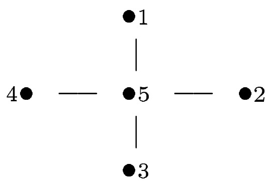
Alina Bucur, 7/10

First example of an affine kind of group. Extended Dynkin diagram of D_4, related to the 4-th moment.



$(5 \longrightarrow w)$

Heuristically (see WMD by Bump Brubaker, Chinta, Friedberg, Hoffstein) we can make a Dirichlet series by including a symbol for every joined pair of vertices

$$\sum \frac{\left(\frac{d}{m_1}\right)\left(\frac{d}{m_2}\right)\left(\frac{d}{m_3}\right)\left(\frac{d}{m_4}\right)}{m_1^{s_1} m_2^{s_2} m_3^{s_4} m_4^{s_4} d^w} = \sum \frac{L(s_1, \chi_d) L(s_2, \chi_d) L(s_3, \chi_d) L(s_4, \chi_d)}{d^w}$$
$$= \sum \frac{L(w, \chi_{m_1 m_2 m_3 m_4})}{m_1^{s_1} m_2^{s_2} m_3^{s_4} m_4^{s_4}}$$

Functional equations:

$$(\boldsymbol{s}, w) \longmapsto \left(1 - s_1, s_2, s_3, s_4, w + s_1 + \frac{1}{2}\right)$$
$$(\boldsymbol{s}, w) \longmapsto \left(w + s_1 + \frac{1}{2}, w + s_2 + \frac{1}{2}, w + s_3 + \frac{1}{2}, w + s_4 + \frac{1}{2}, 1 - w\right)$$

These generate an infinite group of functional equations. The multiple DS cannot have analytic continuation to all s, w. (Picture)

We need the correction polynomials (d_0 squarefree)

$$\sum_{d = d_0 d_1^2} \frac{L(s_1, \chi_{d_0}) L(s_2, \chi_{d_0}) L(s_3, \chi_{d_0}) L(s_4, \chi_{d_0}) P(\boldsymbol{s}, d_0)}{d^w} =$$
$$\sum \frac{L(w, \chi_{m_0}) Q(w; m_1, m_2, m_3, m_4)}{m_1^{s_1} m_2^{s_2} m_3^{s_4} m_4^{s_4}}$$

where m_0 is the squarefree part of $m_1 m_2 m_3 m_4$.

(Discussion of the fact that the Gamma factor for $L(s, \chi_d)$ depends on the sign of d.)

$$P(\boldsymbol{s}, p^{2\alpha}) = \sum_{0 \leqslant k_i \leqslant 4\alpha} a^{(2\alpha)}_{k_1, k_2, k_3, k_4} p^{-k_1 s_1 - k_2 s_2 - k_3 s_3 - k_4 s_4}.$$

The coefficients are mostly determined except the middle term, for example:

$$a^{(2)}_{2,2,2,2}.$$

Still there is a philosophy that can help us to pick these. We ask that the residue

$$\begin{aligned}\mathrm{Res}_{w=1} Z(\boldsymbol{s}, w) &= \prod_{l=0}^{\infty} \prod_{j=1}^{4} \zeta(2l(s_1+\ldots+s_4)+2s_j-2l) \\ &\quad \zeta(2(l+1)(s_1+\ldots+s_4)+2s_j-2l-1) \\ &\quad \prod_{1\leqslant j_1<j_2\leqslant 4} \zeta(l(s_1+\ldots+s_4)+s_{j_1}+s_{j_2}-l).\end{aligned}$$

This converges when $s_1+s_2+s_2+s_4>1$. It has the right group of functional equations, and if we let $s_4 \longrightarrow \infty$ it has the limit

$$\zeta(2s_1)\zeta(2s_2)\zeta(2s_3)\zeta(2s_1+2s_2+2s_3-1)\zeta(s_1+s_2)\zeta(s_1+s_3)\zeta(s_2+s_3).$$

This is the residue of the corresponding multiple Dirichlet series for the third moment, which is an ordinary Weyl group multiple Dirichlet series. Hence this is the natural candidate for the residue.

In the case of a rational function field, this can be accomplished. Over $\mathbb{Q}$ this description is conjectural.

In the function field case, we can write

$$\begin{aligned}\mathrm{Res}_{w=1} Z(\boldsymbol{s}, w) &= \sum c(\boldsymbol{n}, m) q^{-n_1 s_1 - n_2 s_2 - n_3 s_3 - n_4 s_4 - m w} \\ &= \sum \frac{L(\boldsymbol{s}, \chi_d)}{d^w}.\end{aligned}$$

Now we claim that if we impose these conditions:

- This residue
- Functional equation
- Polar divisor

then it is the right multiple Dirichlet series. Note that the residue is

$$\sum_{m_1 m_2 m_3 m_4 \text{ is a square}} \frac{Q(1, \boldsymbol{m})}{m^{\boldsymbol{s}}}$$

so the residue controls the undetermined coefficients.

There are two candidate functions that are constructed; these are then shown to be the same.

- $Z_{\mathrm{res}}(\boldsymbol{s}, w) = \sum c(\boldsymbol{n}, m) q^{-n_1 s_1 - n_2 s_2 - n_3 s_3 - n_4 s_4 - m w}$ is a power series with $c(0, \cdots, 0) = 1$ and it has the correct residue.
- $Z_{P,Q}$ is a multiple DS described as above with particular P and Q, related to the 4-th moment.

The following is carried out:

1. Show that a power series (*) is uniquely determined by these conditions.
2. Produce $Z_{\mathrm{res}}(\boldsymbol{s}, w)$ that satisfies the conditions and continues to the region

$$\mathrm{re}(s_1+s_2+s_3+s_4+2w) > 3.$$

3. Uniqueness implies that $Z_{P,Q} = Z_{\text{res}}$.

The uniqueness in (1) is:

Lemma. *Let* $\sum c(\boldsymbol{n},\ m) q^{-n_1 s_1 - n_2 s_2 - n_3 s_3 - n_4 s_4 - m w}$ *be convergent to a meromorphic function when*

$$\text{re}(s_1 + s_2 + s_3 + s_4 + 2w) > 6 \tag{1}$$

be such that $c(0, \cdots, 0) = 1$ *and* res_w *is the function described above, with the right polar divisor and functional equation. Then it is uniquely determined. "Polar behavior" menas*

$$D(\boldsymbol{s}, w) = \prod_\gamma (1 - q^{2-2w})|_\gamma$$

and $Z_{\text{res}} \cdot D = \tilde{Z}_{\text{res}}$ *has analytic continuation to (1).*

Proof is by induction on m. Assume that the coefficients $\tilde{c}(\boldsymbol{n}, m)$ with $m < m_0$ are known. Then we may use the functional equation to relate

$$\alpha_1 \colon \tilde{c}(n_1, \cdots, n_4, m) \longleftrightarrow \tilde{c}(m - n_1 + 1, n_2, \cdots, n_4, m),$$

$$\beta \colon \tilde{c}(n_1, \cdots, n_4, m) \longleftrightarrow \tilde{c}(\boldsymbol{n}, n_1 + n_2 + n_3 + n_4 - m + 1).$$

There is a scattering matrix but the scattering is by lower order terms, and by induction these are determined. β and induction hypothesis determines all coefficients with

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 - m_0 + 1 &\leqslant m_0 - 1 \\ n_1 + n_2 + n_3 + n_4 &\leqslant 2m_0 - 2. \end{aligned}$$

Permuting the coefficients (which are symmetric) we may assume

$$n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4.$$

Assume that we know $\tilde{c}$ for $\boldsymbol{n}' < \boldsymbol{n}$ (lexicographic order).

$$\tilde{c}(\boldsymbol{n}, m_0) \longrightarrow \tilde{c}(m_0 - n_1 + 1, \cdots)$$

determined for $m_0 + n_1 + 1 \leqslant n_1 - 1$ or $m_0 \leqslant 2n_1 - 2$. After all this, the only undetermined coefficients are $c(n, n, n, n, 2n)$ and $c(n, n, n, n-1, 2n)$, and these we may determine by the residue. When we specialize $w \longrightarrow 1$,

$$\sum_{m \geqslant 0} c(n_1, \cdots, n_4, m) q^{-m}$$

is the coefficient of $q^{-n_1 s_1 - n_2 s_2 - n_3 s_3 - n_4 s_4}$ in $\tilde{Z}_{\text{res}}(\boldsymbol{s}, 1)$.

Now we want bounds

$$\tilde{c}(\boldsymbol{n}, m) = O\Big(q^{(n_1 + n_2 + n_3 + n_4 + m)/2} \Big).$$

Really what is proved is

$$\tilde{c}(\boldsymbol{n}, m) = O\Big(q^{(n_1 + n_2 + n_3 + n_4 + m)/2 + A\sqrt{n_1 + \ldots + n_4} + m} \Big).$$

The proof is combinatorial.