Project: Zorn's Lemma

May 24, 2010

The problems in this last problem set will be graded by me, and treated like a take-home exam. You may discuss the problems with myself or with Robert Hough. You may email me for help or hints if you get stuck. You may also discuss the problems with each other but **you may not show another student your written work** for the project.

This is **due Wednesday May 26.**

Zorn's Lemma is a statement equivalent to the Axiom of Choice. It is particularly useful for applications in mathematics, though less intuitive than the Axiom of Choice.

Zorn's Lemma is proved on Page 142 of the book. There is a misprint in that a reference to "Chapter 7" should apparently be "Chapter 6."

I think it is interesting to be able to give proofs of both Zorn's Lemma and the well-ordering theorem that do not use transfinite induction. That is the goal of this project.

Let S be a partially ordered set. A *chain* in S is a linearly ordered subset I. This means that any two elements of I are comparable. If I is a chain, then an *upper bound* for I is an element $b \in S$ such that $b \geqslant x$ for all $x \in I$. An element x of S is called *maximal* if there is no $y \in S$ such that $y > x$.

Theorem. (Zorn's Lemma) *Let S be a partially ordered set. Suppose that every chain in S has an upper bound. Then S has a maximal element.*

The maximal element may not be unique. The following argument is essentially the same as in the book.

Proof (Version 1) We argue by contradiction. Assume that every chain has an upper bound and that S has no maximal element.

Let α be any ordinal that is greater than the cardinality of S. We will construct a sequence x_β of elements of S indexed by ordinals $\beta < \alpha$ such that if $\beta < \gamma < \alpha$ then $x_\beta < x_\gamma$.

We do this by transfinite recursion. If x_β are all constructed for $\beta < \gamma$, then we define x_γ as follows. By assumption the chain $\{x_\gamma | \gamma < \beta\}$ has an upper bound, that is, some $z \geqslant x_\beta$ for all $\beta < \gamma$. This step uses the axiom of choice: we are actually taking $z = c(B)$ where c is the choice function and B is the nonempty set of upper bounds of $\{x_\gamma | \gamma < \beta\}$.

Now z is not maximal, so we may choose x_γ such that $x_\gamma > z$. Again, we are using the Axiom of Choice.

Now we have a contradiction since the elements $\{x_\beta | \beta < \alpha\}$ are all distinct. The cardinality of this set is the cardinality of α, which is greater than $|S|$. □

1 A Result about Well-Ordered Sets

The result in this section does not use the Axiom of Choice. We will need well-ordered sets, but we will not need any properties beyond the definition. If S is a linearly ordered set, an *initial segment* of S is a subset $I \subset S$ such that if $x \in I$ and $y \in S$ and $y \leqslant x$ then $y \in I$. (The book also assumes that $I \neq S$ as part of the definition of an initial segment, but this is unimportant.) If $x \in S$ let $S[x] = \{y \in S | y < x\}$. This is an initial segment.

Theorem 1 *Let Ξ be a nonempty partially ordered set. Let Π be the set of chains in Ξ. There cannot exist a mapping $\phi : \Pi \longrightarrow \Xi$ such that when $S \in \Pi$:*

$$\phi(S) > y \text{ for all } y \in S. \tag{1}$$

You will supply the proof in the following Problems 1–5.

Suppose that such a ϕ exists. Let Σ be the set of well-ordered subsets L of Ξ such that:

$$\text{If } x \in L \text{ then } x = \phi(L[x]).$$

Problem 1 Prove that the empty set is in Σ and that if $S \in \Sigma$ then $S \cup \{\phi(S)\} \in \Sigma$. Therefore Σ is nonempty and has no largest element. (Order Σ by inclusion.)

Problem 2 Show that if $T \in \Sigma$ and S is an initial segment of T then either $S = T$ or $\phi(S) \in T$.

Problem 3 Show that if $L, M \in \Sigma$ then either $L \subseteq M$ or $M \subseteq L$. **Hint:** if not, let x be the smallest element of $L - M$ and let y be the smallest element of

$M - L$. Show that $L[x] \subseteq M$ and $M[y] \subseteq L$. Show that $L[x] \cap M[y]$ is an initial segment of both L and M. Let $v = \phi(L[x] \cap M[y])$. Use Problem 2 to prove that $v \in L[x] \cap M[y]$ and get a contradiction to (1).

Problem 4 Show that if $L, M \in \Sigma$ and $L \subseteq M$ then L is an initial segment of M.

Problem 5 Let S be the union of elements of Σ. Show that $S \in \Sigma$. Explain why this contradicts Problem 1 and proves the theorem.

2 Zorn's Lemma

Problem 6 Suppose that Ξ is a set in which every chain has an upper bound, but in which Ξ has no maximal element. Show that a map ϕ satisfying (1) exists and so deduce Zorn's Lemma.

You will need the Axiom of Choice.

3 The Well-Ordering Theorem

Cantor stated that every set can be well-ordered. However the first satisfactory proof was given by Zermelo, who formulated the Axiom of Choice in order to prove it. This is proved in the book, but we can prove it easily using Zorn's Lemma.

Problem 7 Prove using Zorn's Lemma if A is any set then A may be well-ordered. **Hint:** Let Ξ be the set of pairs $(S, \leqslant_S)$ where S is a subset of A and $\leqslant_S$ is a well-ordering of S. Define a partial order $\preccurlyeq$on Ξ by $(S, \leqslant_S) \preccurlyeq (T, \leqslant_T)$ if S is an initial segment of T and whenever $x, y \in S$ we have $x \leqslant_S y$ if and only if $x \leqslant_T y$.