Gautam Chinta, 7/9

The objective is to talk about the joint work with Gunnells, but I will start with the simplest case.

1 Instructive example

- $n = 2$ Quadratic Weyl group multiple Dirichlet series.

  $A_2$ Dynkin diagram: $\circ \longrightarrow \circ$

  $Z(s, w) = \sum_{d, m} \left( \frac{d}{m} \right) d^{-w} m^{-s}$

  originally studied by Siegel as the Mellin transform of $GL_2^{(2)}$ Eisenstein series. Later this was taken up by Goldfeld and Hoffstein. The interest is that the coefficients of the Eisenstein series are quadratic L-functions. This may be seen directly from the description above by collecting the coefficients of $m$:

  $Z(s, w) = \sum_{d, m} d^{-w} L(s, \chi_d)$.

  Under the functional equation $L(s, \chi_d) = d^{\frac{1}{2} - s} L(1 - s, \chi_d)$ which leads to

  $Z(s, w) = Z\left(1 - s, s + w - \frac{1}{2}\right)$.

  Similarly

  $Z(s, w) = Z\left(s + w - \frac{1}{2}, 1 - w\right)$.

  These two functional equations generate a group of order 6.

- Italy won the world cup.

  We want

  $Z(s, w) = \sum_d \sum_m \frac{\chi_{d_0}(\hat{m}) a(d, m)}{d^w m^s}$

  where

  $d = d_0 d_1^2$, $d_0$ squarefree

  and $\hat{m}$ is the part of $m$ relatively prime to $d_0$ and $a(d, m)$ is the multiplicative weight factor to be defined. In the context of the previous talk

  $\chi_{d_0}(\hat{m}) a(d, m) \longleftrightarrow H(d, m)$. 

1
Multiplicativity:

\[ a(d,m) = \prod_{p^a \mid d \text{ and } p^b \mid m} a(p^a, p^b). \]

Note: \((d,m) \rightarrow \chi_{d_0}(m) a(d,m)\) “twisted mult.”

Returning to

\[ \sum_d \frac{1}{d^w} \sum_m \frac{\chi_{d_0}(m) a(d,m)}{m^s} \]

the inner sum is

\[ \prod_p \left( 1 + \frac{a(d, p) \chi_{d_0}(p)}{p^s} + \ldots \right) = L(s, \chi_{d_0}) P_d(s). \]

Want (for the inner sum)

\[ L(s, \chi_{d_0}) P_d(s) \rightarrow d^{\frac{1}{2}-s} L(1-s, \chi_{d_0}) P_d(1-s). \]

We know the functional equation of \( L \) and so

\[ P_d(s) = d^{1-2s} P_d(1-s). \]

Euler product form \( P_d(s) \):

\[ P_d(s) = \prod_{p^{2k+1} \mid d} P_{p^{2k+1}}(s) \cdot \prod_{p^{2k} \mid d} P_{p^{2k}}(s, \chi_{d_0}(p)). \]

Want to know

\[ P_{p^{2k+1}}(s) = (p^{2k})^{\frac{1}{2}-s} P_{p^{2k+1}}(1-s), \]

\[ P_{p^{2k}}(s) = (p^{2k})^{\frac{1}{2}-s} P_{p^{2k}}(1-s). \]

Next re-express FE above in terms of the \( a(d,m) \), p-part of \( L(s, \chi_d) \). Case 1:

\[ p^{2k+1} \parallel d, \quad \text{so } p \parallel d_0. \]

**Principle:** Suppose that

\[ f(x) = x^n f\left( \frac{1}{x} \right), \quad f \text{ entire.} \]

Then \( f \) is a polynomial in \( x \) of degree \( n \).

We extract the \( p \)-part

\[ G(x, y) = \sum_{k,l=0}^{\infty} a(p^k, p^l) x^k y^l. \]

Break this into two parts, \( G = G_0 + G_1 \) where \( G_0 \) is the part where \( k \) is even and \( G_1 \) is the part where \( k \) is odd. We get.

\[ G_0(x, y) = G_0\left( x y \sqrt{p}, \frac{1}{p y} \right) \left( 1 - \frac{1}{p y} \right) \]
and can do FE for 
\[(x, y) \mapsto \left( xy\sqrt{p}, \frac{1}{py} \right) .\]

Note
\[
G_0(x, y) = \frac{G(x, y) + G(-x, y)}{2},
\]
\[
G_1(x, y) = \frac{G(x, y) - G(-x, y)}{2}.
\]

Conclusion:
\[
G(x, y) = \frac{1 - \frac{1}{py}}{1 - y} \left[ \frac{G(xy\sqrt{p}, \frac{1}{py}) + G(-xy\sqrt{p}, \frac{1}{py})}{2} \right] + \frac{1}{y\sqrt{p}} \left[ \frac{G(xy\sqrt{p}, \frac{1}{py}) - G(-xy\sqrt{p}, \frac{1}{py})}{2} \right].
\]

Similarly with \( x \) and \( y \) reversed. We also have limiting conditions
\[
G(x, 0) = \frac{1}{1 - x} \quad G(0, y) = \frac{1}{1 - y}.
\]

2 Reformulation

We define an action of the Weyl group \( W \) on power series in \( x \) and \( y \).
\[
W = S_3 = \langle \sigma_1, \sigma_2 | \sigma_1^2 = \sigma_2^2 = 1, (\sigma_1\sigma_2)^3 = 1 \rangle.
\]

Define
\[
(f|\sigma_2)(x, y) = \frac{1 - \frac{1}{py}}{1 - y} \left[ \frac{f(xy\sqrt{p}, \frac{1}{py}) + f(-xy\sqrt{p}, \frac{1}{py})}{2} \right] + \frac{1}{y\sqrt{p}} \left[ \frac{f(xy\sqrt{p}, \frac{1}{py}) - f(-xy\sqrt{p}, \frac{1}{py})}{2} \right],
\]

and similarly for \( \sigma_1 \). It may be verified that this is an action. This requires checking that the braid relation \((\sigma_1\sigma_2)^3 = 1\) is satisfied.

Let
\[
\Delta = (1 - px^2)(1 - py^2)(1 - p^2x^2y^2) = \prod_{\alpha \in \Phi^+} (1 - p^{d(\alpha)}x^{2\alpha}).
\]

This is not the normalizing factor of the Eisenstein series. Then
\[
\frac{\Delta(x)}{\Delta(u, x)} = j(w, x) = \text{sign}(w)p^{d(\alpha)}x^{2\alpha}.
\]
Then

\[ G(x, y) = \sum_{w \in W} \frac{(j | w) j(w, x)}{\Delta(x)}. \]

This is similar to the Weyl character formula. \( G(x, y) \) is a rational function with denominator

\[ (1 - x^2)(1 - y^2)(1 - px^2y^2) = \prod_{\alpha \in \Phi^+} (1 - p^{l(\alpha)-1}x^{2\alpha}). \]

Actually the denominator is

\[ (1 - x)(1 - y)(1 - px^2y^2), \]

but it is proper to multiply both the numerator and denominator by \((1 + x)(1 + y)\).

3 Problems

1. Is \( G \) unique? Uniqueness has been proved for \( A_r \) \((r \leq 5)\) and \( D_4 \).

2. What happens for infinite root systems? We don’t expect analytic continuation to all \( \mathbb{C}^r \), but we do expect continuation to the “Tits cone”. We don’t expect uniqueness without imposing more conditions.

3. One can investigate the case of an affine root system when \( p = 1 \). Is there a product structure in the affine case? Are there generalizations of Macdonald’s identities?

4. General \( n \)

5. Connection to \( G \times T \)

6. Function field.