Theta Representations Of Odd Orthogonal Groups

This is a report on three papers of **Daniel Bump, Solomon Friedberg and David Ginzburg.**

- A Rankin-Selberg integral using the automorphic minimal representation of SO(7), J. Ramanujan Math. Soc. **15** (2000), no. 2, 81–124.
- Small representations for odd orthogonal groups, Int. Math. Res. Not. 2003, no. 25, 1363–1393.
- Lifting Automorphic Representations on Double Covers of Orthogonal Groups (Preprint).

These may be found at `http://math.stanford.edu/~bump`.

- Slides for this talk are also posted.

Metaplectic Covers

If G is a **simply-connected**, F local containing $\mu_r = r$-th roots of unity, Matsumoto constructed a **metaplectic cover** $\tilde{G}(F)$.

$$1 \longrightarrow \mu_r \longrightarrow \tilde{G}(F) \longrightarrow G(F) \longrightarrow 1$$

If G is **not** simply connected, we can still construct $\tilde{G}$ given enough roots of unity. **Example:** Let

$G = \mathrm{SO}_{2n+1} = \{g \in \mathrm{GL}_{2n+1} \,|\, g \cdot J \cdot {}^t g = J\}$ (split orthogonal group)

$$\text{where } J = J_{2n+1} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

Construct a cover by pulling back the **4-fold** cover of SL_{2n+1}:

$$\begin{tikzcd}
1 \arrow[r] & \mu_4 \arrow[r] \arrow[d, equal] & \widetilde{SO}_{2n+1}(F) \arrow[r] \arrow[d] & SO_{2n+1}(F) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_4 \arrow[r] & \widetilde{SL}_{2n+1}(F) \arrow[r] & SL_{2n+1}(F) \arrow[r] & 1
\end{tikzcd}$$

As we will explain, this is **morally** a **double cover**. Let

$$\mathrm{SN}\colon \mathrm{SO}_{2n+1}(F) \longrightarrow F^\times/(F^\times)^2$$

be the spinor norm. The class of $\widetilde{\mathrm{SO}}_{2n+1}$ in $H^2(\mathrm{SO}_{2n+1},\, \mu_4)$ has order 4 but a representative cocyle σ satisfies

$$\sigma(g,h)^2 = (\mathrm{SN}(g), \mathrm{SN}(h))_2 \qquad \text{(quadratic Hilbert symbol)}$$

Thus the cover has degree 2 when restricted to the (large) kernel $\mathrm{SO}_{2n+1}(F)'$ of the spinor norm map.

Small Representations

Let F be a local field, G a semisimple algebraic group over F. Write G for $G(F)$. The **character** of a rep'n π of G is a distribution χ_π defined by

$$\langle \chi_\pi, \varphi \rangle = \operatorname{tr} \pi(\varphi), \qquad \pi(\varphi)v = \int_G \varphi(g)\pi(g)v\, dg,$$

for $\varphi \in C_c^\infty(G)$. Harish-Chandra showed χ_π is a locally integrable function on the regular semisimple elements. It typically blows up at the identity.

Let $\mathfrak{g} = \mathrm{Lie}(G)$. A **coadjoint orbit** is an orbit of the adjoint representation on the dual $\mathfrak{g}^*$.

- Kirillov (1960) and others emphasized, a rep'n of G determines a coadjoint orbit, and is "measured" by it.
- Coadjoint orbits are ordered: $\mathcal{O}_1 \leqslant \mathcal{O}_2$ if $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$. Excluding $\{0\}$ there is a unique smallest orbit $\mathcal{O}_{\min}$.
- (Roughly) the Fourier transform of $\chi_\pi \circ \exp$ is a distribution on $\mathfrak{g}^*$ reflecting the asymptotics of χ_π near 1. There is (often) a unique largest coadjoint orbit $\mathcal{O}_\pi$ in its support.
- A representation with $\mathcal{O}_\pi = \mathcal{O}_{\min}$ is **minimal**. Many groups have minimal representations.
- Sometimes the minimal representation is actually a rep'n of a covering group. The **Weil representation** of Sp_{2n} is minimal but is is actually a representation of $\widetilde{\mathrm{Sp}_{2n}}$.
- Odd orthogonal groups SO_{2n+1} with $2n+1 \geqslant 9$ are an exception. **They have no minimal representations**.

Weil, Kazhdan and Savin

Let $G = \mathrm{Sp}_{2n}$. The **Weil representation** $\omega = \omega_n$ is a representation of the double cover $\widetilde{\mathrm{Sp}}_{2n}(F)$ acting on the Schwartz space $\mathcal{S}(F^n)$. It splits into odd and even parts ω^+ and ω^-. These are minimal. Combining them, ω is really a representation of $\widetilde{\mathrm{Sp}}_{2n} \times O_1$ where the orthogonal group $O_1 \cong \mathbb{Z}/2\mathbb{Z}$.

We can embed $\mathrm{Sp}_{2n} \times O_r \hookrightarrow \mathrm{Sp}_{2nr}$. The restriction of ω_{nr} to $\mathrm{Sp}_{2n} \times O_r$ was used by Weil (1964) to explain classical work of Siegel. In Weil's constuction it is naturally a representation of

$$(\mathrm{Sp}_{2n} \times O_r) \rtimes H_{2nr+1}$$

where H_{2nr+1} is a $2nr+1$ dimensional **Heisenberg group**.

Minimal rep'ns of semisimple $G \neq \mathrm{SL}_n$ were given in a uniform way by Kazhdan and Savin (1989) when G is **simply-laced** (all roots have same length). This includes **even** orthogonal groups.

Example. Let $G = \mathrm{SO}_8$. G has a parabolic subgroup with Levi factor $M = \mathrm{GL}_2 \times \mathrm{SO}_4$. The unipotent radical H_9 is a 9-dimensional Heisenberg group.

$$P = \left\{ \left(\begin{array}{|c|c|c|} \hline \mathrm{GL}_2 & * & * \\ \hline 0 & \mathrm{SO}_4 & * \\ \hline 0 & 0 & * \\ \hline \end{array} \right) \right\}.$$

Let $M^0 = \mathrm{SL}_2 \times \mathrm{SO}_4 = \mathrm{Sp}_2 \times \mathrm{SO}_4$. Induce ω_4 from $M^0 H_9$ to $MH_9 = P$. **Kazhdan (1989) proved this representation extends to $G = \mathrm{SO}_8$**. This construction was extended to general simply-laced groups by Kazhdan and Savin.

Roskies, Torasso and Sabourin

For nonsimply-laced groups, minimal rep'ns were constructed by Savin for G_2 (1993) and Rumelhart for F_4 (1997). For SO_7 a minimal rep'n was constructed independently (~1997) by Roskies and Torasso (p-adic) and by Sabourin (archimedean).

Let $G = \mathrm{SO}_7$. G has a parabolic subgroup with Levi factor $M = \mathrm{GL}_2 \times \mathrm{SO}_3$. The unipotent radical H_7 is a 7-dimensional Heisenberg group.

$$P = \left\{ \left(\begin{array}{|c|c|c|} \hline \mathrm{GL}_2 & * & * \\ \hline 0 & \mathrm{SO}_3 & * \\ \hline 0 & 0 & * \\ \hline \end{array} \right) \right\}.$$

Let $M^0 = \mathrm{SL}_2 \times \mathrm{SO}_3 = \mathrm{Sp}_2 \times \mathrm{SO}_3$. Induce ω_4 from $M^0 H_7$ to $MH_9 = P$, then extend this representation to G.

- The action $\pi(w)$ of a Weyl group element $w \notin P$ is an L^2 Fourier transform; Roskies and Sabourin checked generators and relations. The Schwartz space is not preserved by $\pi(w)$ so the smooth vectors are not characterized.
- This phenomenon applies to other examples beginning with Kazhdan's SO_8 construction.
- For **even** orthogonal groups the metaplectic double cover of Sp_{4nr} splits over $\mathrm{Sp}_{2n} \times \mathrm{SO}_{2r}$. Thus the minimal rep'n of SO_8 constructed by Kazhdan and Savin is a rep'n of SO_8.
- For **odd** orthogonal groups the cover does not split over $\mathrm{Sp}_{2n} \times \mathrm{SO}_{2r+1}$. Thus the Roskies–Sabourin–Torasso rep'n is of $\widetilde{\mathrm{SO}}_7$ and μ_4 must lie in F.

Applications

The importance of the minimal representation in automorphic forms comes from two facts.

- On the adele group, **the minimal representation can be realized as an automorphic representation**. For the Weil representation, this was done by Weil (1964). For the Kazhdan-Savin representations, it was done by Ginzburg, Rallis and Soudry (1997).
- The smallness of the representation translates to the **vanishing of many Fourier coefficients**.

This leads to two classes of applications.

- **Theta correspondence**. Let H and H' be reductive subgroups of G. Assume that H is the centralizer of H' and conversely. There is a strong tendency for the restriction of the minimal representation π_{min} of G to H' to give a **correspondence**. This means that in the list of representations $\pi_i \otimes \pi_i'$ of irreducible representations of $H \times H'$ which occur as quotients of π_{min} there are no repetitions among the π_i or π_i'. Consequently $\pi_i \longleftrightarrow \pi_i'$ is a bijection between some representations of H and some representations of H'. For automorphic forms, this gives some functorial liftings.
- **Rankin-Selberg constructions**. The use of theta functions in Rankin-Selberg constructions of L-functions also relies on the vanishing of Fourier coefficients.

The Weil representation has been used in both classes of applications. We review some examples.

Theta Correspondence

Some examples of applications of $\pi_{\min}$.

- The Weil representation was introduced to explain classical work of Siegel exploiting the dual pair $\mathrm{Sp}_{2n} \times O_r$ in Sp_{2nr}. The Weil representation gives a lifting of automorphic forms
$$O_r \longleftrightarrow \begin{cases} \mathrm{Sp}_{2rn} & \text{if } r \text{ is even;} \\ \widetilde{\mathrm{Sp}}_{2rn} & \text{if } r \text{ is odd.} \end{cases}$$
(Weil, Howe, Rallis-Schiffmann.)
- Theta correspondences using $\pi_{\min}$ on other G begin with Kazhdan (1989), who used the theta representation of SO_8 to give a theta lift
$$R_{K/F}\,\mathrm{GL}_1 \longleftrightarrow \mathrm{GL}_3$$
where $[K\colon F]=3$ and $R_{K/F}$ is groundfield reduction.
- Reductive dual pairs in exceptional groups were classified by Rubenthaler, leading to many potential applications.

Constructions of L-functions

- Beginning with Shimura's construction of the symmetric square L-function, there are many examples of small representations used to construct L-functions.
- In many cases the representation used is not minimal but another "small" representation.
- An example using $\pi_{\min}$: the Roskies-Sabourin-Torasso rep'n of SO_7 was used by Bump, Friedberg and Ginzburg to construct an L-function of degree 14 on SO_7.

Odd Orthogonal Groups

Vogan observed that there no minimal representation for SO_{2n+1} if $2n+1 \geqslant 9$. We can understand this fact by noting that the Kazhdan-Savin-Roskies-Sabourin-Torasso method **fails** for SO_9.

$$\text{Let} \quad P = \left\{ \left(\begin{array}{|c|c|c|} \hline \mathrm{GL}_2 & * & * \\ \hline 0 & \mathrm{SO}_5 & * \\ \hline 0 & 0 & * \\ \hline \end{array} \right) \right\}.$$

We would proceed as follows. We start with the Weil representation of $(\mathrm{SL}_2 \times \mathrm{SO}_5) \rtimes H_{11} = (\mathrm{Sp}_2 \times \mathrm{SO}_5) \rtimes H_{11}$. We then induce this to $P = (\mathrm{SL}_2 \times \mathrm{SO}_5) \rtimes H_{11}$ and try to extend it.

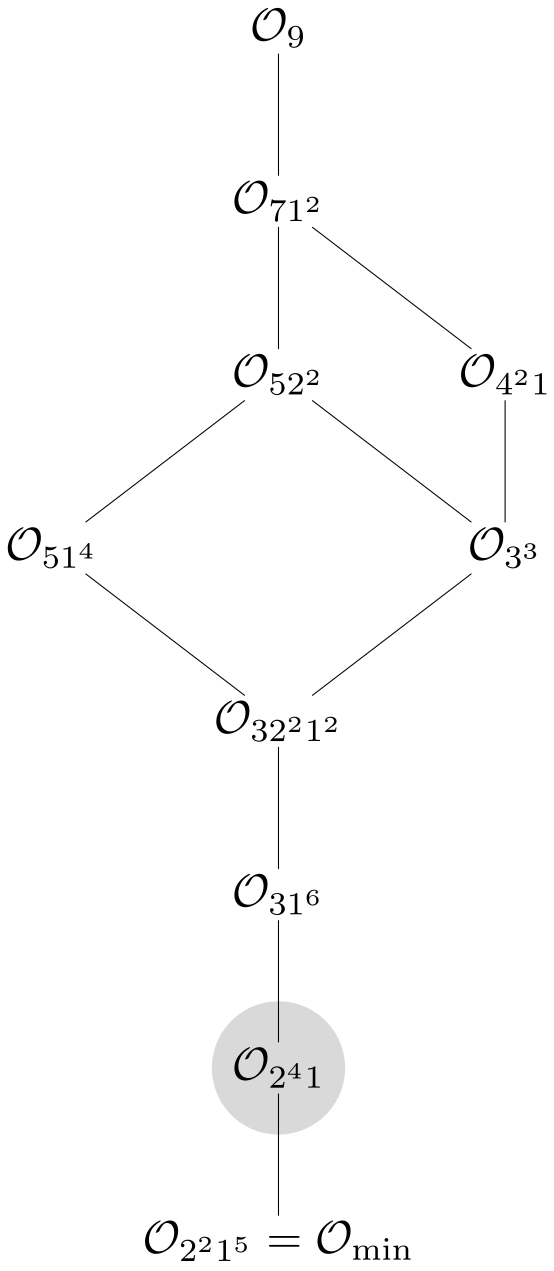
But **this doesn't work** for the following reason. Regarding the Weil rep'n as a projective rep'n of $(\mathrm{SL}_2 \times \mathrm{SO}_5) \rtimes H_{11}$ the associated cover splits over SO_5 but not over $\widetilde{\mathrm{SL}}_2$. But SO_5 contains an SO_9-conjugate of SL_2 so the rep'n cannot be extended to SO_9.

The goal on SO_{2n+1} should therefore be not the construction of $\pi_{\min}$ (which does not exist) but π such that $\mathcal{O}_\pi$ is another "small" coadjoint orbit (i.e. low in the partial order). The smallness of the representation will be reflected in the **vanishing of Fourier coefficients** needed for applications.

Let λ be a partition of $2n+1$. Assume that **every even part occurs an even number of times**. Then there is coadjoint orbit $\mathcal{O}_\lambda$. (The elementary divisors of a representative $X \in \mathfrak{g} \cong \mathfrak{g}^*$ can be read off from the partition.)

Example: $\mathbf{SO_9}$

Here are the nonzero coadjoint orbits of SO_9. We have circled the orbit $\mathcal{O}_{2^41}$ corresponding to a rep'n of $\widetilde{\mathrm{SO}}_9$ that is small but not minimal.



Whether or not it's small enough to be considered the "theta representation" naturally depends on finding applications.

For general n we will construct a rep'n π of $\widetilde{\mathrm{SO}}_{2n+1}$ such that

$$\mathcal{O}_\pi = \begin{cases} \mathcal{O}_{2^n1} & n \text{ even;} \\ \mathcal{O}_{2^{n-1}3} & n \text{ odd.} \end{cases}$$

Kazhdan and Patterson

We will construct the representation π of $\widetilde{\mathrm{SO}}_{2n+1}$ as a quotient of a principal series representation. This construction is very similar to Kazhdan and Patterson (1984). So before we consider SO_{2n+1} we consider GL_n.

We assume $\mu_r \subset F$. Let $\tilde{G}$ denote a metaplectic r-fold cover of GL_n. Let T be the diagonal torus in $G = \mathrm{GL}_n$, and and let $\tilde{T}$ the preimage of T in $\tilde{G}$. Let (ρ, V) of $\tilde{T}$, which are finite-dimensional, and induced from a subgroup of finite index. (Thus $\tilde{T}$ is "almost" abelian.) We extend ρ to $\tilde{B}$, the cover of the standard upper triangular Borel subgroup $B = TN$ of G.

Let $I(\rho) = \{f\colon \tilde{G} \longrightarrow \mathbb{C} \,|\, f(bg) = \delta^{1/2}(b)\,\rho(b)\,f(g)\}$. We have an action of $\tilde{G}$ on this space by right translation. We are interested in knowing when it is reducible.

If $w \in W$ is a Weyl group element, then there is an **intertwining integral operator** $T_w\colon I(\rho) \longrightarrow I({}^w\rho)$ defined by

$$(T_w f)(g) = \int_{(N \cap w N w^{-1}) \backslash N} f(w^{-1} n g)\, dg.$$

Basic Principle. The reducibility of the representation $I(\rho)$ is controlled by these intertwining operators.

The center $Z(\tilde{T})$ acts by scalars in ρ. Let ω_ρ be the central character. Then ρ is called **exceptional** if for every simple root α

$$\omega_\rho\left(i_\alpha\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\right) = |t|^{1/r} \qquad (\text{recall}\, r = \text{cover degree})$$

where i_α: $\mathrm{SL}_2 \longrightarrow \mathrm{GL}_n$ is the embedding along α, lifted to the metaplectic group, when $i_\alpha\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} \in Z(\tilde{T})$, for example when t is an r-th power. If this is true, Kazhdan and Patterson showed that if w_0 is the long element of W and s_α is a simple reflection, the composite

$$I(\rho) \xrightarrow{T_{w_0}} I({}^{w_0}\rho) \xrightarrow{T_{s_\alpha}} I({}^{s_\alpha w_0}\rho)$$

is zero.

This implies that if $\Theta_{\widetilde{\mathrm{GL}}_n}$ is the image of T_{w_0} then the Jacquet module of $\Theta_{\widetilde{\mathrm{GL}}_n}$ (with respect to the Borel subgroup) is an irreducible representation of $\tilde{T}$. By contrast, the Jacquet module of $I(\rho)$ is the direct sum of $n!$ constituents:

$$\mathrm{Jac}(I(\rho)) = \bigoplus_{w \in W} {}^w\rho$$

So $\Theta_{\widetilde{\mathrm{GL}}_n}$ is much smaller than $I(\rho)$.

Moreover this construction has a global analog in taking the residue of an Eisenstein series at the exceptional point. Thus at each place $\Theta_{\widetilde{\mathrm{GL}}_n}$ is the constituent of a global automorphic representation.

The $\mathbf{SO_{2n+1}}$ Construction

Now let $G = \mathrm{SO}_{2n+1}$. We assume that $\mu_4 \subset F$ but all covers will be "morally" double covers. When $\Theta_{\widetilde{\mathrm{GL}}_k}$ appears in the sequel it is understood that $\widetilde{\mathrm{GL}}_k$ is the double cover.

The representation $\Theta_{\widetilde{\mathrm{SO}}_{2n+1}}$ can be constructed like $\Theta_{\widetilde{\mathrm{GL}}_k}$ as a quotient of a principal series representation.

- If $2n+1=3$ the cover splits over $\widetilde{\mathrm{SO}}_3$. In this case $\Theta_{\widetilde{\mathrm{SO}}_3}$ is trivial, i.e. one-dimensional.
- If $2n+1=5$ then (as algebraic groups) $\mathrm{SO}_5 \cong \mathrm{PGSp}_4$. In this case $\Theta_{\widetilde{\mathrm{SO}}_5}$ is "essentially" the even Weil representation.
- If $2n+1=7$ then $\Theta_{\widetilde{\mathrm{SO}}_7}$ is the representation constructed by Roskies, Sabourin and Torasso.
- Let P be a standard maximal parabolic subgroup of G. The Levi factor is $\mathrm{GL}_k \times \mathrm{SO}_{2(n-k)+1}$ for some k. The Jacquet module with respect to this parabolic subgroup, applied to $\Theta_{\widetilde{\mathrm{SO}}_{2n+1}}$ produces $\Theta_{\widetilde{\mathrm{GL}}_k} \otimes \Theta_{\widetilde{\mathrm{SO}}_{2(n-k)+1}}$.

The smallness of $\Theta_{\widetilde{\mathrm{SO}}_{2n+1}}$ will be reflected in the vanishing of many Fourier coefficients of the global automorphic rep'n.

Specifically the coadjoint orbits $\mathcal{O}_\lambda$ correspond to a class of periods in a way that we describe. We have $\mathcal{O}_\pi = \mathcal{O}_{\lambda_0}$ where

$$\lambda_0 = \begin{cases} 2^n 1 & \text{if } n \text{ is even;} \\ 2^{n-1} 3 & \text{if } n \text{ is odd,} \end{cases}$$

and the periods corresponding to $\mathcal{O}_\lambda$ vanish **unless** $\lambda \leqslant \lambda_0$. This is a reflection of the smallness of the representation.

Local Vanishing

Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition of $2n+1$ in which each even part occurs with even multiplicity. We will describe a 1-parameter subgroup h_λ of SO_{2n+1}. For each $0 < k \in \mathbb{Z}$ let

$$h_k(t) = \begin{pmatrix} t^{k-1} & & & \\ & t^{k-3} & & \\ & & \ddots & \\ & & & t^{1-k} \end{pmatrix}.$$

Then the one parameter subgroup of GL_{2n+1} described by

$$t \longmapsto \left(\begin{array}{|c|c|c|c|} \hline h_{\lambda_1}(t) & & & \\ \hline & h_{\lambda_2}(t) & & \\ \hline & & \ddots & \\ \hline & & & h_{\lambda_r}(t) \\ \hline \end{array}\right)$$

does not lie in SO_{2n+1} but it is conjugate in GL_{2n+1} to a one-parameter subgroup $h_\lambda(t)$ of SO_{2n+1}.

Let G_λ be the centralizer in SO_{2n+1} of h_λ. For $0 < k \in \mathbb{Z}$ let

$$\mathfrak{u}_k = \{X \in \mathfrak{so}_{2n+1} \mid \mathrm{ad}(h(t))X = t^k X\}.$$

Then $\bigoplus_{k \geqslant 2} \mathfrak{u}_k$ is the Lie algebra of a subgroup N_λ of upper triangular matrices of SO_{2n+1}. The group G_λ acts on N_λ and hence on

$$X^*(N_\lambda) = X^*(N_\lambda / [N_\lambda, N_\lambda]).$$

Let ψ_λ be a character of N_λ **in general position** and let

$$H_\lambda = \mathrm{stab}(G_\lambda, \psi) \cdot N_\lambda.$$

Then ψ_λ extends to a character of H_λ.

Example. In SO_5 Let $\lambda = 31^2$. Then

$$h_\lambda(t) = \begin{pmatrix} t^2 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & t^{-2} \end{pmatrix}, \qquad N_\lambda = \left\{ v = \begin{pmatrix} 1 & x & y & z & * \\ & 1 & & & -z \\ & & 1 & & -y \\ & & & 1 & -x \\ & & & & 1 \end{pmatrix} \right\}.$$

The centralizer $G_\lambda \cong \mathrm{GL}_1 \times \mathrm{SO}_3$. A character

$$\psi_\xi \colon v \longmapsto \alpha x + \beta y + \gamma z$$

of $N_\lambda \cong F^3$ is parametrized by $\xi = (\alpha, \beta, \gamma)$ with SO_3 acting by the standard representation and GL_1 by scalars.

There are three types of orbits of $\mathrm{GL}_1 \times \mathrm{SO}_3$ on ξ: the zero vector, the isotropic vectors (such as $\xi = (1, 0, 0)$) and orbits of vectors of nonzero length. If F is not algebraically closed, the $\mathrm{GL}_1 \times \mathrm{SO}_3$ action on the anisotropic vectors is not transitive, but any orbit of any anisotropic vector is in "general position." Denoting the stabilizer of ξ by SO_2^ξ,

$$H_\lambda = \mathrm{SO}_2^\xi \cdot N_\lambda.$$

Returning to the general case,

Theorem. Unless $\mathcal{O}_\lambda \leqslant \mathcal{O}_{\lambda_0} = \mathcal{O}_\pi$, there exists no nonzero linear functional Λ on the space V_π of π such that

$$\Lambda(\pi(n)v) = \psi_\lambda(n)\,\Lambda(v), \qquad n \in N_\lambda, v \in V_\pi.$$

Global Vanishing Periods

Globally the theta representation is automorphic. It can be realized as a residue of an Eisenstein series. The local result has a global analog in the vanishing of many Fourier coefficients.

Example. On SO_5, with $\lambda = 31^2$ as before, the local representation $\Theta_{\widetilde{\mathrm{SO}}_5}$ is a local constituent of an automorphic representation. Let $\alpha, \beta, \gamma \in F$ such that the length $\alpha\gamma - \beta^2$ of the vector $\xi = (\alpha, \beta, \gamma)$ is nonzero. Then if θ is an automorphic form in the global automorphic representation $\Theta_{\widetilde{\mathrm{SO}}_5}$ we have

$$\int_{(\mathbb{A}/F)^3} \theta\left(\begin{pmatrix} 1 & x & y & z & * \\ & 1 & & & -z \\ & & 1 & & -y \\ & & & 1 & -x \\ & & & & 1 \end{pmatrix} g\right) \psi(\alpha x + \beta y + \gamma z)\, dx\, dy\, dz = 0.$$

The vanishing of such periods is sufficient for a class of applications: theta lifting.

Let F be a global field and $\mathbb{A}$ its ring of adeles. Let π be a genuine cuspidal rep'n of $\widetilde{\mathrm{SO}}_{2k+1}$. If $h \in \widetilde{\mathrm{SO}}_{2m}(\mathbb{A})$ let

$$\tilde{f}(h) = \int_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta_{2k+2m+1}(h, g)}\, d g,$$

Here $\varphi_\pi(g)$ denotes a general vector in the space of π and similarly, $\theta_{2k+2m+1}(r)$ denotes a general function in the space of $\theta_{2k+2m+1}$. Denote this rep'n of $\widetilde{\mathrm{SO}}_{2m}$ by $\theta_{2k+2m+1}(\pi)$.

Properties of the Lifting

Theorem 1. *Let π be a cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Suppose the lift $\theta_{2k+2m-1}(\pi)$ is zero. Then the lift $\theta_{2k+2m+1}(\pi)$ is a cuspidal represention of $\widetilde{\mathrm{SO}}_{2m}(\mathbb{A})$.*

Thus in a tower of lifts to covers of even orthogonal groups, the first nonzero lift is cuspidal. This corresponds to a well-known result of Rallis for dual pairs in $\widetilde{\mathrm{Sp}}_{2n}$.

Moreover, the next result (probably not best possible) shows that eventually some lift is nonzero.

$$\begin{tikzcd}
 & \vdots \\
 & \widetilde{SO}_{2m+2} \\
\widetilde{SO}_{2k+1} \arrow[ur] \arrow[r] \arrow[dr] & \widetilde{SO}_{2m} \\
 & \widetilde{SO}_{2m-2} \\
 & \vdots
\end{tikzcd}$$

Theorem 2. *Let π be a cuspidal representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Then $\theta_{10k+1}(\pi)$ is nonzero. In other words, every π lifts nontrivially to an automorphic represention on $\widetilde{\mathrm{SO}}_{8k}(\mathbb{A})$.*

Theorem 3. *The representation $\theta_{4k+3}(\pi)$ has a nonzero Whittaker model if and only if the representation π has a nonzero Bessel model.*

Here the **Bessel model** is with respect to a Bessel subgroup which has a large unipotent radical and SO_2 as its Levi factor. The statement of this theorem is not precise – see the paper for a precise statement.

Functoriality

- Rallis (1982) showed in many cases that theta lifts of classical automorphic representations are functorial in the sense of Langlands.
- Savin (1988) computed the Iwahori Hecke algebras of metaplectic groups. His computation suggests an extension of the L-group formalism to metaplectic groups. Therefore the notion of Langlands' functoriality may also extend.
- We show that the lift $\widetilde{\mathrm{SO}}_{2k} \longrightarrow \widetilde{\mathrm{SO}}_{2k+1}$ is functorial.

We recall that if G is a reductive algebraic group then the dual group is defined; the complex points of the dual group form the connected component ${}^LG^\circ$ of the L-group. Assuming for simplicity that G is split and semisimple, the Weyl group of G and ${}^LG^\circ$ are the same; the long roots of G correspond to short roots of ${}^LG^\circ$; and the center of G is isomorphic to the fundamental group of ${}^LG^\circ$ and conversely.

If we replace G by a metaplectic group, and if we accept the modification of the L-group suggested by Savin's paper, the statement about the long and short roots being interchanged must be modified.

L-Groups of Metaplectic Groups

Let $\tilde{G}$ be the r-fold cover.

- Assume the type of G is B_n, C_n or F_4. The long and short roots are interchanged if r is odd; otherwise the Cartan types of G and ${}^L\tilde{G}^\circ$ are the same.
- Assume G is of type G_2. The long and short roots are interchanged if r is not a multiple of 3. In any case the Cartan type of ${}^L\tilde{G}^\circ$ is also G_2.

For $G = \mathrm{SO}_{2k+1}$, the L-group is thus

$$ {}^L\tilde{G}^\circ = \begin{cases} \mathrm{Sp}_{2k}(\mathbb{C}) & \text{if } r \text{ is odd;} \\ \mathrm{SO}_{2k+1}(\mathbb{C}) & \text{if } r \text{ is even.} \end{cases} $$

Thus if $r = 2$ we have ${}^L\widetilde{\mathrm{SO}}_n^\circ \cong \mathrm{SO}_n$, whether n is odd or even. Embedding ${}^L\widetilde{\mathrm{SO}}_n^\circ \hookrightarrow {}^L\widetilde{\mathrm{SO}}_{n+1}^\circ$, we see **there should be functorial lifts from automorphic forms on $\widetilde{\mathrm{SO}}_n$ to $\widetilde{\mathrm{SO}}_{n+1}$.**

Since we have constructed a theta lift, it is natural to ask whether it fits this lifting philosophy.

The L-group formalism should parametrize spherical representations of $\tilde{G}$ by conjugacy classes in ${}^L\widetilde{G^\circ}$. Let F be a nonarchimedean local field. Let $\tilde{\pi}_{2n+1}$ be a spherical representation of $\widetilde{\mathrm{SO}}_{2n+1}(F)$. There exist unramified characters $\chi_1, \cdots, \chi_n$ such that $\tilde{\pi}_{2n+1}$ is parabolically induced from

$$\begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1 & & & \\ & & & & t_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix} \longmapsto \prod_{i=1}^{n} \chi_i(t_i).$$

The conjugacy class corresponding to $\tilde{\pi}_{2n}$ in $\mathrm{SO}_{2n}(\mathbb{C}) = {}^L\widetilde{\mathrm{SO}}_{2n}^\circ$ is

$$\begin{pmatrix} \chi_1(\varpi) & & & & & & \\ & \ddots & & & & & \\ & & \chi_n(\varpi) & & & & \\ & & & 1 & & & \\ & & & & \chi_n(\varpi)^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \chi_1(\varpi)^{-1} \end{pmatrix}.$$

- This would not be the right parametrization were the cover of **odd** degree. In that case, we would take a conjugacy class in $\mathrm{Sp}_{2n}(\mathbb{C})$.
- There is a similar parametrization a rep'n $\tilde{\pi}_{2n}$ for $\widetilde{\mathrm{SO}}_{2n}$.
- We say that $\tilde{\pi}_{2n+1}$ is a **functorial lift** of $\tilde{\pi}_{2n}$ if their conjugacy classes correspond under the inclusion $\widetilde{\mathrm{SO}}_{2n}(\mathbb{C}) \longrightarrow \widetilde{\mathrm{SO}}_{2n+1}(\mathbb{C})$.

Functoriality Theorem

First, there is a local statement:

Theorem 4. *If* $\tilde{\pi}_{2n} \times \tilde{\pi}_{2n+1}$ *occurs in the restriction of* Θ_{4n+1} *to* $\widetilde{\mathrm{SO}}_{2n} \times \widetilde{\mathrm{SO}}_{2n+1}$ *then* $\tilde{\pi}_{2n+1}$ *is a functorial lift of* $\tilde{\pi}_{2n}$*.*

This implies a global result:

Theorem 5. *Let* F *be a global field and* $\mathbb{A}$ *its adele ring. If* $\tilde{\pi}_{2n}$ *is an automorphic representation of* $\widetilde{\mathrm{SO}}_{2n}(\mathbb{A})$ *and* $\tilde{\pi}_{2n} \cong \otimes_v \tilde{\pi}_{2n,v}$*, and if* $\tilde{\pi}_{2n+1} = \otimes_v \tilde{\pi}_{2n+1,v}$ *is any irreducible constitutent of* $\theta_{4n+1}(\tilde{\pi}_{2n})$*, then* $\tilde{\pi}_{2n+1,v}$ *is a lift of* $\tilde{\pi}_{2n,v}$ *for almost all* v*.*

There should be similar statements for the lifts $\widetilde{\mathrm{SO}}_{2n-1} \longrightarrow \widetilde{\mathrm{SO}}_{2n}$ but we have not investigated these.