi\right)dt.$$

For a modular form K_∞ is a product of Gammas, but for Maass forms, K_∞ is a complicated integral. We have an assymptotic formula. The Hecke character χ involves some parameters t_ν that satisfy

$$\sum_{\nu|\infty}\alpha_\nu t_\nu=0, \qquad \alpha_n=\begin{cases}1 & \nu \text{ real}\\ 2 & \nu \text{ complex.}\end{cases} \tag{1}$$

If $\nu|\infty$ is complex

$$K_\nu\left(\frac{1}{2}+it,w,\chi\right)\sim A(0,\mu_{f,\nu})(1+l_\nu^2+4(t+t_\nu)^2)^{-w}.$$

If $\nu|\infty$ is real

$$K_\nu\left(\frac{1}{2}+it,w,\chi\right)\sim B(0,\mu_{f,\nu})(1+|t+t_\nu|)^{-w}.$$

These formulas can be found in a paper of Q. Zhang.

Now we want to simplify the kernel and obtain an asymptotic formula for the moment. For $\chi\in\hat{C}_0$ let

$$K_\chi(t,w)=\prod_{\substack{\nu|\infty\\ \nu \text{ real}}}(1+|t+t_\nu|)^{-w}\prod_{\substack{\nu|\infty\\ \nu\text{ complex}}}(1+l_\nu^2+4(t+t_\nu)^2)^{-w}.$$

In other words, we replace the kernel by its main term in the asymptotic formula. (We cannot produce this kernel by choosing ϕ.) We define

$$Z(w)=\sum_{\chi\in\hat{C}_{0,S}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^2K_\chi(t,w)dt.$$

Theorem. *We have*

$$\sum_{\chi\in\hat{C}_{0,S}}\int_{\mathcal{J}_\chi(x)}\left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^2dt\sim cx(\log(x))^{r_1+r_2},$$

where $\mathcal{J}_\chi(x)=\{t\in\mathbb{R}|K_\chi(t,-1)<x\}$.

Let us analyze a bit. One might be surprised that x is just to the power 1 here but this is a matter of normalization. When $\chi=1$, $\mathcal{J}_\chi(x)\sim x^{1/[k:\mathbb{Q}]}$. As χ varies, $\mathcal{J}_\chi(x)$ will shrink and the sum is essentially finite. To see this we use (1). The number of terms is of the order $x^{[k:\mathbb{Q}]-1}$.

Convexity breaking

It is conjectured that

$$\int_{\mathcal{J}_1(x)} \left|L\left(\frac{1}{2}+it,f\right)\right|^2 dt \sim cx^{1/[k:\mathbb{Q}]}(\log(x))^{\cdots}.$$

Although this is not provable, the method is still enough to break convexity.

Assuming

(1) $Z(w)$ has mero continuation to a region $\mathrm{re}(w) < 1-c$, and that the only pole in this region is at $w=1$, with order r_1+r_2-1, and

(2) $Z(w)$ has polynomial growth in this region in vertical strips.

Part (2) is not proved, but Adrian expects to do it. Then

$$\sum_{\chi\in\hat{C}_{0,S}} \int_{\mathcal{J}_\chi(x)} \left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^2 dt \sim xP(\log(x)) + O(x^\theta)$$

where P is a polynomial and $\theta<1$. The argument goes as follows.

The first step is to prove a mean value result. Let

$$S(x) = \sum_{\chi\in\hat{C}_{0,S}} \int_{\mathcal{J}_\chi(x)} \left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^2 dt$$

and let $E(x)=S(x)-xP(\log(x))$. Then

$$\int_0^x E(u)^2 du \ll x^{2\delta+1}. \tag{2}$$

The second step invokes the following

Lemma. (Ivic and Motohashi) *If*

$$\int_0^x E(u)^2 du \ll x^{2\delta+1}$$

then

$$E(x) \ll x^{\frac{2\delta+3}{3}}.$$

To get the mean value result, make use of

$$\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{x^w}{w}dw = \begin{cases} 1 & \text{if } x>1 \\ 0 & \text{if } 0<x<1 \end{cases} + O_\varepsilon\left(x^{1+\varepsilon}\min\left(1,\frac{1}{T\log(x)}\right)\right).$$

Integrate around a rectangle with vertices $1+\varepsilon-iT$, $1+\varepsilon+iT$, $\delta+iT$, $\delta-iT$

$$\frac{1}{2\pi i}\int_{\text{rectangle}} Z(w)\frac{x^w}{w}dw = xP(\log(x)) + \text{error}.$$

Split the integral into four parts.

$$\lim_{T\longrightarrow\infty}\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}Z(w)\frac{x^w}{w}dw=S(x).$$

Using Phragmen-Lindelöf we can choose $\delta<1$ such that $Z(w)\ll|\mathrm{Im}(w)|^{\text{very small power}}$. What we need is to choose δ so that $Z(w)/w$ is L^2, i.e.

$$\int_{\delta-i\infty}^{\delta+i\infty}\left|\frac{Z(w)}{w}\right|^2dw<\infty.$$

By the Plancherel formula

$$\int_{\delta-i\infty}^{\delta+i\infty}\left|\frac{Z(w)}{w}\right|^2dw\leqslant\int_0^{\infty}E(u)^2u^{-2\delta-1}du.$$

(Titchmarsh argument.)

$$\int_0^{\infty}E(u)^2u^{-2\delta-1}du=O(1)$$

and so

$$\int_0^{x}E(u)^2u^{-2\delta-1}du\ll1.$$

$$x^{-2\delta-1}\int_{x/2}^{x}E(u)^2\ll\int_{x/2}^{x}E(u)^2u^{-2\delta-1}du\ll1$$

which gives (2).

Now using the Lemma of Ivic and Motohashi, we have

$$E(x)\ll x^{\theta},\qquad\theta=\frac{2\delta+3}{3}<1$$

and we can use this to break convexity.

$$S(x)=\sum_{\chi\in\hat{C}_{0,S}}\int_{\mathcal{J}_\chi(x)}\left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^2dt\sim xP(\log(x))+O(x^{\theta})$$

with $\theta<1$. If $x'<x$ $\mathcal{J}_\chi(x')\subset\mathcal{J}_\chi(x)$. We consider

$$S(x+H)-S(x).$$

(We will eventually take $H=x^{\theta}$.) In this difference we are left with positive terms. That is, any term that appears in $S(x+H)$ appears in $S(x)$. We get

$$\int_{\mathcal{J}_\chi(x+H)\backslash\mathcal{J}_\chi(x)}\left|L\left(\frac{1}{2}+it,f\right)\right|^2dt<S(x+H)-S(x).$$

Now

$$\mathcal{J}_1(x)\backslash\mathcal{J}_1(x)=\{t\in\mathbb{R}|\,x<(1+|t|)^{r_1}(1+4t^2)^{r_2}<x+H\},$$

so with $d=[k:\mathbb{Q}]$

$$\int_{(x/4^{r_2})^{1/d}}^{((x+H)/4^{r_2})^{1/d}-1}\left|L\left(\frac{1}{2}+it,f\right)\right|^2dt<\int_{\mathcal{J}_\chi(x+H)\backslash\mathcal{J}_\chi(x)}\left|L\left(\frac{1}{2}+it,f\right)\right|^2dt\ll H^{1+\varepsilon}$$

as long as

$$x^{\theta} \ll H < \frac{x}{10}.$$

We can replace x by $4^{r_2}x$ and H by $4^{r_2}H$ and we get

$$\int_{x^{1/d}}^{(x+H)^{1/d}} \left|L\left(\frac{1}{2}+it, f\right)\right|^2 dt \ll H^{1+\varepsilon}.$$

$$(x+H)^{1/d} = x^{1/d}\left(1+\frac{H}{x}\right)^{1/d}.$$

This gives

$$\int_{x^{1/d}}^{x^{1/d}(1+\frac{\Delta}{x})-1} \left|L\left(\frac{1}{2}+it, f\right)\right|^2 dt \ll \Delta^{1+\varepsilon}.$$

Take

$$T = x^{1/d}, \qquad M = x^{\frac{1}{d}-1}\Delta.$$

$$\int_T^{T+M} \left|L\left(\frac{1}{2}+it, f\right)\right|^2 dt \ll (T^{d-1}M)^{1+\varepsilon} \ll T^{d\theta(1+\varepsilon)}, \qquad T^{d\theta+1-d} \ll M \ll T^{d\theta+1-d}.$$

Now by an argument of Anton Good this implies that

$$L\left(\frac{1}{2}+it, f\right) \ll t^{d\theta/2+\varepsilon}.$$

Convexity is

$$L\left(\frac{1}{2}+it, f\right) \ll t^{d/2+\varepsilon}.$$

By the same argument you can break convexity in the Grössencharacter parameter.

Concluding remark

What allows us to break convexity is

$$\sum_{\chi \in \hat{C}_{0,S}} \int_{\mathcal{J}_\chi(x^{[k:\mathbb{Q}]})} \left|L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 dt \sim c x^{[k:\mathbb{Q}]}(\log(x))^{r_1+r_2} + O(x^{\theta_k}).$$

If we had an exponent $x^{2[k:\mathbb{Q}]}$ we couldn't break convexity since

$$\frac{[k:\mathbb{Q}]}{\theta_k} \longrightarrow 1.$$