Brubaker, 7/9

- Talk (Brubaker) 10–11:30
- Lunch 11:30–1
- Organizational meet 1–1:30
- Break
- Talk (Gautam) 1:45–3:15

1 WMD 1 and WMD 2

Φ: reduced root system of rank r, $n \gg 0$ (dep. of Φ)

$$Z(s_1, \cdots, s_r; \Phi; n) = \sum_{C_1, \cdots, C_r} \frac{H(C_1, \cdots, C_r)}{\mathbb{N}C_1^{2s_1} \cdots \mathbb{N}C_r^{2s_r}}.$$

Where $\mathfrak{o}_S$ is the ring of S-integers in the number field $F \supset \mu_n$ (n-th roots of unity). In practice it helps to assume that $F \supset \mu_{2n}$. It is asssumed that S contains the primes dividing $n \subset$ the archimedean places and enough others that the class number of $\mathfrak{o}_S$ is 1.

Q: what can be done without $F \supset \mu_n$? Mention of Goldfeld-Hoffstein-Patterson construction and Diaconu-Tian.

Highlights:

- Explicit twisted multiplicativity that reduces the description of H to prime powers, so we must define
$$H(p^{k_1}, \cdots, p^{k_r}).$$
The twisted multiplicativity does not make Z into an Euler product.
- There is one nonzero $H(p^{k_1}, \cdots, p^{k_r})$ for each element of the Weyl group associated to Φ for each element of the Weyl group associated to Φ.
- These coefficients contain products of Gauss sums.
- The series $Z(s_1, \cdots, s_r)$ has meromorphic continuation to all $\mathbb{C}^r$ and possesses functional equations that generate a group isomorphic to the Weyl group of Φ.
- Explicit polar divisor.

So far all this is a review of what was already known in Bretton Woods.

Q2: Can one list properties that H is required to have (locally) in order to have the functional equations?

- These are expected to agree with (1, ⋯, 1)-th Fourier-Whittaker coefficients of minimal parabolic Eisenstein series on metaplectic n-fold covers of ${}^LG$ (Langlands dual group of $G \longleftrightarrow \Phi$.
- WMD4 in part constructs similar Dirichlet series to $Z(s_1, \cdots, s_r)$ with additional data corresponding to $m_1, \cdots, m_r \in \mathfrak{o}_S$. These are conjectured to be $(m_1, \cdots, m_r)$-th Fourier-Whittaker coefficients of metaplectic Eisenstein series, for $n \gg 1$.

Related to Q2, Gautam argued that the uniqueness that characterizes Z in the untwisted case ($m_i = 1$) will break down in the twisted case and asks what further conditions beyond functional equations are needed to uniquely characterize these.

The integer n occurs in the Gauss sums that make up the coefficients $H(p^{k_1}, \cdots, p^{k_r})$, e.g.

$$g(p^l, p^r) = \sum_{a \bmod p^r} \left(\frac{a}{p^r}\right)_n \psi\left(\frac{p^l a}{p^r}\right),$$

where $(\text{-})_n$ is the n-th power residue symbol and ψ is an additive character of $F_S/\mathfrak{o}_S$, where $F_S = \prod_{v \in S} F_v$.

2 WMD3

Compute FW coefficients of Eisenstein series on GL(3).

$$\int_{F_S/\mathfrak{o}_S} E\left(w\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\right)\psi(-m_1 x_1 - m_2 x_2)\, dx_1 dx_2 dx_3.$$

(In the actual paper BBFH used a congruence Eisenstein series and integrated over $\mathbb{C}/\mathfrak{o}$ instead of $F_S/\mathfrak{o}_S$.) This produces

$$\sum_{C_1, \cdots, C_r} \frac{H(C_1, C_2; m_1, m_2)}{\mathbb{N}C_1^{2s_1} \mathbb{N}C_2^{2s_2}}$$

and leads to an explicit description (see *Metaplectic Eisenstein series on* GL(3), one of the companion papers to WMD3. The result is sums of products of Gauss sums.

Example: GL(3), $(m_1, m_2) = (1, 1)$. Want to describe $H(p^{k_1}, p^{k_2}; 1, 1)$. The nonzero coefficients are at the seven locations $(0,0), (1,0), (1,0), (1,2), (2,1), (2,2)$ as in the following diagram:

	2		*	*
k_2	1	*	*	*
	0	*	*	
		0	1	2
			k_1	

Here are the coefficients:

(k_1,k_2)	$H(p^{k_1},p^{k_2})$
$(0,0)$	1
$(1,0)$	$g(1,p)$
$(0,1)$	$g(1,p)$
$(1,2)$	$g(1,p)\,g(p,p^2)$
$(2,1)$	$g(1,p)\,g(p,p^2)$
$(2,2)$	$g(1,p)^2 g(p,p^2)$

Such diagrams resemble weight-multiplicity diagrams for heighest weight representations of GL(3). $\rho =$ Weyl vector $= \frac{1}{2}$ sum of the positive roots. Inspired by this observation, BBFH led to the Gelfand-Tsetlin description of the coefficients

3 Gelfand-Tsetlin description

By a *Gelfand-Tsetlin pattern* we mean a triangular array of integers

$$\mathfrak{T}=\left\{\begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{array}\right\} \tag{1}$$

where the rows interleave; that is, $a_{i-1,j-1} \geqslant a_{i,j} \geqslant a_{i-1,j}$. We will say that the pattern is *strict* if each row is strictly decreasing.

Example, when $\Phi = A_2$ (relevant to GL(3))

$$\left\{\begin{array}{ccccc} 5 & & 4 & & 0 \\ & 3 & & 2 & \\ & & 3 & & \end{array}\right\}$$

When $\Phi = A_1$ (relevant to n-fold GL(2)) GT patterns with top row $\{k+1,\ 0\}$ parametrize p^k coefficient of the Eisenstein series. This coefficient is

$$Z(s;p^l) = \sum_d \frac{g(p^l,d)}{\mathbb{N}d^{2s}}, \qquad H(d,p^l) = g(p^l,d) = \sum_{a \bmod p^r} \left(\frac{a}{p^l}\right)_n \psi\left(\frac{da}{p^l}\right).$$

By multiplicativity we only need to know $g(p^l,p^k)$. Let us consider the case where $l=0$. There are only two possible values of k, $k=0$ and $k=1$.

$$\left\{\begin{array}{ccc} 1 & & 0 \\ & 0 & \end{array}\right\} \longleftrightarrow g(1,1) = 1\text{-coefficient}$$

$$\left\{\begin{array}{ccc} 1 & & 0 \\ & 1 & \end{array}\right\} \longleftrightarrow g(1,p) = p\text{-coefficient}$$

Q3: What connections between Hecke operators and coefficients of metaplectic Eisenstein series? Goldfeld asked whether the Plucker computation can be done using Hecke operators. We don't think so but the effect of the Hecke operators definitely needs to be thought about.

The recipe from WMD3 is now described. We define

$$G(\mathfrak{T}) = \prod_{j \geqslant i \geqslant 1} \gamma(i, j),$$

where

$$\gamma(i, j) = \begin{cases} g\left(p^{s_{ij} - a_{ij} + a_{i-1, j-1} - 1}, p^{s_{ij}}\right) & \text{if } a_{ij} > a_{i-1, j}, \\ \mathbb{N} p^{s_{ij}} & \text{if } a_{ij} = a_{i-1, j}, \end{cases}$$

$$s_{ij} = \sum_{k=j}^{r} a_{ik} - \sum_{k=j}^{r} a_{i-1, k}. \tag{2}$$

Thus we are associating one factor $\gamma(i, j)$ to each entry of the pattern below the top row. An example was done.

The evidence for the GT description was reviewed. This part of the lecture is not recorded in these notes. (See WMD3.)

4 Open problems

- Take description for A_r ($r > 2$) and try to prove the meromorphic continuation and functional equation using combinatorial methods. (Nat Thiem has done this for $r = 2$).
- Make a conjecture for other root systems. (An analog of GT patterns is known – "Proctor patterns"
- Explore how GT patterns enter into FW coefficients by analyzing the geometry of the associated flag variety and se how the individual contributions are parmetrized by GT patterns.
- Prove that this description is equivalent to that of Chinta and Gunnells.