Hoffstein, 7/10

There are these mysterious objects that people would like to understand, and they are defined like this.

$$\frac{\tau^{(n)}(m)}{\mathbb{N}m^{1/2n}} = \mathrm{Res}_{s=\frac{1}{2}+\frac{1}{n}} \frac{G(m,d)}{\mathbb{N}d^s}, \qquad G(m,d) = \frac{1}{\sqrt{\mathbb{N}d}} \sum \left(\frac{\alpha}{d}\right)_n e\left(\frac{m\alpha}{d}\right).$$

More generally

$$G_j(m,d) = \frac{1}{\sqrt{\mathbb{N}d}} \sum \left(\frac{\alpha}{d}\right)_n^j e\left(\frac{m\alpha}{d}\right), \qquad G = G_1.$$

We are of course over a field containing the n-th roots of unity. We denote

$$D^{(n)}(s;m) = \mathrm{Res}_{s=\frac{1}{2}+\frac{1}{n}} \frac{G(m,d)}{\mathbb{N}d^s}.$$

This is a Fourier coefficient of an Eisenstein series on the metaplectic n-fold cover of GL(2), and its residue $\tau(m)$ is thus the Fourier coefficient of the residue, which is the theta function $\theta^{(n)}$. The fact that the Eisenstein series has a pole can be read off from the constant term, and from this it can be deduced that $\tau(m)$ is nonzero for some m.

The Eisenstein series is an eigenfunction of the Hecke operators, and so θ is also. For example, if $n=2$ and $m=1$, we want to compute

$$\mathrm{Res}_{s=1} \frac{G(1,d)}{\mathbb{N}d^s}.$$

In this quadratic case, $G(1,d)$ is essentially 1 and this is the Dedekind zeta function. By contrast, if $n=3$, the answer is much harder and it was figured out by Patterson that

$$\tau^{(3)}(p) = \overline{G(1,p)} \qquad \text{(cubic Gauss sum.)}$$

Patterson used a delicate argument involving a converse theorem, but it was pointed out later by Deligne that one could also prove this as a consequence of the uniqueness of Whittaker models – which is true for $\theta^{(n)}$ when $n=2$ or 3 but not when $n>3$. This result was used by Patterson and Heath-Brown to disprove the Kummer conjecture.

Discussion of the second moment of the Gauss sums, and the identity

$$G(\chi)^3 = J(\chi,\chi) = \left(\frac{\pi}{|\pi|}\right)\mu(\pi)$$

was invoked to explain why these are problematic.

Returning to Deligne's point, this can be expressed in classical language using the Hecke operators. In general

$$\lambda_p = p^{1/2} + p^{-1/2}$$

$$\tau^{(n)}(mp^j) = G_{j+1}^{(n)}(m,p)\tau^{(n)}(mp^{n-2-j}), \qquad 0 \leqslant j \leqslant n-1,$$

where $(m,p)=1$. If we normalize $\tau(1)=1$ we have

$$\tau^{(n)}(mp^{n-1}) = 0, \qquad \tau^{(n)}(p^{n-2}) = G_{n-1}(1,p) = \overline{G(1,p)}.$$

(The last identity is true if -1 is an n-th power, and it is useful to assume this.)

In general

$$\tau(mp^n) = \tau(m)\mathbb{N}p^{1/2},$$

and this fact, known as the "periodicity theorem" is valid even if m and p are not assumed to be coprime.

When $n=4$, uniqueness of Whittaker functions fails, and the coefficients $\tau(m)$ are only partly known. Toshiaki Suzuki as well as Patterson worked on this.

$$\begin{aligned}\tau(1) &= 1\\ \tau(p) &= ?\\ \tau(p^2) &= \overline{G^{(4)}(1,p)}\\ \tau(p^3) &= 0.\end{aligned}$$

If p and q are distinct primes

$$\tau^{(4)}(pq) = G_2^{(4)}(q,p)\tau^{(4)}(pq).$$

Now $G_2^{(4)}(q,p)$ is actually a quadratic Gauss sum and can be evaluated:

$$G_2^{(4)}(q,p) = \left(\frac{q}{p}\right)_2.$$

This means that $\tau^{(4)}(pq)=0$ if q is a nonresidue.

The Patterson conjecture is a conjectured identity of two Rankin-Selberg Dirichlet series:

$$\zeta(4s-1)\sum\frac{\tau(m)^2}{\mathbb{N}m^s} = \left(\zeta(4s-1)\sum\frac{G(1,m)}{\mathbb{N}m^s}\right)^2.$$

The evidence is that this implies identities between the (partially determined) coefficients $\tau^{(4)}$ that are consistent with everything that is already known to be true; on the other hand, the gamma factors and locations of the poles are consistent. If m is square-free, the conjecture implies

$$\tau(m)^2 = \sum_{m_1m_2=m} G(1,m_1)G(1,m_2).$$

This gives

$$\begin{aligned}\tau(p)^2 &= 2\overline{G(1,p)}\\ \tau(pq) &= 2\overline{G(1,pq)}\left[1+\left(\frac{p}{q}\right)+\left(\frac{q}{p}\right)+1\right].\end{aligned}$$

Note that if $(p/q)=-1$ this is equivalent to the previously mentioned vanishing.

Bump and Hoffstein gave related conjectures that are different from the Patterson conjecture. For the first order theta function, let $\theta_r^{(n)}$ be the theta function on the n-cover of GL_r. BH conjectured

$$L(s,\theta_2^{(4)}\times\theta_3^{(4)}) = L(s,\overline{\theta_2^{(4)}}\times\theta_1^{(4)});$$

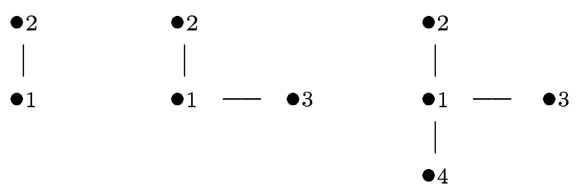
the right-hand side should be interpreted as just the Mellin transform $L(s,\theta_2^{(4)})$.

Patterson's conjecture implies the BH conjecture for θ_2, but unlike the Patterson conjecture, the BH has a substantial generalization to other general linear groups.

If $n=5$, very little is known but D_4 gives a potential attack.

$$\begin{aligned} \tau(1) &= 1 \\ \tau(p) &= ? \\ \tau(p^2) &= ? \\ \tau(p^3) &= \overline{G^{(4)}(1,p)} \\ \tau(p^4) &= 0. \end{aligned}$$

We think if D_4 as the third in a progression of Dynkin diagrams, A_2, A_3 and D_4.



A_2 A_3 D_4

For A_2 and the $n=3$ cover, consider the following

$$\begin{aligned} \sum \frac{\left(\frac{m_1}{m_2}\right)\overline{G(1,m_1)}\,\overline{G(1,m_2)}}{m_1^{s_1}m_2^{s_2}} &= \sum \frac{\overline{G(1,m_1)}\,\overline{G(m_1,m_2)}}{m_1^{s_1}m_2^{s_2}} \\ &= \sum \frac{\overline{G(1,m_1)}}{m_1^{s_1}} \sum_{m_2} \frac{\overline{G(m_1,m_2)}}{m_2^{s_2}}. \end{aligned}$$

The residue

$$\sum_{m_1} \frac{\overline{G(1,m_1)}}{m_1^{s_1}} \tau^{(3)}(m_1)$$

has a pole at $s_1 = \frac{5}{6}$, which makes it plausible that the argument of $\tau^{(3)}(m_1)$ and the argument of $\overline{G(1,m_1)}$ are opposite. In this case, we can verify this because the value of $\tau^{(3)}(m_1)$ is known. However the same idea is applicable in cases where such information is not available.

Turning next to A_3 and $n=4$,

$$\sum \frac{\overline{\left(\frac{m_1}{m_2}\right)\left(\frac{m_1}{m_3}\right)}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}} = Z(s_1,s_2,s_3).$$

Taking the residue at s_2 and $s_3 = \frac{3}{4}$ gives

$$\sum \frac{G(1,m_1)\tau(m_1)^2}{m_1^{s_1+1/4}}$$

and the fact that this has a pole at $s_1 = 3/4$ gives evidence for the Patterson conjecture.

Now turning to D_4. We will get an application if $n = 5$.

$$\sum \frac{\overline{\left(\frac{m_1}{m_2}\right)\left(\frac{m_1}{m_3}\right)\left(\frac{m_1}{m_4}\right)}}{m_1^{s_1} m_2^{s_2} m_3^{s_3} m_4^{s_4}} G(1, m_1) G(1, m_2) G(1, m_3) G(1, m_4) = Z(s_1, s_2, s_3, s_4).$$

The residue is at $s_2 = s_3 = s_4 = \frac{1}{2} + \frac{1}{n}$. When $n = 5$

$$\sum \frac{G(1, m_1)\tau(m_1)^3}{m_1^{s_1 + 3/2n}} = \sum \frac{G(1, m_1)\tau(m_1)^3}{m_1^{s_1 + 3/10}}.$$

There appears to be a pole to the right of 7/10 but this is believed not to be a real pole. There is a question about whether the Eisenstein series on SO_8 has a pole at that location; even if it does, the Dirichlet series might not have a pole if the residue of the Eisenstein series does not have a Whittaker model.

From the multiple Dirichlet series point of view, we set $s_1 = s_2 = s_3 = s$ and ask for the functional equations of

$$Z(s, w) = \sum \frac{\sum (G(m, d)/d^s)^3 G(1, m)}{m^w}.$$

(From now on $n = 5$). There is a pole of order 3 at $s = 7/10$. The normalizing factor involves $\Gamma_5(s)^3$ and $\Gamma_5(w)$, where Γ_5 is the Gamma function needed by the Kubota Dirichlet series (see WMD1). The functional equations are

$$Z(s, w) \longmapsto Z\left(1 - s, w + 3s - \frac{3}{2}\right), \qquad \text{or} \qquad Z\left(s + w - \frac{1}{2}, 1 - w\right).$$

We iterate the functional equations several times:

$$\begin{aligned} Z(s, w) &= Z\left(s + w - \frac{1}{2}, 1 - w\right) \\ &= Z(w + 2s - 1, 3 - 2w - 3s) \\ &= Z\left(2 - 2s - w, w + 3s - \frac{3}{2}\right) \\ &= Z\left(s, \frac{5}{2} - 3s - w\right). \end{aligned}$$

As a sanity check we can go one more time and get $Z(1 - s, 1 - w)$, but the last expression is what we need.

$$\mathrm{Res}^3_{s = \frac{7}{10}} Z(s, w) \longrightarrow \mathrm{Res}^3_{s = \frac{7}{10}} Z(s, w).$$

The notation means that since s is really three variables $s_1 = s_2 = s_3$ this is not really the residue but the coefficient of $(s - \frac{7}{10})^{-3}$ in the Taylor expansion. We needed

$$\Gamma_5(s + w - 1/2)\Gamma_5(w)\Gamma_5(3s + 2w - 2)\Gamma_5(w + 2s - 1)^3 \Gamma_5\left(s + 3s - \frac{3}{2}\right)$$

in the normalizing factor (and actually one more gamma function).

$$\mathrm{Res}^3_{s=\frac{7}{10}} Z(s,w) = \sum \frac{\tau^{(5)}(m)^3 G(1,m)}{m^{w+3/10}} = L_5(w+3/10)$$

where

$$L_5(u) = \sum \frac{\tau^{(5)}(m)^3 G(1,m)}{m^u}.$$

There is a reality check at this point.

Returning to $Z(s,w)$ there are poles at

$$\begin{aligned} s = \frac{7}{10}, \frac{3}{10}\,(\text{triple}), &\quad s+w-\frac{1}{2} = \frac{7}{10}, \frac{3}{10}, \\ s = \frac{7}{10}, \frac{3}{10}, &\quad 3s+2w-2 = \frac{7}{10}, \frac{3}{10}, \\ w+2s-1 = \frac{7}{10}, \frac{3}{10}, &\quad w+3s-\frac{3}{2} = \frac{7}{10}, \frac{3}{10}, \\ w+\frac{1}{5} = \frac{7}{10}, \frac{3}{10}, &\quad w+\frac{2}{5} = \frac{7}{10}, \frac{3}{10} \\ \frac{3}{5}+2w-\frac{1}{2} = & \end{aligned}$$

These notes will be completed when Jeff finishes straightening this out.