Gautam Chinta, 7/9

The objective is to talk about the joint work with Gunnells, but I will start with the simplest case.

1 Instructive example

- $n=2$ Quadratic Weyl group multiple Dirichlet series.

$$A_2 \text{ Dynkin diagram:} \qquad \circ - \circ$$

$$Z(s,w)=\sum_{d,m}\left(\frac{d}{m}\right)d^{-w}m^{-s}$$

  originally studied by Siegel as the Mellin transform of $\mathrm{GL}_2^{(2)}$ Eisenstein series. Later this was taken up by Goldfeld and Hoffstein. The interest is that the coefficients of the Eisenstein series are quadratic L-functions. This may be seen directly from the description above by collecting the coefficients of m:

$$Z(s,w)=\sum_{d,m} d^{-w}L(s,\chi_d).$$

  Under the functional equation $L(s,\chi_d)=d^{\frac{1}{2}-s}L(1-s,\chi_d)$ which leads to

$$Z(s,w)=Z\left(1-s,s+w-\frac{1}{2}\right).$$

  Simlarly

$$Z(s,w)=Z\left(s+w-\frac{1}{2},1-w\right).$$

  These two functional equations generate a group of order 6.

- Italy won the world cup.

We want

$$Z(s,w)=\sum_{d}\sum_{m}\frac{\chi_{d_0}(\hat{m})a(d,m)}{d^w m^s}$$

where

$$d=d_0d_1^2, \qquad d_0 \text{ squarefree}$$

and $\hat{m}$ is the part of m relatively prime to d_0 and $a(d,m)$ is the multiplicative weight factor to be defined. In the context of the previous talk

$$\chi_{d_0}(\hat{m})a(d,m)\longleftrightarrow H(d,m).$$

Multiplicativity:

$$a(d,;m) = \prod_{\substack{p^\alpha \| d \\ p^\beta \| m}} a(p^\alpha, p^\beta).$$

Note: $(d,m) \longmapsto \chi_{d_0}(\hat{m})a(d,m)$ "twisted mult."

Returning to

$$\sum_d \frac{1}{d^w} \sum_m \frac{\chi_{d_0}(\hat{m})a(d,m)}{m^s}$$

the inner sum is

$$\prod_p \left(1 + \frac{a(d,p)\,\chi_{d_0}(\hat{p})}{p^s} + \ldots\right) = L(s, \chi_{d_0})\, P_d(s).$$

Want (for the inner sum)

$$L(s, \chi_{d_0})\, P_d(s) \longleftrightarrow d^{\frac{1}{2}-s} L(1-s, \chi_{d_0})\, P_d(1-s).$$

We know the functional equation of L and so

$$P_d(s) = d_1^{1-2s} P_d(1-s).$$

Euler product form $P_d(s)$:

$$P_d(s) = \prod_{p^{2k+1} \| d} P_{p^{2k+1}}(s) \cdot \prod_{p^{2k} \| d} P_{p^{2k}}(s, \chi_{d_0}(p)).$$

Want to know

$$P_{p^{2k+1}}(s) = (p^{2k})^{\frac{1}{2}-s} P_{p^{2k+1}}(1-s),$$

$$P_{p^{2k}}(s) = (p^{2k})^{\frac{1}{2}-s} P_{p^{2k1}}(1-s).$$

Next re-express FE above in terms of the $a(d,m)$. p-part of $L(s, \widehat{\chi_d})$. Case 1:

$$p^{2k+1} \| d, \qquad \text{so } p \| d_0.$$

Principle: *Suppose that*

$$f(x) = x^n f\left(\frac{1}{x}\right), \qquad f \text{ entire.}$$

Then f is a polynomial in x of degree n.

We extract the p-part

$$\mathcal{G}(x,y) = \sum_{k,l=0}^{\infty} a(p^k, p^l) x^k y^l$$

Break this into two parts, $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1$ where $\mathcal{G}_0$ is the part where k is even and $\mathcal{G}_1$ is the part where k is odd. We get.

$$\mathcal{G}_0(x,y) = \mathcal{G}_0\left(xy\sqrt{p}, \frac{1}{py}\right)\left(1 - \frac{1}{py}\right)$$

and can do FE for

$$(x,y)\longmapsto\left(xy\sqrt{p},\frac{1}{py}\right).$$

Note

$$\begin{aligned} G_0(x,y) &= \frac{G(x,y)+G(-x,y)}{2},\\ G_1(x,y) &= \frac{G(x,y)-G(-x,y)}{2}. \end{aligned}$$

Conclusion:

$$\begin{aligned} \mathcal{G}(x,y) &= \frac{1-\frac{1}{py}}{1-y}\left[\frac{\mathcal{G}\left(xy\sqrt{p},\frac{1}{py}\right)+\mathcal{G}\left(-xy\sqrt{p},\frac{1}{py}\right)}{2}\right]\\ &+ \frac{1}{y\sqrt{p}}\left[\frac{\mathcal{G}\left(xy\sqrt{p},\frac{1}{py}\right)-\mathcal{G}\left(-xy\sqrt{p},\frac{1}{py}\right)}{2}\right]. \end{aligned}$$

Similarly with x and y reversed. We also have limiting conditions

$$\begin{aligned} \mathcal{G}(x,0) &= \frac{1}{1-x}\\ \mathcal{G}(0,y) &= \frac{1}{1-y}. \end{aligned}$$

2 Reformulation

We define an action of the Weyl group W on power series in x and y.

$$W=S_3=\langle\sigma_1,\sigma_2|\sigma_1^2=\sigma_2^2=1,(\sigma_1\sigma_2)^3=1\rangle.$$

Define

$$\begin{aligned} (f|\sigma_2)(x,y) &= \frac{1-\frac{1}{py}}{1-y}\left[\frac{f\left(xy\sqrt{p},\frac{1}{py}\right)+f\left(-xy\sqrt{p},\frac{1}{py}\right)}{2}\right]\\ &+ \frac{1}{y\sqrt{p}}\left[\frac{f\left(xy\sqrt{p},\frac{1}{py}\right)-f\left(-xy\sqrt{p},\frac{1}{py}\right)}{2}\right], \end{aligned}$$

and similarly for σ_1. It may be verified that this is an action. This requires checking that the braid relation $(\sigma_1\sigma_2)^3=1$ is satisfied.

Let

$$\Delta=(1-px^2)(1-py^2)(1-p^2x^2y^2)=\prod_{\alpha\in\Phi^+}(1-p^{l(\alpha)}\boldsymbol{x}^{2\alpha}).$$

This is *not* the normalizing factor of the Eisenstein series. Then

$$\frac{\Delta(\boldsymbol{x})}{\Delta(w\boldsymbol{x})}=j(w,\boldsymbol{x})=\mathrm{sign}(w)p^{d(\alpha)}\boldsymbol{x}^{2\alpha}.$$

Then

$$\mathcal{G}(x,y) = \frac{\sum_{w \in W} (j|w)\, j(w, \boldsymbol{x})}{\Delta(\boldsymbol{x})}.$$

This is similar to the Weyl character formula. $\mathcal{G}(x,y)$ is a rational function with denominator

$$(1-x^2)(1-y^2)(1-px^2y^2) = \prod_{\alpha \in \Phi^+} (1 - p^{l(\alpha)-1}\boldsymbol{x}^{2\alpha}).$$

Actually the denominator is

$$(1-x)(1-y)(1-px^2y^2),$$

but it is proper to multiply both the numerator and denominator by $(1+x)(1+y)$.

3 Problems

1. Is $\mathcal{G}$ unique? Uniqueness has been proved for A_r $(r \leqslant 5)$ and D_4.

2. What happens for infinite root systems? We don't expect analytic continuation to all $\mathbb{C}^r$, but we do expect continuation to the "Tits cone". We don't expect uniqueness without imposing more conditions.

3. One can investigate the case of an affine root system when $p = 1$. Is there a product structure in the affine case? Are there generalizations of Macdonalds identities?

4. General n

5. Connection to $G \times T$

6. Function field.