Chinta, 7/11

Last time:
Φ an irreducible reduced root system (simply laced) of rank r.
(1) Defined an action of W on rational functions in $x_1, \cdots, x_r$.
(2) Showed that the p-parts of $\Phi^{(2)}$ are invariant under this action.
(3) Construted an invariant rational function.

Our goals:
(i) Discuss function field MDS
(ii) Suggest a genearlization of (2) to higher order covers.

1 Function field basics

$k = \mathbb{F}_q(t)$. If $f \in \mathbb{F}_q[t]$, $|f| = q^{\deg(r)}$.

$$\zeta(s) = \sum_{\substack{f \neq 0 \\ \text{monic}}} \frac{1}{|f|^s} = \left(1 - q^{1-s}\right)^{-1} = \sum_{\substack{p \neq 0 \\ \text{monic, irreducible}}} \left(1 - \frac{1}{|p|^s}\right).$$

Functional equations: $\zeta^*(s) = \zeta(s)(1 - q^{-s})^{-1} = q^{2s-1}\zeta^*(s)$. Let

$$\left(\frac{f}{g}\right) = \left(\frac{f}{g}\right)_2 = \chi_f(g).$$

Define

$$L(s, \chi_f) = \sum_{\substack{g \neq 0 \\ \text{monic}}} \frac{\chi_f(g)}{|g|^s} = \sum_{\substack{p \neq 0 \\ \text{monic, irreducible}}} \left(1 - \frac{\chi_f(p)}{|p|^s}\right).$$

$$L(s, \chi_f) = \begin{cases} q^{s-1} \frac{1 - q^{-s}}{1 - q^{s-1}} |f|^{\frac{1}{2} - s} L(1 - s, \chi_f) & \text{if } \deg(f) \text{ is even;} \\ (q|f|)^{\frac{1}{2} - s} q^{2s-1} L(1 - s, \chi_f) & \text{if } \deg(f) \text{ is odd.} \end{cases}$$

2 $A_2^{(2)}$ MDS: $Z(s_1, s_2)$.

Roughly

$$Z(s_1, s_2) = \sum_{c_1, c_2} \frac{\left(\frac{c_1}{c_2}\right)}{|c_1|^{s_1} |c_2|^{s_2}} = Z(x, y), \qquad x = q^{-s_1}, y = q^{-s_2}.$$

We abuse notation and use the same letter Z. And:

$$Z(s_1,s_2;i,j)=\sum_{\substack{\deg(c_1)\equiv i \bmod 2\\ \deg(c_2)\equiv j \bmod 2}} \frac{\left(\frac{c_1}{c_2}\right)}{|c_1|^{s_1}|c_2|^{s_2}}=Z(x,y).$$

Functional equations for these *global objects* over a function field

$$\begin{aligned} Z(x,y;q) &= \frac{1}{qx^2}\frac{1-x}{1-\frac{1}{qx}}\left(\frac{Z\left(\frac{1}{qx},xy\sqrt{q}\right)+Z\left(\frac{1}{qx},-xy\sqrt{q}\right)}{2}\right)\\ &+ \frac{1}{x\sqrt{q}}\left(\frac{Z\left(\frac{1}{qx},xy\sqrt{q}\right)-Z\left(\frac{1}{qx},-xy\sqrt{q}\right)}{2}\right).\end{aligned}$$

Compare from the previous lecture the functional equations of the p-parts of the global A_2 object

$$\begin{aligned} \mathcal{G}(x,y;p) &= \frac{1-\frac{1}{py}}{1-y}\left[\frac{\mathcal{G}\left(xy\sqrt{p},\frac{1}{py}\right)+\mathcal{G}\left(-xy\sqrt{p},\frac{1}{py}\right)}{2}\right]\\ &+ \frac{1}{y\sqrt{p}}\left[\frac{\mathcal{G}\left(xy\sqrt{p},\frac{1}{py}\right)-\mathcal{G}\left(-xy\sqrt{p},\frac{1}{py}\right)}{2}\right].\end{aligned}$$

Example:

$$\mathcal{G}(x,y;p)=\frac{1-xy}{(1-x)(1-y)(1-px^2y^2)}$$

and the function field MDS is

$$Z(x,y;q)=\frac{1-q^2xy}{(1-qx)(1-qy)(1-q^3x^2y^2)}.$$

We can write

$$\mathcal{G}(x,y;p)=Z(px,py;p^{-1}),\qquad Z(x,y;p)=\mathcal{G}(px,py;p^{-1}).$$

If we had the uniqueness assertion discussed from the previous lecture, we would know analogous identities for A_r.

Problem. Examples in higher genus. There is one example in Fisher-Friedberg.

3 Higher degree: $A_2^{(n)}$

It is a little complicated to state the functional equation. We break $Z(s_1,s_2;A_2^{(n)})$ down as

$$\sum_{i,j} Z(s_1,s_2;i,j;A_2^{(n)})$$

where

$$Z(s_1, s_2; i, j; A_2^{(n)}) = \sum_{\substack{\deg(c_1) \equiv i \bmod 2 \\ \deg(c_2) \equiv j \bmod 2}} \frac{\overline{\left(\frac{c_1}{c_2}\right)_n} g(1, c_1) g(1, c_2)}{|c_1|^{s_1} |c_2|^{s_2}}.$$

We change the normalization from previous lectures, and also from the quadratic case, where the Gauss sums were suppressed.

$$\begin{aligned} Z(x, y; i, j) &= P_{ij}(x) Z\left(\frac{1}{q^2 x}, qxy; i, j\right) \\ &\quad + Q_{ij}(x) Z\left(\frac{1}{q^2 x}, qxy; j+1-i, j\right), \end{aligned}$$

where

$$P_{ij}(x) = -(qx)^{1-(-2i+j+1)n} \frac{q-1}{1-q^{n+1}x^n}$$

where $(\alpha)_n \equiv \alpha \bmod n$ and $0 \leqslant \alpha \leqslant n-1$, and

$$Q_{ij}(x) = -\tau(\varepsilon^{2i-j-1})(qx)^{1-n} \frac{1-q^n x^n}{1-q^{n+1}x^n}$$

where

$$\tau(\varepsilon^i) = \sum_{a \bmod q} \varepsilon^i\left(\left(\frac{a}{q}\right)_n\right) e\left(\frac{a}{q}\right)$$

where

$$\varepsilon\colon \mu_n \longrightarrow \mathbb{C}^\times$$

is a fixed homomorphism. (Hoffstein writes τ_i as $g_i(1, t)$.) We can compute

$$Z(x, y; A_2^{(n)}) = \frac{1 + \tau_1 qx + \tau_1 qy + \tau_1\tau_2 q^3 xy^2 + \tau_1\tau_2 q^3 x^2 y + \tau_1^2 \tau_2 q^4 x^2 y^2}{(1-q^{n+1}x^n)(1-q^{n+1}y^n)(1-q^{2n+1}x^n y^n)}.$$

A miracle occurs when we compare this to the p-part of the $A_2^{(n)}$ MDS: this is

$$1 + g(1,p)x + g(1,p)y + g(1,p)g(p,p^2)x^2 y + g(1,p)g(p,p^2)xy^2 + g(1,p)^2 g(p,p^2)x^2 y^2.$$

Indeed, if we map

$$\begin{aligned} \tau_i &\longmapsto g_i(1,p)/p \\ q &\longmapsto 1/p \\ x, y &\longmapsto p^2 x, p^2 y \end{aligned}$$

we get an exact correspondence.

We can define an action of W on rational functions in $x_1, \cdots, x_r$, then average over the Weyl group as in the Weyl character formula, and conjecture that this sum agrees with the Gelfand-Tsetlin conjecture. Evidence: this is known for $A_2^{(n)}$ and for $A_r^{(2)}$ when $r \leqslant 5$. It is known for $A_3^{(n)}$ when $n = 3$ or 4.

Problem. Prove glo bal functional equations using invariance of local computation under W.