Math 263: Infinite-Dimensional Lie Algebras

Our text will be Kac, *Infinite-Dimensional Lie Algebras* (third edition) though I will also rely on Kac and Raina, *Bombay Lectures on Highest Weight Representations of Infinite-Dimensional Lie Algebras*. I will assume you have access to the first text but not the second. Unfortunately Kac' book is long and dense. These notes are an attempt to make a speedy entry. Most relevant for the first part of these notes are Chapters's 7 and 14 of Kac's book.

1 Examples of Lie Algebras

Given an associative algebra A, we have two Lie algebras, $\mathrm{Lie}(A)$ and $\mathrm{Der}(A)$.

- $\mathrm{Lie}(A) = A$ as a set, with bracket operation $[x, y] = xy - yx$.
- $\mathrm{Der}(A)$ is the set of derivations of A. It is a subalgebra of $\mathrm{Lie}(\mathrm{End}(A))$. For this it is not necessary that A be associative.

If X is a vector field on a manifold M, then X may be be identified with the map $C^\infty(M) \longrightarrow C^\infty(M)$ which is differentiation in the direction X. By the Leibniz rule, $X(fg) = X(f)\ g + fX(g)$. So X is a derivation of $C^\infty(M)$ and in fact every derivation of $A = C^\infty(M)$ arises from a vector field. Thus the space vector fields on M may be identified with $\mathrm{Der}(A)$. It is an infinite-dimensional Lie algebra.

If G is a Lie group, then the vector space $\mathfrak{g}$ of left invariant vector fields is isomorphic to the tangent space $T_1(G)$ of G at the identity. This is obviously closed under the bracket operation. This is the Lie algebra of G.

Given a vector field X on a manifold M, there is a family of integral curves. That is, (at least if M is compact) there should be defined a map $\phi: M \times (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\phi(m, 0) = m$ and

$$Xf(m) = \frac{d}{dt} f(\phi(m,t))|_{t=0}$$

and more generally

$$Xf(\phi(u,m)) = \frac{d}{dt} f(\phi(t+u,m))|_{t=0}.$$

If G is a Lie group, and $X \in \mathfrak{g}$, the Lie algebra, then X is a left-invariant vector field, and it may be deduced that the integral curve has the form

$$\phi(g,t) = g \cdot \exp(tX)$$

where $\exp: \mathfrak{g} \longrightarrow G$ is a map, the *exponential map*. We also denote $\exp(X) = e^X$. So

$$Xf(g) = \frac{d}{dt} f(g \cdot e^{tX})|_{t=0}.$$

If $\pi: G \longrightarrow \mathrm{GL}(V)$ is a representation, then we obtain a representation of $\mathfrak{g}$ by

$$d\pi(X)v = \frac{d}{dt} \pi(e^{tX})v|_{t=0}.$$

We have

$$d\pi([X,Y]) = d\pi(X)\circ d\pi(Y) - d\pi(Y)\circ d\pi(X).$$

2 Attitude

For Lie groups, statements about group theory may be translated into statements about Lie algebras. For example, the classification of irreducible representations of a semisimple complex or compact Lie group is (at least in the simply-connected case) the same as the classification of the irreducible representations of its Lie algebra. (Representations of a complex Lie group are assumed analytic, and representations of a compact Lie group are assumed continuous. Representations of a complex Lie algebra are assumed $\mathbb{C}$-linear, and representationsions of a real Lie algebra are assumed $\mathbb{R}$-linear.

Thus to a first approximation, the finite-dimensional theories of Lie groups and Lie algebras are nearly equivalent. The theory of Lie groups is superior for supplying intuition, but the theory of Lie algebras is simpler since the operation is linear.

In the infinite-dimensional case, the fact that the theory of Lie algebras is simpler is decisive. Still, we may use the theory of groups to supply intuition, so we may preface a discussion with some optional group theory.

For example, let us motivate the definition of an invariant bilinear form with the group-theoretic definition. We assume that we have an action of a Lie algebra on a vector space V, that is, a map $\pi\colon \mathfrak{g} \longrightarrow \mathrm{End}(V)$ such that $\pi([X,Y]) = \pi(X)\circ\pi(Y) - \pi(Y)\circ\pi(X)$. Then by an *invariant bilinear form* we mean a bilinear map $B\colon V\times V \longrightarrow \mathbb{C}$ such that

$$B(X\cdot v, w) + B(v, X\cdot w) = 0. \tag{1}$$

Here $X\cdot v$ means $\pi(X)v$. Why is this the right definition? Begin with the well-motivated and obviously correct notion for a Lie group:

$$B(g\cdot v, g\cdot w) = B(v,w).$$

Take $g = e^{tX}$ and differentiate with respect to t:

$$0 = \frac{d}{dt}B(e^{tX}\cdot v, e^{tX}\cdot w).$$

By the chain rule, if $F(t,u)$ is a smooth function of two variables then

$$\frac{d}{dt}F(t,t) = \frac{\partial}{\partial t}F(t,u)|_{u=t} + \frac{\partial}{\partial u}F(t,u)|_{u=t}$$

and applying this to $F(t,u) = B(e^{tX}\cdot v, e^{uX}\cdot w)$ we obtain (1).

3 Central Extensions and Derivations

As with groups, we may consider extensions of Lie algebras, that is, equivalence classes of short exact sequences

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{q} \longrightarrow 0.$$

Given an extension, there are two cases where there is associated an action of $\mathfrak{q}$ on $\mathfrak{k}$. This means a homomorphism from $\mathfrak{q}$ to the Lie algebra $\mathrm{Der}(\mathfrak{k})$ of derivations of $\mathfrak{k}$. The first is the case where $\mathfrak{k}$ is abelian. The second is where the extension is split. In both these cases, the Lie algebra $\mathfrak{g}$ may be recovered from $\mathfrak{k}$, $\mathfrak{q}$ and some additional data.

- If $\mathfrak{k}$ is abelian, then $\mathfrak{k}$ is a $\mathfrak{q}$-module, and the extra data needed to describe $\mathfrak{g}$ is a 2-cocycle. The special case where the $\mathfrak{q}$-module structure is a 2-cocycle. For simplicity we will only discuss (below) the case where the $\mathfrak{q}$-module structure on $\mathfrak{k}$ is trivial, or equivalently that the image of $\mathfrak{k}$ in $\mathfrak{g}$ is central.
- If the extension is split, then the only data needed to describe $\mathfrak{g}$ is the homomorphism $\mathfrak{q} \longrightarrow \mathrm{Der}(\mathfrak{k})$.

As an important special case, we are interested in increasing the dimension of a Lie algebra by one. That is, we want to consider extensions:

$$0 \longrightarrow \mathbb{C} \cdot c \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow \mathbb{C} \cdot d \longrightarrow 0$$

where $\mathbb{C} \cdot c$ and $\mathbb{C} \cdot d$ are one-dimensional abelian Lie algebras. In the first case we may assume that c is central, that is, $[c, X] = 0$ for all $X \in \mathfrak{g}$. In the second case, we assume only that $\mathfrak{g}'$ is an ideal, that is, $[\mathfrak{g}, \mathfrak{g}'] \subset \mathfrak{g}'$.

3.1 Central extensions

Let $\mathfrak{a}$ be an abelian Lie algebra. We're mainly interested in the case where $\mathfrak{a}$ is one-dimensional. We wish to classify central extensions

$$0 \longrightarrow \mathfrak{a} \longrightarrow \tilde{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \longrightarrow 0\,.$$

We choose a section s: $\mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}$, which is a linear map (not a homomorphism) such that $p \circ s = 1_{\mathfrak{g}}$. Then if $X, Y \in \mathfrak{g}$ consider

$$\phi(X, Y) = s([X, Y]) - [s(X), s(Y)].$$

Clearly $\phi(X, Y)$ is in the kernel of p, which we may identify with $\mathfrak{a}$. We observe that ϕ is skew-symmetric and satisfies the *cocycle identity*

$$\phi([X, Y], Z) + \phi([Y, Z], X) + \phi([Z, X], Y) = 0.$$

Proof. We have

$$\phi([X, Y], Z) = s([X, Y], Z) - [s([X, Y]), s(Z)].$$

Since $s([X, Y]) - [s(X), s(Y)] = \phi(X, Y) \in \mathfrak{a}$, we may rewrite this

$$\phi([X, Y], Z) = s([X, Y], Z) - [[s(X), s(Y)], s(Z)].$$

The cocycle identity now follows from the Jacobi identities in $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. $\square$

A skew-symmetric map that satisfies the cocycle identity is called a *2-cycle*. Let $Z_2(\mathfrak{g}, \mathfrak{a})$ be the space of 2-cocycles.

We see that every extension produces a 2-cocycle, and conversely given a 2-cocycle we may reconstruct $\tilde{\mathfrak{g}}$ as follows. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$ with the bracket operation

$$[\![X+\delta, Y+\varepsilon]\!] = [X, Y] + \delta + \varepsilon + \phi(X, Y)$$

Then the 2-cocycle condition implies the Jacobi identity.

We have some freedom of choice in the section s. We could vary it by an arbitrary linear map $f\colon \mathfrak{g} \longrightarrow \mathfrak{a}$. This changes the cocycle by the map

$$\psi(X, Y) = f([X, Y]).$$

A cocycle of this form is called a *coboundary*. If $B_2(\mathfrak{g}, \mathfrak{a})$ is the space of coboundaries then we see that central extensions are classified by $H_2(\mathfrak{g}, \mathfrak{a}) = Z_2(\mathfrak{g}, \mathfrak{a})/B_2(\mathfrak{g}, \mathfrak{a})$.

3.2 Adjoining a derivation

A derivation is a local analog of an automorphism. Thus, let A be an algebra, whether associative or not. A map $D\colon A \longrightarrow A$ such that $D(xy) = D(x)\, y + x\, D(y)$ is a *derivation*. Exponentiating it:

$$e^D = 1 + D + \frac{1}{2}D^2 + \cdots$$

then satisfies $e^D(xy) = e^D(x)e^D(y)$.

$$(x + D(x) + \frac{1}{2}D^2(x) + \ldots.)$$

If $\mathfrak{g}$ is a Lie algebra, then $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is the *adjoint representation*, defined by

$$\mathrm{ad}(X)\, Y = [X, Y].$$

It follows from the Jacobi identity that $\mathrm{ad}(X)$ is a derivation. The map ad is a Lie algebra homomorphism, that is, $\mathrm{ad}([X, Y]) = \mathrm{ad}(X)\mathrm{ad}(Y) - \mathrm{ad}(Y)\mathrm{ad}(X)$. This also follows from the Jacobi identity.

Suppose that $\mathfrak{k}$ is an ideal of $\mathfrak{g}$ of codimension 1, and let $d \in \mathfrak{g} - \mathfrak{k}$. Then $\mathrm{ad}(d)$ induces a derivation of $\mathfrak{k}$. Conversely, suppose that $\mathfrak{g}$ is given, together with a derivation D. Then we may construct a Lie algebra $\mathfrak{g}$ that is the direct sum of $\mathfrak{g}$ and a one-dimensional space $\mathbb{C} \cdot d$, where the Lie bracket is extended to $\mathfrak{k}$ by requiring that $[D, X] = -[X, D] = d(X)$. The Jacobi identity follows from the fact that D is a derivation.

More generally, given a pair $\mathfrak{k}$, $\mathfrak{q}$ of Lie algebras and a homomorphism $\alpha\colon \mathfrak{q} \longrightarrow \mathrm{Der}(\mathfrak{k})$ we may construct a Lie algebra $\mathfrak{g}$ which contains copies of both $\mathfrak{q}$ and $\mathfrak{k}$. The subalgebra $\mathfrak{k}$ is an ideal, and if $X \in \mathfrak{q}$ and $Y \in \mathfrak{k}$, then $\alpha(Y)X = [Y, X]$.

4 Virasoro Algebra

Let us consider the Lie algebra $\mathfrak{w}$ of (Laurent) polynomial vector fields on the circle $\mathbb{T} = \{t \in \mathbb{C}^\times \| t| = 1\}$, or equivalently, polynomial vector fields on $\mathbb{C}^\times$. This is the Lie algebra of the (infinite-dimensional) group of diffeomorphisms of the circle. It has a basis

$$\mathfrak{d}_i = t^{1-i}\frac{d}{dt}.$$

These satisfy the commutation relation

$$[\mathfrak{d}_i, \mathfrak{d}_j] = (i - j)\mathfrak{d}_{i+j}.$$

Let N be an integer.

Proposition 1. *Let*

$$\phi_N(\mathfrak{d}_i, \mathfrak{d}_j) = \begin{cases} i^N & \text{if } i = -j \\ 0 & \text{otherwise}, \end{cases}$$

extended by linearity to a map $\mathfrak{w} \times \mathfrak{w} \longrightarrow \mathbb{C}$. If $N = 1$ or 3 then ϕ_N is a 2-cocycle. (We are regarding $\mathbb{C}$ as a one-dimensional abelian Lie algebra.

Proof. If $N$is odd then ϕ_N is skew-symmetric, and we need

$$\phi_N([\mathfrak{d}_i, \mathfrak{d}_j], \mathfrak{d}_k) + \phi_N([\mathfrak{d}_k, \mathfrak{d}_i], \mathfrak{d}_j) + \phi_N([\mathfrak{d}_j, \mathfrak{d}_k], \mathfrak{d}_i) = 0.$$

The left-hand side is trivially zero unless $i + j + k = 0$, so we assume this. Then the left-hand side equals

$$(i - j)k^N + (j - k)i^N + (k - i)j^N = (i - j)(-i - j)^N + (i + 2j)i^N - (j + 2i)j^N.$$

This is zero if $i + j + k = 0$ and $N = 1$ or 3. □

It may be seen that ϕ_1 and ϕ_3 span $Z_2(\mathfrak{w}, \mathbb{C})$, but ϕ_1 is a coboundary. The space $H^2(\mathfrak{w}, \mathbb{C})$ is one-dimensional. There is a unique isomorphism class of Lie algebras that are nontrivial central extensions of $\mathfrak{w}$ by $\mathbb{C}$. (Each element of $H^2(\mathfrak{w}, \mathbb{C})$ determines a unique equivalence class of extensions, but two inequivalent extensions may be isomorphic as Lie algebras.)

The choice of extension is traditionally to take the cocycle $\frac{1}{12}(\phi_3 - \phi_1)$. Thus we obtain the *Virosoro algebra* with basis $\mathfrak{d}_i, c$, where c is central, and

$$[\mathfrak{d}_i, \mathfrak{d}_j] = \begin{cases} (i - j)\mathfrak{d}_{i+j} + \frac{1}{12}(i^3 - i) & \text{if } i = -j, \\ (i - j)\mathfrak{d}_{i+j} & \text{otherwise.} \end{cases}$$

5 Loop algebras

In this section we will describe affine Kac-Moody Lie algebras following Chapter 7 of Kac, *Infinite-dimensional Lie algebras.* This illustrates both methods of enlarging a Lie algebra: first by making a central extension, then adjoining a derivation.

If $\mathfrak{g}$ is a Lie algebra and A is a commutative associative algebra, then $A \otimes \mathfrak{g}$ is a Lie algebra with the bracket

$$[a \otimes X, b \otimes Y] = ab \otimes [X, Y].$$

Let $\mathfrak{g}$ be a complex Lie algebra that admits an ad-invariant symmetric bilinear form, which we will denote $(|)$. Recalling that $\text{ad}: \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g})$ is the map $\text{ad}(X)Y = [X, Y]$, this means

$$([X, Y]|Z) = -(Y|[X, Z]).$$

Since the form is symmetric, this implies

$$([X,Y]|Z)=([Y,Z]|X). \tag{2}$$

For example, let G_0 be a compact Lie group. Then G_0 acts on itself by conjugation, inducing a representation Ad of G_0 on left-invariant vector fields, that is, on its Lie algebra $\mathfrak{g}_0$. The representation of $\mathfrak{g}_0$ on itself induced by Ad is ad. Since Ad: $G_0 \longrightarrow \mathrm{GL}(\mathfrak{g}_0)$ is a representation of a compact Lie group on a finite-dimensional real vector space, it admits an invariant bilinear form, and by the argument in Section 2, it follows that this gives an invariant positive definite symetric bilinear form on $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is actually a real Lie algebra, we then pass to the complexified Lie algebra $\mathfrak{g}=\mathbb{C}\otimes\mathfrak{g}_0$ to obtain a complex Lie algebra.

Let $\mathcal{L}=\mathbb{C}[t,t^{-1}]$ be the commutative algebra of Laurent polynomials. Then $\mathcal{L}(\mathfrak{g})=\mathcal{L}\otimes\mathfrak{g}$ is an infinite-dimensional Lie algebra. We will show how to obtain a central extension $\mathfrak{g}'$ of it. The residue map Res: $\mathcal{L}\longrightarrow\mathbb{C}$ is defined by

$$\mathrm{Res}\left(\sum_n c_n t^n\right)=c_{-1}.$$

We will also define

$$\phi(P,Q)=\mathrm{Res}(P'Q), \qquad P'=\frac{dP}{dt}.$$

We have

$$\mathrm{Res}(P')=0$$

for any Laurent polynomial. Thus (using $'$ to denote the derivative) since $(PQ)'=P'Q+PQ'$ and $(PQR)'=P'QR+PQ'R+PQR'$ we obtain

$$\phi(P,Q)=-\phi(Q,P)$$

and

$$\phi(PQ,R)+\phi(QR,P)+\phi(RP,Q)=0. \tag{3}$$

These relations are similar to the conditions that must be satisfied by a 2-cocycle. So we define ψ: $\mathcal{L}(\mathfrak{g})\times\mathcal{L}(\mathfrak{g})\longrightarrow\mathbb{C}$ by

$$\psi(a\otimes X, b\otimes Y)=(X|Y)\phi(a,b).$$

This bilinear form is clearly skew-symmetric. Here is the proof of the cocycle relation:

$$\psi([a\otimes X,b\otimes Y],c\otimes Z)=([X,Y]|Z)\phi(ab,c).$$

Using (2) and (3) we have

$$\begin{aligned}\psi([a\otimes X,b\otimes Y],c\otimes Z)+\psi([b\otimes Y,c\otimes Z],a\otimes X)+\psi([c\otimes Z,a\otimes X],b\otimes Y) &= \\ ([X,Y]|Z)[\phi(ab,c)+\phi(bc,a)+\psi(ca,b)] &= 0.\end{aligned}$$

Let $\mathfrak{g}'$ be the central extension of $\mathcal{L}(\mathfrak{g})$ determined by this 2-cocycle. We may further enlarge $\mathfrak{g}'$ by adjoining a derivation: use the derivation tdt of $\mathcal{L}$ (acting by zero on the adjoined one-dimensional central element. Let $\hat{\mathfrak{g}}$ be the resulting Lie algebra.

If $\mathfrak{g}$ is a simple complex Lie algebra, then $\hat{\mathfrak{g}}$ is an affine Kac-Moody Lie algebra.

6 Projective Representations and Central Extensions

It is well-known that there is a relationship between projective representations of a group and central extensions. We briefly recall this before considering the analogous situation for Lie algebras. Suppose that $\pi: G \to GL(V)$ is a projective representation of a group G; that is, a map that induces a homomorphism $G \to \mathrm{Aut}(\mathbb{P}^1(V))$, the group of automorphisms of the corresponding projective space. Concretely, this means that

$$\pi(g)\,\pi(h) = \phi(g,h)\,\pi\,(g\,h)$$

for some map $\phi: G \times G \to \mathbb{C}$. Now the map ϕ satisfies the cocycle condition

$$\phi\,(g_1\,g_2, g_3)\;\phi(g_1, g_2) = \phi\,(g_1, g_2\,g_3)\;\phi(g_2, g_3),$$

which means that we may construct a central extension of G. Let $\tilde{G}$ be the group which as a set is $G \times \mathbb{C}^\times$ with the multiplication

$$(g,\epsilon)(h,\delta) = (g\,h, \epsilon\,\delta\,\phi(g,h)).$$

The cocycle condition implies that this group law is associative. we may define a representation $\tilde{\pi}: \tilde{G} \to GL(V)$ by

$$\tilde{\pi}\,(g,\epsilon)\,v = \epsilon\,\pi(g)\,v,$$

and this is a true representation. Therefore every projective representation of G may be lifted to a true representation of a covering group, that is, a central extension determined by a cocycle describing the projective representation.

Now let us consider the corresponding construction for a Lie algebra representation. A representation of a Lie algebra $\mathfrak{g}$ is a linear map $\pi: \mathfrak{g} \to \mathrm{End}(V)$ such that

$$\pi([X,Y])\,v = \pi(X)\,\pi(Y)\,v - \pi(Y)\,\pi(X)\,v, \qquad X, Y \in \mathfrak{g}.$$

The corresponding notion of a *projective representation* requires a map $\phi: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ such that

$$\pi([X,Y])\,v = \pi(X)\,\pi(Y)\,v - \pi(Y)\,\pi(X)\,v + \phi(X,Y)\,v, \qquad X, Y \in \mathfrak{g}. \tag{4}$$

Lemma 2. *In (4), ϕ is a 2-cocycle.*

Proof. It is straightforward to check that $\phi(X,Y) = -\phi(Y,X)$. We have

$$\begin{aligned}\pi([[X,Y],Z]) = {} & \pi(X)\,\pi(Y)\,\pi(Z) - \pi(Y)\,\pi(X)\,\pi(Z) - \pi(Z)\,\pi(X)\,\pi(Y) \\ & + \pi(Z)\,\pi(Y)\,\pi(X) + \phi([X,Y],Z)\;I_V.\end{aligned}$$

Summing cyclicly and using the Jacobi relation gives

$$\phi([X,Y],Z) + \phi([Y,Z],X) + \phi([Z,X],Y) = 0.$$

□

Remark 3. As a special case, let $\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ be a true representation. We obtain a projective representation by perturbing it. Thus, let $\lambda\colon \mathfrak{g} \to \mathbb{C}$ be an arbitrary linear functional, and define $\pi'(X) = \pi(X) - \lambda(X)$. Then π' is a projective representation with $\phi(X, Y) = \lambda([X, Y])$. This cocycle is a coboundary.

7 Fermions

The Dirac equation describes the electron or other particles of spin $\frac{1}{2}$. Such a particle is a fermion, meaning that the wave equation is skew-symmetric, changing sign when particles are interchanged. Therefore no two particles can occupy the same state. (This is the *Pauli exclusion principle*.) In the simplest case, electrons are solutions to the Dirac equation characterized by energy (which is quantized) but no other quantum number. There is thus one solution with energy k for each integer k. Thus if v_k is a vector in a suitable Hilbert space V representing such a solution, a solution representing the superposition of several electrons might be represented by an element $v_{k_1} \wedge v_{k_2} \wedge \cdots$ of the exterior algebra on V.

A difficulty with the Dirac equation is the existence of solutions with negative energy. Particles with negative energy are unphysical and yet they are predicted by the theory. Dirac postulated that the negative energy solutions are fully populated. Thus the ground state is the "vacuum vector"

$$|0\rangle = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \cdots$$

If one electron is promoted to a state with higher energy, we might obtain a state (for example)

$$v_3 \wedge v_0 \wedge v_{-1} \wedge v_{-3} \wedge v_{-4} \cdots$$

The v_3 then appears as an electron with energy 3. The missing particle v_{-2} represents the *absence* of a usually present negative energy particle, that is, a positive energy particle whose charge is the opposite of the charge of the electron. This energetic state represents an electron-positron pair. This scheme avoids the physical appearance of negative energy solutions.

We are led to consider the space of semi-infinite wedges, spanned by elements of the form

$$v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots$$

where (due to the skew-symmetry of the exterior power) we may assume that

$$i_m > i_{m-1} > i_{m-2} > .$$

We will call such a vector a *semi-infinite monomial*. The integer m is chosen so that $i_{-k} + k = 0$ for sufficiently large k. The integer m is the excess of electrons over positrons, so m is called the *charge* of the monomial. Let $\mathfrak{F}^{(m)}$ be the span of all monomials of charge m, and let $\mathfrak{F}$ be the direct sum of the $\mathfrak{F}^{(m)}$. The space $\mathfrak{F}$ is called the *Fermionic Fock space*.

We will denote

$$|m\rangle = v_m \wedge v_{m-1} \wedge \cdots \in \mathfrak{F}^{(m)}. \tag{5}$$

This is the *vacuum vector* in $\mathfrak{F}^{(m)}$.

Let $\mathfrak{gl}_\infty$ be the Lie algebra spanned by elements E_{ij} with $i, j \in \mathbb{Z}$. This is an associative ring (without unit) having multiplication

$$E_{ij}\, E_{kl} = \delta_{jk}\, E_{il}, \qquad \delta = \text{Kronecker delta},$$

and as usual the associative ring becomes a Lie algebra with bracket $[X, Y] = XY - YX$. Elements may be thought of as infinite matrices as follows: we will represent an element $\sum a_{ij} E_{ij} \in \mathfrak{gl}_\infty$, where all but finitely many a_{ij} as a matrix (a_{ij}) indexed by $i, j \in \mathbb{Z}$.

The Lie algebra $\mathfrak{gl}_\infty$ acts on the v_k by $E_{ij}\, v_k = \delta_{jk}\, v_i$. Then there is an action on the seminfinite monomials by

$$X\,(v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots) = \sum_k\, v_{i_m} \wedge \cdots \wedge v_{i_{k+1}} \wedge X(v_{i_k}) \wedge v_{i_{k-1}} \wedge \cdots . \tag{6}$$

In accordance with the Leibnitz rule, we have applied the matrix X to a single v_k, then summed over all k.

The Lie algebra $\mathfrak{gl}_\infty$ has an enlargement which we will denote $\bar{\mathfrak{a}}$. This is again obtained from an associative algebra, namely we now consider (a_{ij}) where now we allow an infinite number of nonzero entries, but we require that there exists a bound N such that $a_{ij} = 0$ when $|i - j| > N$. This is sufficient to imply that in the definition of matrix multiplication, $(a_{ij}) \cdot (b_{ij}) = (c_{ij})$ where $c_{ij} = \sum_k a_{ik} b_{kj}$, the summation over k is finite.

The action of $\mathfrak{gl}_\infty$ on semi-infinite monomials does not extend to $\bar{\mathfrak{a}}$. However it *almost* does so extend. Applying (6) formally gives

$$\sum_{\substack{n\\ |n| \leqslant N}} \left[\sum_{\substack{i,j,k\\ i-j=n\\ i_k=j}} a_{ij}\, v_{i_m} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_i \wedge v_{i_{k+1}} \wedge \cdots \right]. \tag{7}$$

The inner sum is finite for all n *except* $n = 0$. The reason is that if $n \neq 0$ then there will only be finitely many i_k such that replacing v_{i_k} by v_{i_k+n} will not cause a repeated index, and $v_{i_m} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_{i_k+n} \wedge v_{i_{k+1}} \wedge \cdots$ is treated as zero unless the indices are all distinct.

However the $n = 0$ the inner sum is

$$\sum_{i_k} a_{i_k, i_k} v_{i_m} \wedge \cdots \wedge v_{i_{k+1}} \wedge v_{i_k} \wedge v_{i_{k-1}} \wedge \cdots$$

which is problematic.

We may address this problem by making use of Remark 3. We will perturb the representation $\pi\colon \mathfrak{gl}_\infty \longrightarrow \mathrm{End}(\mathfrak{F})$ defined by the above action (so $Xv = \pi(X)v$) by introducing a linear functional λ on $\mathfrak{g} = \mathfrak{g}$, then let

$$\pi'(X)v = \pi(X) - \lambda(X) I_{\mathfrak{F}}.$$

Thus π' is only a projective representation, but if λ is chosen correctly it has the advantage that it extends to $\bar{\mathfrak{a}}$.

We consider the linear functional $\lambda\colon \mathfrak{gl}_\infty \longrightarrow \mathbb{C}$ defined on basis vectors by

$$\lambda(E_{ij}) = \begin{cases} 1 & \text{if } i = j < 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the term $n=0$ in (7) becomes

$$\sum_{i_k \geqslant 0} a_{i_k, i_k} v_{i_m} \wedge \cdots \wedge v_{i_{k+1}} \wedge v_{i_k} \wedge v_{i_{k-1}} \wedge \cdots$$

which is a finite sum even when $(a_{ij}) \in \bar{\mathfrak{a}}$.

Now the cocycle $\phi(X, Y) = \lambda([X, Y])$ is a coboundary on $\mathfrak{gl}_\infty$. It is given by

$$\phi(E_{ij}, E_{ji}) = \begin{cases} 1 & \text{if } i < 0, j \geqslant 0 \\ -1 & \text{if } i \geqslant 0, j < 0, \\ 0 & \text{otherwise} \end{cases}$$

while $\phi(E_{ij}, E_{kl}) = 0$ unless $j = k$ and $i = l$. Let $\mathfrak{a}$ be the corresponding central extension of $\bar{\mathfrak{a}}$. Then we obtain a representation of $\mathfrak{a}$ on $\mathfrak{F}$.

8 The Heisenberg Lie Algebra and the Stone-von Neumann Theorem

Let $\mathfrak{H}_\infty$ be the Heisenberg Lie algebra, which has a basis p_i, q_i ($i = 1, 2, 3, \cdots$) and c. The vector c is central, while

$$[p_i, q_i] = c. \tag{8}$$

These elements have nicknames coming from quantum mechanics: q_i are "position," p_i are "momentum." This is because if we have independent particles, with corresponding space-like variables x_1, x_2, x_3, $\cdots$ the position and momentum operators on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by

$$(q_i f)(x_1, x_2, \cdots) = x_i f, \qquad p_i = -i\hbar \frac{\partial}{\partial x_i}$$

give a representation of $\mathfrak{H}_\infty$ with c action by $-i\hbar$. Here $\hbar$ is Planck's constant. There is a certain symmetry since the Schwartz space is invariant under the Fourier transform, which intertwines the position and momentum operators.

The representation extends to tempered distributions. To obtain a purely algebraic analog, we may restrict the representation to the space of polynomials, which are not in the Schwartz space, but nevertheless are tempered distributions. The constant polynomial is annihilated by the position operators, and this corresponds to the "vacuum vector." Interpreted as a physical state, it corresponds to a function with definite (zero) momentum but completely indeterminate position. The symmetry is now lost since the Fourier transform of a polynomial is a tempered distribution but not a polynomial, or even a function. Still, the polynomial representation is an algebraic substitute for the topological one.

Abstractly, given a representation of $\mathfrak{H}_\infty$, we call a nonzero vector v a *vacuum vector* if it is annihilated by the p_i. If $cv = \lambda v$ then we say λ is the *eigenvalue* of v. In the physics literature the vacuum vector is denoted $|0\rangle$ (following Dirac).

Similarly, let a be any constant, and consider the representation of $\mathfrak{H}_\infty$ on polynomials with q_i being multiplication by x_i and $p_i = \lambda \frac{\partial}{\partial x_i}$, and λ being the eigenvalue of c. This $\mathfrak{H}_\infty$-module is denoted R_λ.

Let H be the Heisenberg Lie group, which is a central extension of $\mathbb{R}^{2n}$, based on a symplectic form, which may be regarded as a 2-cocycle. Concretely, H is $\mathbb{R}^{2n}\times\mathbb{T}$ (where $\mathbb{T}$ is the group of complex numbers of absolute value 1) with multiplication

$$(x_1,\cdots,x_n,y_1,\cdots,y_n,t)(x_1',\cdots,x_n',y_1',\cdots,y_n',t') = \big(x_1+x_1',\cdots,y_1+y_1',\cdots,tt'\exp\big(\sum x_iy_i'-y_ix_i'\big)\big).$$

The Lie algebra of H is $\mathfrak{H}_\infty$. The classical Stone-von Neumann theorem says that H admits a unique irreducible unitary representation with any given nontrivial central character. The following result is an algebraic analog.

Proposition 4. *Let V be any irreducible $\mathfrak{H}_\infty$-module. Suppose that V is generated by a vacuum vector with eigenvalue λ. Then $V\cong R_\lambda$.*

Proof. See Kac, Corollary 9.13 on page 162. In order to see why this is true, consider the abelian “Lagrangian” subalgebra $\mathfrak{H}_+$ generated by the q_i. Since $\mathfrak{H}_+$ is abelian, its universal enveloping algebra $U(\mathfrak{H}_+)$ is the polynomial ring $A=\mathbb{C}[q_1,q_2,\cdots]$. We will see that if v_0 is the vacuum vector, then $V=A\cdot v_0$. Indeed, given any polynomial $P(q_1,q_2,\cdots)$ we have

$$[p_i,P(q_1,q_2,\cdots)]=\frac{\partial P}{\partial q_i}(q_1,q_2,\cdots)c.$$

Since p_i annihilates v_0 we see that

$$p_iP(q_1,q_2,\cdots)v_0=[p_i,P(q_1,q_2,\cdots)]v_0=\lambda\frac{\partial P}{\partial q_i}(q_1,q_2,\cdots)v_0.$$

This shows that $A\cdot v_0$ is closed under the action of the p_i as well as the q_i, and moreover gives an algorithm for computing the action of the p_i. $\square$

9 The Harmonic Oscillator

The results of this section are, strictly speaking, not needed for the discussion of the Boson-Fermion correspondence, but they are useful to consider since we will be introduced to the raising and lowering operators. Moreover, the Proposition of this section is almost the same as Proposition 4, and the idea of the proof is the same.

Let us consider the one-dimensional Heisenberg Lie algebra $\mathfrak{H}_3$, spanned by P, Q and $\hbar$ subject to the conditions

$$[P,Q]=-i\hbar,\qquad[\hbar,P]=[\hbar,Q]=0.$$

Given any nonzero complex number λ, this has a representation on the Schwartz space $\mathcal{S}(\mathbb{R})$ in which

$$Q\,f(x)=x\,f(x),\qquad Pf=i\lambda\frac{\partial f}{\partial x},\qquad\hbar f=\lambda f. \tag{9}$$

This representation extends to tempered distributions, and it has an algebraic submodule $\mathbb{C}[x]$ consisting of all polynomial functions. Polynomials are not Schwartz functions, but they are tempered distributions. In some sense, this representation on $\mathbb{C}[x]$ is an algebraic version of the analytic representation on $\mathcal{S}(\mathbb{R})$. Let us call it $V(\lambda)$.

The polynomial 1 is a vacuum vector since it is annihilated by P; we will denote it as $|0\rangle$. Here is an analog of Proposition 4.

Proposition 5. *Let $\lambda \neq 0$. Then any irreducible representation of $\mathfrak{H}_3$ that has a vacuum vector v_0, and such that $\hbar v_0 = \lambda v_0$, is isomorphic to the module described by (9).*

Proof. By a simple change of basis, we may reduce to the case $\lambda = i$. Thus if we normalize so that $\hbar = 1$ this means that

$$[Q, P] = i.$$

Let us intoduce two operators

$$a = \frac{1}{\sqrt{2}}(Q + iP), \qquad a^* = \frac{1}{\sqrt{2}}(Q - iP). \tag{10}$$

These operators satisfy

$$[a, a^*] = -i[Q, P] = 1. \tag{11}$$

The vacuum vector $|0\rangle$, we have already noted is a notation for the constant function 1. Define recursively

$$v_0 = |0\rangle, \qquad v_{k+1} = \frac{1}{\sqrt{k+1}} a^* v_k. \tag{12}$$

(The constant $1/\sqrt{k+1}$ is included to make the v_{k+1} orthonormal, though we will not check that they have norm 1.) Let $N = a^* a$. It is clearly a Hermitian operator. We claim that

$$N v_k = k v_k. \tag{13}$$

Indeed, if $v = v_0$ then since $a v_0 = 0$ we have (13) when $k = 0$. Assume it is true for some k. Then using (11) we have

$$N a^* v_k = a^* a a^* = a^*(a^* a + 1) v_k = a^*(N + 1) v_k = (k + 1) a^* v_k.$$

Dividing by $\sqrt{k+1}$ we obtain (13) with k increased to $k + 1$, so (13) follows by induction.

The v_k are eigenvalues of the self-adjoint operator N with distinct eigenvalues, so they are linearly independent. Now consider the space

$$\bigoplus_{k=0}^{\infty} \mathbb{C} v_k. \tag{14}$$

This space is clearly closed under the action of a^*, and the effect of a^* on the v_k is given in (12). Let us show that the effect of a on v_k may also be computed. We will show recursively that

$$a v_k = \sqrt{k} v_{k-1}.$$

Indeed, we have

$$a v_k = \frac{1}{\sqrt{k}} a a^* v_{k-1} = \frac{1}{\sqrt{k}} (N + 1) v_{k-1} = \frac{1}{\sqrt{k}} k v_{k-1}.$$

We see that (14) is closed under both a and a^*, hence under both P and Q. By irreducibility, we see that the v_k are a basis, and moreover, the effect of P and Q are determined, so the module is unique up to isomorphism. $\square$

The operators (10) are called (respectively) *annihilation* and *creation*. As we will now explain, they act on the stationary states of the quantum mechanical harmonic oscillator, changing the energy level. The operator a^* creates a quantum of energy, and the operator a destroys a quantum of energy. The operator $H = \frac{1}{2}(a a^* + a^* a)$ is the Hamiltonian of the harmonic oscillator. Concretely, this is the differential operator

$$P^2 + Q^2 = -\frac{d^2}{dx^2} + x^2$$

corresponding to a quadratic potential $U(x) = x^2$. The eigenfunctions are the v_k, which turn out to be $v_0 = e^{-x^2/2}$ times Hermite polynomials. Since $H = N + \frac{1}{2}$, the energy levels are $\frac{1}{2}$, $\frac{3}{2}, \frac{5}{2}, \cdots$. The creation and annihilation operators shift up and down between the levels.

10 Heisenberg representation on the Fermionic Fock space

Let Λ_n be the shift operator acting on the space V spanned by vectors v_j ($j \in \mathbb{Z}$):

$$\Lambda_n(v_j) = v_{j-n}.$$

If $j \neq 0$ we adapt the representation to an action on $\mathfrak{F}$ by:

$$\Lambda_n(v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots) = \sum_k v_{i_1} \wedge \cdots \wedge \Lambda_n(v_{i_k}) \wedge \cdots \qquad . \tag{15}$$

The k-th term is interpreted to be 0 if $i_k - n = i_l$ for some $l \neq k$ (since then v_{i_l} appears twice). Under our assumption that $j \neq 0$ it follows that this is zero for all but finitely many k.

If $j = 0$ this definition doesn't make sense because the sum (15) is infinite, but we define Λ_0 to act by the scalar m on $\mathfrak{F}^{(m)}$.

We also make use of raising and lowering operators which we now define. Define

$$\psi_j(\xi) = v_j \wedge \xi.$$

Thus if $\xi = v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots$ then

$$\psi_j(\xi) = \begin{cases} 0 & \text{if } j = i_k \text{ for any } k, \\ (-1)^{m-k} v_{i_m} \wedge \cdots \wedge v_{i_k} \wedge v_j \wedge v_{i_{k-1}} \wedge \cdots & \text{if } i_k > j > i_{k-1} \text{ for some } k. \end{cases}$$

Let ψ_j^* be the adjoint of ψ_j with respect to the inner product such that the semiinfinite monomials are an orthonormal basis. Thus ψ_j^* removes v_j from the monomial if it can be found; otherwise it gives zero. Concretely with $\xi = v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots$

$$\psi_j^*(\xi) = \begin{cases} (-1)^{m-k+1} v_{i_m} \wedge \cdots \wedge v_{i_{k-1}} \wedge \hat{v_j} \wedge v_{i_{k+1}} & \text{if } i_k = j, \\ 0 & \text{if } j \notin \{i_m, i_{m-1}, \cdots\}. \end{cases}$$

The "hat" over v_j means that this vector is omitted.

Proposition 6. *The operators Λ_j and Λ_{-j} are adjoints. We have*

$$[\Lambda_j, \psi_n] = \psi_{n-j}, \qquad [\Lambda_j, \psi_n^*] = -\psi_{n+j}^*, \tag{16}$$

$$[\Lambda_n, \Lambda_m] = n\delta_{n,-m} I_{\mathfrak{F}}. \tag{17}$$

Proof. To see that Λ_j and Λ_{-j} are adjoints, observe that each preserves the orthogonal subspaces $\mathfrak{F}^{(m)}$. Moreover if $\xi = v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots$ and $\xi' = v_{j_m} \wedge v_{j_{m-1}} \wedge \cdots$ then $\langle \Lambda_j \xi, \xi' \rangle$ is zero unless $i_k = j_k$ for all but one value $k = k_0$ and $i_{k_0} - j = j_{k_0}$, in which case it is 1; and $\langle \xi, \Lambda_{-j} \xi' \rangle$ is the same.

For the first equation in (16), using the Leibnitz rule

$$[\Lambda_j, \psi_n]\xi = \Lambda_j(v_n \wedge \xi) - v_n \wedge \Lambda_j(x) = \Lambda_j(v_n) \wedge \xi = v_{n-j} \wedge \xi = \psi_{n-j}(\xi).$$

Taking adjoints gives the second equation in (16).

Now to prove $[\Lambda_n, \Lambda_m] = n\delta_{n,-m} I_{\mathfrak{F}}$, let us first show that applying $\Lambda_n \Lambda_m$ to a vacuum vector. For definiteness, let us use $|0\rangle$ as in (5), though there would be no difference for any other vacuum. Let us consider first the case where $n = -m$. We will show that

$$\Lambda_n \Lambda_{-n} |0\rangle - \Lambda_{-n} \Lambda_n |0\rangle = n|0\rangle.$$

We may assume that $n > 0$, since if $n = 0$ this is trivial, and the cases n and $-n$ are equivalent. In this case, it is easy to see that $\Lambda_n |0\rangle = 0$ since every term in (15) will have a repeated v_i. On the other hand,

$$\Lambda_{-n} |0\rangle = \sum_{k=0}^{n-1} (-1)^{k-1} v_k \wedge v_0 \wedge v_{-1} \wedge \cdots \wedge \widehat{v_{k-n}} \wedge \cdots$$

where the hat means the vector v_{k-n} has been omitted. Applying Λ_n to each term produces just $|0\rangle$, for n terms altogether.

We now know that

$$[\Lambda_n, \Lambda_{-n}]\xi = n\xi \tag{18}$$

when $\xi = |0\rangle$ is the vacuum. Next let us show that if (18) is true for a vector ξ then it is true for $\psi_j \xi$ for any j. Indeed

$$[\Lambda_n, \Lambda_{-n}]\psi_j \xi - n\psi_j \xi = [\Lambda_n, \Lambda_{-n}]\psi_j \xi - \psi_j [\Lambda_n, \Lambda_{-n}]\xi.$$

But by the Jacobi identity

$$[[\Lambda_n, \Lambda_{-n}], \psi_j] = [\Lambda_n, [\Lambda_{-n}, \psi_j]] - [\Lambda_{-n}, [\Lambda_n, \psi_j]]$$

and by (16) this is zero. Now (17) follows since every semiinfinite monomial can be built up from a vacuum by applications of ψ_j for various j. $\square$

11 Boson-Fermion Correspondence

Let $\mathfrak{H}$ be the Lie algebra spanned by elements L_i $(i \in \mathbb{Z})$ and a central element $\hbar$ subject to the relations

$$[L_m, L_n] = m\delta_{m,-n}\hbar.$$

In addition to $\hbar$ there is a second central element L_0. The basis elements L_n, $\frac{1}{n}L_n$, $\hbar$ obviously satisfy the same relations (8) as the p_i, q_i and c, so $\mathfrak{H} \cong \mathfrak{H}_\infty \oplus \mathbb{C}L_0$. Thus the algebraic Stone-von-Neumann result is true in the following form: let λ: $Z(\mathfrak{H}) \longrightarrow \mathbb{C}$ be a linear functional on the center $Z(\mathfrak{H})$, which is the vector space spanned by $\hbar$ and L_0, and assume that $\lambda(\hbar)$ is nonzero. Define a vacuum vector to be one annihilated by the L_j with $j > 0$. Then there exists a unique isomorphism class of irreducible representations generated by a vacuum vector such that $Z(\mathfrak{H})$ acts by the linear functional λ.

Let $\mathcal{B}$ be the ring of polynomials $\mathbb{C}[x_1, x_2, \cdots]$. For every m we have an action of r_m: $\mathfrak{H} \longrightarrow \mathrm{End}(\mathcal{B})$ on $\mathcal{B}$ in which $L_0 = m I_{\mathcal{B}}$ and if $k > 0$

$$L_k = \frac{\partial}{\partial x_k}, \qquad L_{-k} = k x_k,$$

with $\hbar$ acting by the scalar 1.

On the other hand, we have the action on $\mathfrak{F}^{(m)}$ in which $L_k \longrightarrow \Lambda_k$ which by Proposition 6 gives a representation. Due to the algebraic Stone-von-Neumann theorem, there is a unique intertwining map σ_m: $\mathfrak{F}^{(m)} \longrightarrow \mathcal{B}$ that intertwines the action on $\mathfrak{F}^{(m)}$ with r_m, and which maps the vacuum vector $|m\rangle$ to the vacuum vector in $\mathcal{B}$, which is the polynomial 1. We combine these into an intertwining map σ: $\mathfrak{F} \longrightarrow \hat{\mathcal{B}} = \mathbb{C}[q, q^{-1}] \otimes \mathcal{B}$ that sends $|m\rangle$ to q^m, where now $\hat{\mathcal{B}}$ is an $\mathfrak{H}$-module with $\mathfrak{H}$ acting by σ_m on the coset $q^m \mathcal{B}$, that is, on $q^m \otimes \mathcal{B}$.

Let u be a generator of a ring $\mathbb{C}[[u]]$ of formal power series. Combine the raising and lowering operators into series:

$$\psi(u) = \sum_{j \in \mathbb{Z}} u^j \psi_j, \qquad \psi^*(u) = \sum_{j \in \mathbb{Z}} u^{-j} \psi_j^*.$$

Note that applied to a semi-infinite monomial, all but finitely many ψ_j with $j < 0$ produce 0, and similarly all but but finitely many ψ_j^* with $j > 0$ produce 0. Hence $\psi(u)$ and $\psi^*(u)$ are meaningful. If u is an indeterminate, they map $\mathfrak{F} \longrightarrow \mathbb{C}((u)) \otimes \mathfrak{F}$, where $\mathbb{C}((u))$ is the field of fractions of $\mathbb{C}[[u]]$, consisting of formal power series $\sum a_n u^n$ where $a_n = 0$ if $n < -N$ for some N.

By (16) we have

$$[\Lambda_n, \psi(u)] = u^n \psi(u), \qquad [\Lambda_n, \psi^*(u)] = -u^n \psi^*(u). \tag{19}$$

By abuse of notation we will write $\mathfrak{F}$ and $\mathcal{B}$ instead of $\mathbb{C}((u)) \otimes \mathfrak{F}$ and $\mathbb{C}((u)) \otimes \mathcal{B}$, so that we may work with formal power series. Let $\Psi(u)$ and $\Psi^*(u)$ be the corresponding endomorphisms of $\mathcal{B}$, that is,

$$\Psi(u) = \sigma \psi(u) \sigma^{-1}, \qquad \Psi^*(u) = \sigma \psi^*(u) \sigma^{-1}.$$

Transferring (19) to $\mathcal{B}$ we have

$$[L_n, \Psi(u)] = u^n \Psi(u), \qquad [L_n, \Psi^*(u)] = -u^n \Psi^*(u). \tag{20}$$

Concretely, $[L_n, \Psi(u)] = u^n \Psi(u)$ means, first taking $n = -j$ where $j > 0$, and then taking $n = j$:

$$[x_j, \Psi(u)] = \frac{u^{-j}}{j} \Psi(u), \qquad \left[\frac{\partial}{\partial x_j}, \Psi(u)\right] = u^j \Psi(u). \tag{21}$$

Proposition 7. *On $q^m\mathcal{B}$ we have*

$$\Psi(u) = u^{m+1} q \cdot \exp\left(\sum_{j=1}^{\infty} u^j x_j\right) \exp\left(-\sum_{j=1}^{\infty} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j}\right), \tag{22}$$

$$\Psi^*(u) = u^{-m} q^{-1} \cdot \exp\left(-\sum_{j=1}^{\infty} u^j x_j\right) \exp\left(\sum_{j=1}^{\infty} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j}\right). \tag{23}$$

Proof. We recall Taylor's theorem in the form

$$f(x+t) = \sum_{j=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dt^n} = \exp\,(t\frac{d}{dx}) f(x).$$

Define

$$T_u f(x_1, x_2, \cdots) = f\left(x_1 + u^{-1}, x_2 + \frac{u^{-1}}{2}, \cdots\right). \tag{24}$$

Then

$$T_u = \exp\left(\sum_{j=1}^{\infty} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j}\right).$$

We consider

$$[x_j, \Psi(u) T_u] = [x_j, \Psi(u)] T_u - \Psi(u) [T_u, u].$$

By (24) we have

$$[T_u, x_j] = \frac{u^{-j}}{j} T_u.$$

Combining this with (21)

$$[x_j, \Psi(u) T_u] = 0.$$

This means that $\Psi(u) T_u$ is free of terms involving $\partial/\partial x_j$, and we may write

$$\Psi(u) T_u = q P(x),$$

a polynomial in x (also involving u as a formal power series). The factor q is clear since Ψ maps $q^m\mathcal{B}$ into $q^{m+1}\mathcal{B}$ but $P(x)$ remains to be determined.

Now consider

$$\exp\left(-\sum_{j=1}^{\infty} u^j x_j\right) \Psi(u) T_u. \tag{25}$$

We have

$$\left[\frac{\partial}{\partial x_j}, \exp\left(-\sum_{j=1}^{\infty} u^j x_j\right)\right] = -u^j \exp\left(-\sum_{j=1}^{\infty} u^j x_j\right).$$

Combining this with (21) and the fact that $\left[\frac{\partial}{\partial x_j}, T_u\right] = 0$ we see that $\frac{\partial}{\partial x_j}$ annihilates (25), and so this is a constant multiple of the vacuum vector. Here "constant" means independent of the x_i. The constant, we have already noted, is q times a factor depending only on u, and since it is independent of the x_i and q, it depends only on u. We evaluate it by applying (25) to the vacuum vector q^m.

To carry this out, consider the following commutative diagram

$$\begin{array}{ccccccccc} q^m\mathcal{B} & \xrightarrow{T_u} & q^m\mathcal{B} & \xrightarrow{\Psi(u)} & q^{m+1}\mathcal{B} & \xrightarrow{\alpha} & q^{m+1}\mathcal{B} & \xrightarrow{q^{-1}} & q^m\mathcal{B} \\ \downarrow\sigma & & \downarrow\sigma & & \downarrow\sigma & & \downarrow\sigma & & \downarrow\sigma \\ \mathfrak{F}^{(m)} & \xrightarrow{T_u'} & \mathfrak{F}^{(m)} & \xrightarrow{\psi(u)} & \mathfrak{F}^{(m+1)} & \xrightarrow{\beta} & \mathfrak{F}^{(m+1)} & \xrightarrow{\kappa} & \mathfrak{F}^{(m)} \end{array} \tag{26}$$

The maps T_u' and κ are defined by the commutativity of the squares the appear in. The map q^{-1} is multiplication by q^{-1}, and α is multiplication by

$$\exp\left(-\sum_{j=1}^{\infty} u^j x_j\right);$$

therefore the third square is commutative if

$$\beta = \exp\left(-\sum_{j=1}^{\infty} \frac{u^j}{j}\Lambda_{-j}\right).$$

Let us argue that

$$\kappa(v_{i_{m+1}} \wedge v_{i_m} \wedge \cdots) = v_{i_{m+1}-1} \wedge v_{i_m-1} \wedge \cdots. \tag{27}$$

The map $\kappa = \sigma\,(q^{-1})\sigma$ is a homomorphism of $\mathfrak{H}_0$ modules, where $\mathfrak{H}_0$ is spanned by the L_i with $i \neq 0$ and $\hbar$. Let κ' be the map defined by (27). We observe that κ' is clearly a homomorphism of $\mathfrak{H}_0$ modules, in view of (15). As $\mathfrak{F}^{(m+1)}$ is irreducible, κ and κ' differ by a constant multiple. But applied to the vacuum $|m+1\rangle$ both maps give $|m\rangle$, proving (27).

We have proven that the top diagram is multiplication by a constant, and therefore so does the bottom map. To compute it, we apply it to $|m\rangle$. The map T_u' takes $|m\rangle$ to $|m\rangle$ since the translation map T_u preserves the vacuum in $q^m\mathcal{B}$, which is the constant function $q^{m\cdot}$. The map $\psi(u)$ takes $|m\rangle$ to $u^{m+1}|m+1\rangle$ plus a sum of other seminfinite monomials orthogonal to $|m+1\rangle$. Evidently applying β then produces $u^{m+1}|m+1\rangle$ plus a sum of semi-infinite monomials orthogonal to it. The κ produces $u^{m+1}|m\rangle$ plus something orthogonal to it. Since we know that the bottom composition applied to $|m\rangle$ gives a multiple of $|m\rangle$, we see that this composition is just multiplication by the constant u^{m+1}, and we have proved (22). The other equation (23) is similar. □

12 The Murnaghan-Nakayama Rule

Let $\alpha_1, \cdots, \alpha_n$ be variables. Let $\Lambda^{(n)}$ be the ring of symmetric polynomials in $\alpha_1, \cdots, \alpha_n$. Particular important elements of $\Lambda^{(n)}$ are the k-th *elementary symmetric polynomials*

$$e_k(\alpha) = e_k^{(n)}(\alpha) = e_k(\alpha_1, \cdots, \alpha_n) = \sum_{i_1 < \cdots < i_k} \alpha_{i_1} \cdots \alpha_{i_k}$$

and the k-th *complete symmetric polynomials*

$$h_k(\alpha) = h_k^{(n)}(\alpha) = h_k(\alpha_1, \cdots, \alpha_n) = \sum_{i_1 \leqslant \cdots \leqslant i_k} \alpha_{i_1} \cdots \alpha_{i_k}.$$

We define $e_0 = h_0 = 1$ and $e_k = h_k = 0$ if $k < 0$.

There is a ring homomorphism $\Lambda^{(n+1)} \longrightarrow \Lambda^{(n)}$ in which $\alpha_{n+1} \longrightarrow 0$. This homomorphism sends $e_k^{(n+1)}$ to $e_k^{(n)}$ and $h_k^{(n+1)}$ to $h_k^{(n)}$, so e_k and h_k have a well-defined meaning in the inverse limit

$$\Lambda = \lim_{\longleftarrow} \Lambda^{(n)}.$$

The ring Λ may be thought of as the ring of symmetric polynomials in infinitely many variables.

Proposition 8. *The ring $\Lambda^{(n)}$ is a polynomial ring $\mathbb{C}[e_1, \cdots, e_n] = \mathbb{C}[h_1, \cdots, h_n]$. Thus the ring Λ is a polynomial ring $\mathbb{C}[e_1, e_2, \cdots] = \mathbb{C}[h_1, h_2, \cdots]$. It admits an involution that sends $e_i \longrightarrow h_i$ and $h_i \longrightarrow e_i$.*

Proof. We first recall why any symmetric polynomial f is a polynomial in $e_1, \cdots, e_n$. By induction, we assume this is true in $\Lambda^{(n)}$ and check that it is true in $\Lambda^{(n+1)}$. Let $f \in \Lambda^{(n)}$ and let f' be the image of f in $\Lambda^{(n)}$ obtained by putting $\alpha_{n+1} = 0$. By induction on n, f' may be expressed as a polynomial F' in $e_1^{(n)}, \cdots, e_k^{(n)}$. Writing $f' = F'\big(e_1^{(n)}, \cdots, e_n^{(n)}\big)$, we observe that $f - F'\big(e_1^{(n+1)}, \cdots, e_n^{(n+1)}\big)$ is in the kernel of the homomorphism $\Lambda^{(n+1)} \longrightarrow \Lambda^{(n)}$, and this implies that it is a multiple of α_{n+1}. Since it is symmetric, it is a multiple of $e_{n+1}^{(n+1)} = \alpha_1 \cdots \alpha_{n+1}$ and we may write $f = F'\big(e_1^{(n+1)}, \cdots, e_n^{(n+1)}\big) + g e_{n+1}^{(n+1)}$ for some symmetric polynomial g. The degree of g is strictly less than that of f, and by induction on degree, g is a polynomial in $e_1^{(n+1)}, \cdots, e_{n+1}^{(n+1)}$, so the same is true of f.

Therefore $\Lambda^{(n)} = \mathbb{C}[e_1, \cdots, e_n]$. We note that $e_1, \cdots, e_n$ are algebraically independent. See Lang's *Algebra*, third edition, page 192 for an algebraic proof valid over any field, or working over the complex numbers observe that if $\phi(e_1, \cdots, e_n) = 0$ where ϕ is a nonzero polynomial, then there are values $a_1, \cdots, a_n \in \mathbb{C}$ such that $\phi(a_1, \cdots, a_n) \neq 0$. Now by the fundamental theorem of algebra we may find $r_1, \cdots, r_n \in \mathbb{C}$ such that

$$(X - r_1) \cdots (X - r_n) = X^n - a_1 X^{n-1} + \cdots \pm a_n$$

so $a_k = e_k(r_1, \cdots, r_n)$. Therefore $\phi(a_1, \cdots, a_n) = \phi(e_1(r_1, \cdots, r_n), \cdots, e_n(r_1, \cdots, r_n)) = 0$, a contradiction, proving that the e_i are algebraically independent.

Thus $\Lambda^{(n)} = \mathbb{C}[e_1, \cdots, e_n]$ is a polynomial ring. There exists a homomorphism $\iota: \Lambda^{(n)} \longrightarrow \Lambda^{(n)}$ such that $\iota(e_k) = h_k$ for $k = 1, \cdots, n$. We will show that $\iota(h_k) = e_k$ when $k \leqslant n$ also. To prove this, define

$$E(t) = \sum_{k=0}^{\infty} e_k t^k, \qquad H(t) = \sum_{k=0}^{\infty} h_k t^k.$$

It is not hard to show that

$$E(t) = \prod_{i=1}^{n} (1 + \alpha_i t), \qquad H(t) = \prod_{i=1}^{n} (1 - \alpha_i t)^{-1},$$

where for the second identity we assume $|t\alpha_i| < 1$ so that we may expand the geometric series. Therefore

$$E(t) H(-t) = 1.$$

This means that

$$e_k - e_{k-1}h_1 + e_{k-2}h_2 - \ldots + (-1)^k h_k = 0 \tag{28}$$

for $k > 0$. This allows us to write h_k recursively as a polynomial in the e_k. Applying ι we obtain

$$h_k - h_{k-1}\iota(h_1) + h_{k-2}\iota(h_2) - \ldots + (-1)^k \iota(h_k) = 0$$

and comparing this relation with (28) we see (using induction on k) that $\iota(h_k) = e_k$, provided $k \leqslant n$.

Taking the limit as $n \longrightarrow \infty$, the corresponding statements are true for Λ. In the limiting case, we no longer need the assumption that $k \leqslant n$, so ι interchanges e_k and h_k for all k. □

In addition to the e_k and h_k let p_k be the k-th *power-sum symmetric polynomial:*

$$p_k(\alpha_1, \cdots, \alpha_N) = \sum \alpha_i^k,$$

or the image of this polynomial in Λ.

Proposition 9. *(i) (Pieri's formula)*

$$h_k s_\lambda = \sum_{\substack{\mu \supset \lambda \\ \mu \backslash \lambda \text{ is a horizontal strip of size } k}} s_\mu,$$

(ii) (Pieri's formula)

$$e_k s_\lambda = \sum_{\substack{\mu \supset \lambda \\ \mu \backslash \lambda \text{ is a vertical strip of size } k}} s_\mu,$$

(iii) (Murnaghan-Nakayama rule)

$$p_k s_\lambda = \sum_{\substack{\mu \supset \lambda \\ \mu \backslash \lambda \text{ is a ribbon of size } k}} (-1)^{\mathrm{ht}(\mu \backslash \lambda)} s_\mu.$$

Here $\mu \supset \lambda$ means that the Young diagram of μ contains that of λ, or alternatively that $\mu_i \geqslant \lambda_i$ for all i. Then we may consider the skew shape $\mu \backslash \lambda$. Its Young diagram is the set theoretic difference between the Young diagrams of μ and λ. To say that $\mu \backslash \lambda$ is a *horizontal strip* means that it has no two boxes in the same column. To say that it is a *vertical strip* means that it has no two boxes in the same row. To say that it is a *ribbon* means that it is connected and contains no 2×2 block.

We will prove (iii), the Murhaghan-Nakayama rule. Pieri's formula (ii) may be proved easily by the same method.

Lemma 10. *Let $\xi: \mathbb{Z} \longrightarrow V$ be a function taking values in a vector space V and let f be a skew-symmetric n-linear map from $\mathbb{C}^n$ to another vector space U. Define*

$$F(\lambda) = f(\xi(\lambda_1), \cdots, \xi(\lambda_n)), \qquad \lambda \in \mathbb{Z}^n.$$

If $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ define

$$S(\lambda) = F(\lambda + \rho) = F(\lambda_1 + n - 1, \lambda_2 + n - 2, \cdots, \lambda_n)$$

where $\rho = (n-1, n-2, \cdots, 0)$. *Let* $\boldsymbol{e}_i = (0, \cdots, 1, \cdots, 0)$ *be the* k*-th standard vector. Assume that* λ *is a partition, so* $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. *Then the nonzero elements of the set*

$$\{S(\lambda + k\boldsymbol{e}_i) | i = 1, \cdots, n\} \tag{29}$$

are precisely the nonzero elements of the set

$$\{(-1)^{\mathrm{ht}(\mu\backslash\lambda)} S(\mu) \mid \mu \supset \lambda \text{ and } \mu\backslash\lambda \text{ is a ribbon of length } k\}.$$

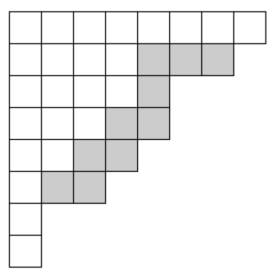
Before giving the proof let us consider an example. Let $\lambda = (8, 4, 4, 3, 2, 1, 1, 1)$ and let $n = 10$. With $\rho = (7, 6, 5, 4, 3, 2, 1, 0)$ we have $\lambda + \rho = (15, 10, 9, 7, 5, 3, 2, 1)$. One of the elements of the set (29) is

$$S(\lambda + 10\boldsymbol{e}_6) = F(15, 10, 9, 7, 5, 13, 2, 1) = (-1)^4 F(15, 13, 10, 9, 7, 5, 2, 1) = S(\mu)$$

where

$$\mu = (8, 7, 5, 5, 4, 3, 1, 1).$$

We confirm that $\mu - \lambda$ is a ribbon with the following diagram:



We can see that the skew diagram is a ribbon.

Proof. We may now explain the reason that the Lemma is true. If $F(\lambda + \rho + k\boldsymbol{e}_i) \neq 0$, then the components of the vector $\lambda + \rho + k\boldsymbol{e}_i$ are distinct, and in particular $\lambda_i + n - i + k$ does not equal $\lambda_j + n - j$ for any $j \neq i$. Thus there is a $j < i$ such that

$$\lambda_{j-1} + n - j + 1 > \lambda_i + n - i + k > \lambda_j + n - j, \tag{30}$$

unless $\lambda_i + n - i - k > \lambda_1 + n - 1$; in the latter case, the following explanation will remain true with $j = 1$. Let $\mu + \rho$ be $\lambda + \rho + k\boldsymbol{e}_i$ with the entries rearranged in descending order. This means that the i-th entry of $\lambda + \rho + k\boldsymbol{e}_i$ is moved forward into the j-th position. Taking into account that $\rho_j = \rho_i + j - i$ this means that $\mu_j = \lambda_i + k - i + j$.

Now (30) may be rewritten

$$\lambda_{j-1} \geqslant \mu_j > \lambda_j, \tag{31}$$

and furthermore if $j < l \leqslant i$, we have moved the $(l-1)$-th entry into the l-th position, which means that

$$\mu_{l+1} = \lambda_l + 1. \tag{32}$$

For l not in the range $j \leqslant l \leqslant i$ we have $\mu_l = \lambda_l$. It is easy to see that the conditions (31) and (32) mean that $\mu\backslash\lambda$ is a ribbon. □

We may now give two examples for Lemma 10. One example will prove the Murnaghan-Nakayama rule. The other is relevant to the Fermion-Boson correspondence.

Let us prove Proposition 9 (iii), the Murnaghan-Nakayama rule. Let $\alpha_1, \cdots, \alpha_n$ be indeterminates, and let $V = \mathbb{C}^n$. Let $\xi: \mathbb{Z} \longrightarrow V$ be the map

$$\xi(k) = (\alpha_1^k, \cdots \alpha_n^k).$$

Let $f: V^n \longrightarrow \mathbb{C}$ be the map sending $v_1, \cdots, v_n$ to $\Delta^{-1}\det(v)$, where v is the matrix with rows v_i and

$$\Delta = \det\,(\alpha_i^{n-j})_{i,j} = \prod_{i<j} (\alpha_i - \alpha_j).$$

Then if λ is a partition then $S(\lambda) = s_\lambda(\alpha_1, \cdots, \alpha_n)$ is the Schur polynomial.

Now consider $p_k s_\lambda$. This equals Δ^{-1} times $\sum_{i=1}^m M_i$, where $M_i = \alpha_i^k s_\lambda$. We will argue that

$$\sum_{i=1}^n M_i = \sum_{i=1}^n M_i', \tag{33}$$

where in the notation of $M_i' = S(\lambda + k\boldsymbol{e}_i)$. Indeed, M_i may be described as follows. Begin with the matrix:

$$\begin{pmatrix} \alpha_1^{\lambda_1+n-1} & \alpha_2^{\lambda_1+n-1} & \cdots & \alpha_n^{\lambda_1+n-1} \\ \alpha_1^{\lambda_2+n-2} & \alpha_2^{\lambda_2+n-2} & \cdots & \alpha_n^{\lambda_2+n-2} \\ \vdots & \vdots & & \vdots \\ \alpha_1^{\lambda_n} & \alpha_2^{\lambda_n} & \cdots & \alpha_n^{\lambda_n} \end{pmatrix}.$$

Increase each exponent in the i-th column by k, then take the determinant. On the other hand, M_i' may be described as follows: increase each exponent in the i-th row by k, then take the determinant. In either case, we obtain the following $n! \cdot n$ terms:

$$\sum_k \sum_{w \in S_n} \sum_{i,j} \alpha_{w(j)}^{\lambda_i + n - i + k\delta_{i,k}},$$

proving (33). Now using the Lemma, we see that $p_k s_l$ equals

$$\sum_{\substack{\mu \supset \lambda \\ \mu \backslash \lambda \text{ is a ribbon of size } k}} (-1)^{\mathrm{ht}(\mu\backslash\lambda)} s_\mu,$$

proving the Murnaghan-Nakayama rule.

13 The Boson-Fermion correspondence (continued)

Let p_k be the k-th power sum symmetric polynomial, either regarded as a polynomial in some fixed set of "hidden variables" $\alpha_1, \cdots, \alpha_n$, or (better, since n will not be fixed) as an element of the ring Λ. We identify these (normalized) with the variables x_k in the Bosonic Fock space:

$$x_k = \frac{1}{k} p_k.$$

Let $S_\lambda(x)$ be the polynomials in the x_i such that $S_\lambda(x)=s_\lambda(\alpha)$ is the Schur polynomial. For example,

$$S_{(1)}=x_1,\qquad S_{(2)}=\frac{1}{2}x_1^2+x_2,\qquad S_{(1,1)}=\frac{1}{2}x_1^2-x_2,\qquad S_{(2,1)}=\frac{1}{3}x_1^3-x_3.$$

Let us just consider the charge 0 part $\mathfrak{F}^{(0)}$ of the Fermionic Fock space. We index the monomials in $\mathfrak{F}^{(0)}$ by partitions: given

$$\xi=v_{i_0}\wedge v_{i_{-1}}\wedge v_{i_{-2}}\wedge\cdots,$$

where $i_0>i_{-1}>\cdots$ and $i_{-k}=-k$ for k sufficiently large, writing $\lambda_k=i_{-k}+k$, we have $\lambda_0\geqslant\lambda_1\geqslant\lambda_2\geqslant\ldots$ and $\lambda_k=0$ eventually, so $\lambda=(\lambda_0,\lambda_1,\cdots)$ is a partition. We write $\xi=\xi(\lambda)$, and then ξ is a bijection of the set of all partitions onto a basis of $\mathfrak{F}^{(0)}$.

Theorem 11. *Let λ be a partition. Then $\sigma(\xi(\lambda))=S_\lambda$.*

Proof. This is another application of Lemma 10. We fix an integer n that is greater than the length of λ. Let $F\colon\mathbb{Z}^n\longrightarrow\mathfrak{F}^{(0)}$ be the map

$$F(\mu_0,\cdots,\mu_{n-1})=\sigma(v_{\mu_0}\wedge v_{\mu_1-1}\wedge v_{\mu_2-2}\wedge v_{\mu_{n-1}-n+1}\wedge v_{-n}\wedge v_{-n-1}\wedge\cdots).$$

This map is skew symmetric, so by the Lemma we have

$$\sum_{j=0}^{n-1}F(\lambda_0,\cdots,\lambda_{j-1}+k,\cdots,\lambda_{n-1})=\sum_{\substack{\mu\supset\lambda\\ \mu\backslash\lambda\text{ is a ribbon of size }k}}(-1)^{\mathrm{ht}(\mu\backslash\lambda)}F(\mu).$$

By the Leibnitz rule (15) the left-hand side is Λ_{-k} applied to $F(\lambda)$, and remembering that in the bosonic picture this is multiplication by $kx_k=p_k$. Thus we have proved

$$kx_kF(\lambda)=\sum_{\substack{\mu\supset\lambda\\ \mu\backslash\lambda\text{ is a ribbon of size }k}}(-1)^{\mathrm{ht}(\mu\backslash\lambda)}F(\mu).\tag{34}$$

On the other hand, we have

$$p_ks_\lambda=\sum_{\substack{\mu\supset\lambda\\ \mu\backslash\lambda\text{ is a ribbon of size }k}}(-1)^{\mathrm{ht}(\mu\backslash\lambda)}s_\mu\tag{35}$$

by the Murnaghan-Nakayama rule.

Now we make use of the following facts about symmetric functions. The ring Λ of symmetric functions is generated by the p_k as a $\mathbb{C}$-algebra, and as a vector space, the s_λ are a basis. Consequently, there is a linear map $\theta\colon\Lambda\longrightarrow\mathbb{C}[x_1,x_2,x_3,\cdots]$ that sends S_λ to $F(\lambda)$. Since we are identifying Λ with $\mathbb{C}[x_1,x_2,x_3,\cdots]$ by identifying p_k with $\frac{1}{k}x_k$, we have, by (34) and (35), $\theta(p_ks_\lambda)=p_kj(s_\lambda)$, and therefore θ is a Λ-module homomorphism $\Lambda\longrightarrow\Lambda$. Since $\theta(1)=1$, it follows that θ is the identity map, which in view of our identifications means $\sigma(\xi(\lambda))=S_\lambda$. $\square$