Averages and Ratios of Characteristic Polynomials
References

1. Keating and Snaith, **RMT and $\zeta\left(\frac{1}{2} + it\right)$**, Comm. Math. Phys. 214 (2000), no. 1, 57–89. Value distribution of $\zeta$ mirrors value distribution of characteristic polynomials. Moments computed using Selberg integral.

2. Conrey and Snaith, **Applications of the L-functions ratios conjectures**. Instead of moments, consider ratios. This will have many applications.
   http://arxiv.org/abs/math.NT/0509480

3. Conrey, Farmer and Zirnbauer, **Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups $U(N)$**, Computes mean values of ratios for classical ensembles using supersymmetry.


5. Bump and Gamburd, **Averages and ratios of characteristic polynomials**, Comm. Math. Phys. (to appear) is the **basis of this talk**.

Representation Theory of $S_k$

Let $\lambda = (\lambda_1, \cdots, \lambda_N)$ be a partition of $k$. Thus $\lambda_i \in \mathbb{Z}$, $\lambda_1 \geq \cdots \geq \lambda_N \geq 0$ and $\sum \lambda_i = k$. Let

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_N} \subseteq S_k.$$  

If $\lambda$ is a partition represent it by a Young diagram. Transposing the diagram of $\lambda$ gives the diagram of the conjugate partition $\lambda'$.

Example. $N = 4$, $\lambda = (2, 2, 1, 0) = (2, 2, 1) = (2^2 1)$.

Thus $\lambda$ is a partition of 5 of length 3 or into 3 parts.

**Theorem.** $\text{Ind}_{S_{\lambda}}^{S_k}(1)$ and $\text{Ind}_{S_{\lambda'}}^{S_k}(\text{sgn})$ have a unique irreducible constituent in common.

**Proof.** Mackey theory + combinatorics. □

- Call this $s_\lambda$.
- Thus the irreducible representations of $S_k$ are parametrized by partitions of $k$. 
The ring $\Lambda$

We define a graded $\mathbb{Z}$-algebra $\Lambda$.

- The homogeneous part $\Lambda_k$ of degree $k$ is the additive group of generalized characters of $S_k$.
- The multiplication is induction. Thus
  \[ S_k \times S_l \hookrightarrow S_{k+l} \]
  so if $\chi$, $\psi$ are characters of $S_k$ and $S_l$ then $\chi \otimes \psi$ is a character of $S_k \times S_l$ which we induce to $S_{k+l}$.
- A $\mathbb{Z}$-basis of $\Lambda_k$ is $\{ s_\lambda | \lambda \text{ is a partition of } k \}$.
- Let $h_k$ and $e_k$ be the trivial and sign characters of $S_k$. Then $\Lambda = \mathbb{Z}[h_1, h_2, \cdots] = \mathbb{Z}[e_1, e_2, \cdots]$.
- We have
  \[
  h_k = s_{(k)} \quad \text{}(k) \text{ is the partition } (k, 0, \cdots, 0)
  \]
  \[
  e_k = s_{(1^k)} \quad \text{}(1^k) \text{ is the partition } (1, 1, \cdots, 1)
  \]
- $\Lambda$ has an automorphism $\iota$ of degree 2 such that $\iota(h_i) = e_i$ and $\iota(e_i) = h_i$. This is the involution.
- On $\Lambda_k$ the involution $\iota$ amounts to tensoring with the sign character.
- The involution corresponds to conjugation of partitions: $\iota(s_\lambda) = \iota(s_{\lambda'})$
The characteristic map

Let

\[ s_\lambda(X_1, \ldots, X_N) = \frac{\text{Alt}(X_1^{\lambda_1+N-1}X_2^{\lambda_2+N-2} \cdots X_N^{\lambda_N})}{\text{Alt}(X_1^{N-1}X_2^{N-2} \cdots X_N^{0})} \]

be the Schur polynomial, where

\[ \text{Alt} = \sum_{\sigma \in S_N} (-1)^{\text{sgn}(\sigma)} \sigma. \]

There is a ring homomorphism (due to Frobenius)

\[ \chi: \Lambda \longrightarrow \mathbb{Z}[X_1, \ldots, X_N]^{S_N}, \]

\[ \chi(h_i) = h_i(X_1, \ldots, X_N) = \sum_{i_1 \leq \cdots \leq i_N} X_{i_1} \cdots X_{i_N}, \]
\[ \chi(e_i) = e_i(X_1, \ldots, X_N) = \sum_{i_1 < \cdots < i_N} X_{i_1} \cdots X_{i_N}, \]
\[ \chi(s_\lambda) = s_\lambda(X_1, \ldots, X_N). \]

(Complete, elementary, Schur symmetric polynomials).

- The homomorphism is surjective but not injective. Thus \( \iota \) does not induce an involution of symmetric polynomials.
- However it is bijective on \( \Lambda_k \) provided \( N \geq k \).
Frobenius-Schur duality

Let $V = \mathbb{C}^N$ be the standard module of $U(N)$. We have commuting actions of $U(N)$ and $S_k$ on $\otimes^k V = V \otimes \cdots \otimes V$ ($k$ times):

$$\sigma: v_1 \otimes \cdots \otimes v_k \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \quad \sigma \in S_k.$$  

Thus we can decompose $\otimes^k V$ simultaneously:

$$\otimes^k V = \sum_{\lambda} \pi_{\lambda} \otimes s_{\lambda} \text{ as } U(N) \times S_k\text{-modules.}$$

- The irreducible representations of $S_k$ that appear are the $s_\lambda$ where $\lambda = (\lambda_1, \cdots, \lambda_N)$ is a partition of $k$ of length $\leq N$.

- The irreducible representations of $U(N)$ that appear are those whose matrix coefficients are homogeneous polynomials of degree $k$.

If $g \in U(N)$ has eigenvalues $\alpha_1, \cdots, \alpha_N$ and $f \in \Lambda$ maps to $f = \text{ch}(f) \in \mathbb{Z}[X_1, \cdots, X_N]^{S_N}$ then

$$\chi_\lambda(g) = f(\alpha_1, \cdots, \alpha_N), \quad \chi_\lambda = \text{character of } \pi_\lambda.$$  

**Multiplication in $\Lambda$ corresponds to tensor production of representations of $U(N)$**. Thus

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda \mu} s_\nu, \quad \pi_\lambda \otimes \pi_\mu = \bigoplus_\nu c^\nu_{\lambda \mu} \pi_\nu.$$  

The $c^\nu_{\lambda \mu}$ are **Littlewood-Richardson coefficients**.
Weights

We consider integer sequences \((\lambda_1, \cdots, \lambda_N)\).

- \(\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N\) is called a **weight** of \(N\).
- Weights correspond to rational characters of the diagonal torus of \(U(N)\):
  \[
  \lambda: \begin{pmatrix} t_1 & \cdots & \cdot & t_N \\ \end{pmatrix} \mapsto t_1^{\lambda_1} \cdots t_N^{\lambda_N}.
  \]
- If \(\lambda_1 \geq \cdots \geq \lambda_N\) the weight is called **dominant**.
- The **height** of \(\lambda\) is the inner product \(\langle \lambda, \rho \rangle\),
  \[
  \rho = (N - 1, N - 2, \cdots, 0).
  \]
- If \(\lambda\) is a dominant weight \(\lambda\), there is a unique irreducible \(\pi_{\lambda}\) of \(U(N)\) containing \(\lambda\), no higher weight. **Highest weight module** (Weyl).
- If \(\lambda_N \geq 0\) the dominant weight \(\lambda = (\lambda_1, \cdots, \lambda_N)\) is a **partition** of \(k = \sum \lambda_i\) into \(\leq N\) parts.
- So dominant weights parametrize irreducible representations of \(U(N)\) ...
- ... and **partitions** of \(k\) parametrize irreducible representations of \(S_k\).
- Weights and partitions overlap, hence give a bijection between some representations of \(U(N)\) and some representations of \(S_k\). **This is the bijection we constructed using** \(\otimes^k \mathbb{C}^N\).
Generating functions

Remember that

\[ h_i(X_1, \ldots, X_N) = \sum_{i_1 \leq \cdots \leq i_N} X_{i_1} \cdots X_{i_N}, \]
\[ e_i(X_1, \ldots, X_N) = \sum_{i_1 < \cdots < i_N} X_{i_1} \cdots X_{i_N}, \]

are the complete and elementary symmetric poly’s.

\[ \sum_{t=0}^{\infty} t^k h_k(\alpha_1, \ldots, \alpha_N) = \prod_{j=1}^{N} (1 - t \alpha_j)^{-1}. \]
\[ \sum_{t=0}^{\infty} t^k e_k(\alpha_1, \ldots, \alpha_N) = \prod_{j=1}^{N} (1 + t \alpha_j). \]

These may be regarded as generating functions for the symmetric and exterior algebras on \( \mathbb{C}^N \).

- The involution \( \iota \) of \( \Lambda \) interchanges

\[ \sum_{t=0}^{\infty} t^k h_k \leftrightarrow \sum_{t=0}^{\infty} t^k e_k \]

- There is no corresponding correspondence of symmetric functions (since \( e_k = 0 \) for \( k > N \)) but roughly we can think of the involution as transforming the two types of generating functions into one another.
Correspondences

If $G$ and $H$ are groups, a correspondence of representations is a bijection between some of the irreducible representations $\pi_\lambda$ of $G$ and some of the irreducible representations $\rho_\lambda$ of $H$. We consider the module

$$\omega = \bigoplus \pi_\lambda \otimes \rho_\lambda \text{ of } G \times H.$$  

- By assumption there are no repetitions among the $\pi_\lambda$ or $\rho_\lambda$.
- In important cases, $\omega$ has a natural construction.
- If $G = U(N)$ and $H = S_k$ this module is $\otimes^k V$.
- Howe discovered that the Weil representation gives correspondences for reductive dual pairs of subgroups of $\text{Sp}(2n)$ or its double cover (the metaplectic group).

**GL($N$, $\mathbb{C}$) × GL($m$, $\mathbb{C}$) duality**

Let $\text{GL}(N, \mathbb{C}) \times \text{GL}(m, \mathbb{C})$ act on $\text{Mat}_{n \times m}(\mathbb{C})$ by left and right multiplication. There is induced an action on the polynomial ring $S(\text{Mat}_{n \times m}(\mathbb{C}))$. This is a correspondence. It induces a correspondence of the maximal compact subgroups $U(N) \times U(M)$ since irreducible rep’s of $U(N)$ correspond bijectively to analytic reps of $\text{GL}(N, \mathbb{C})$. This is a Howe correspondence.
Inner product formulas

**Theorem.** (Peter-Weyl + Schur orthogonality)
Assume $G$ compact. The characters of the irreducible representations span the subspace of $L^2(G)$ consisting of class functions. They are an orthonormal basis.

Thus given a correspondence:

$$
\begin{align*}
G & \quad \quad \quad \quad \quad \quad H \\
\pi_\lambda & \quad \leftrightarrow \quad \rho_\lambda \\
\chi_{\pi_\lambda} \in L^2(G) & \quad \leftrightarrow \quad \chi_{\rho_\lambda} \in L^2(H)
\end{align*}
$$

Let $L^2_\omega(G)$ and $L^2_\omega(H)$ be the span of the characters $\chi_{\pi_\lambda}$ and $\chi_{\rho_\lambda}$. The correspondence determines an isometry

$$L^2_\omega(G) \rightarrow L^2_\omega(H).$$

They may allow us to transfer an inner product computation from $G$ to $H$. Thus if $\phi \in L^2_\omega(G)$ is a class function, and $\psi \in L^2_\omega(H)$ is the corresponding class function on $H$ we have

$$\int_G |\phi(g)|^2dg = \int_H |\psi(h)|^2dh.$$

We will give some examples where the right-hand side is easier to evaluate than the left-hand side, leading to nontrivial results in RMT.
Diaconis and Shashahani

Let $G = U(N)$, $H = S_k$, $\omega = \text{Frobenius-Schur duality}$

**Theorem 1.** If $N \geq k_1 + 2k_2 + \cdots + r k_r$ then

$$\int_{U(N)} |\text{tr}(g)|^{2k_1} |\text{tr}(g^2)|^{2k_2} \cdots |\text{tr}(g^r)|^{2k_r} \, dg = \prod_{j=1}^{r} j^{k_j} k_j! \ .$$

**Proof.** Let $k = k_1 + 2k_2 + \cdots + r k_r$, and let $\lambda$ be the partition of $k$ containing $k_1$ entries equal to 1, $k_2$ entries equal to 2, and so forth. Let $C_\lambda$ be the conjugacy class of permutations $\sigma \in S_k$ of type $\lambda$ (so $\sigma$ has $k_j$ cycles of length $j$ in its decomposition to disjoint cycles). Let $p_\lambda$ be the **conjugacy class indicator** on $S_k$,

$$p_\lambda(g) = \begin{cases} z_\lambda & \text{otherwise} \ , \\ 0 \end{cases}$$

where $z_\lambda = \prod_{j=1}^{r} j^{k_j} k_j! \ .$ The class function

$$g \mapsto \text{tr}(g)^{k_1} \text{tr}(g^2)^{k_2} \cdots \text{tr}(g^r)^{k_r}$$

in $L^2(\omega(G))$ corresponds to the function $p_\lambda \in L^2(\omega(H))$ and so its norm is the same as the norm of $p_\lambda$, i.e. $z_\lambda$.  

Thus $$\int_{U(N)} |\text{tr}(g)|^{2k_1} |\text{tr}(g^2)|^{2k_2} \cdots |\text{tr}(g^r)|^{2k_r} \, dg$$

stabilizes when $N$ is large. Asymptotically the value distribution of $\text{tr}(g), \text{tr}(g^2), \cdots, \text{tr}(g^r)$ is of **independent normal** (i.e. Gaussian) random variables.
**Cauchy Identity**

\[
\sum_{\lambda} s_{\lambda}(\alpha_1, \ldots, \alpha_n)s_{\lambda}(\beta_1, \ldots, \beta_m) = \prod_{i,j} (1 - \alpha_i \beta_j)^{-1}
\]

- Sum is over partitions of length \( \leq N \).
- Fundamental in what we do next.
- Underlying correspondence: \( G, H = U(n), U(m) \)

Take \( n = m \). Consider the action of \( U(n) \times U(n) \) on \( L^2(G) \):

\[
(g, h)f(x) = f(g^{-1}xh)
\]

**Lemma.** If \( G \) is any compact group, \( G \times G \) acts on \( L^2(G) \) and

\[
L^2(G) = \bigoplus_{\pi \in \text{Irr}(G)} \pi \otimes \hat{\pi} \quad \quad \hat{\pi} = \text{contragredient rep’n}.
\]

(This is again Peter-Weyl theorem). For \( G = U(n) \), there is an involution \( \iota: G \rightarrow G \) namely \( \iota g = \iota g^{-1} \) such that \( g \mapsto \pi(\iota g) \) is equivalent to \( \hat{\pi} \). This is because \( g^{-1} \) is conjugate to \( \iota g \). So we modify the action:

\[
(g, h)f(x) = f(\iota g x h)
\]

and in this action:

\[
L^2(G) = \bigoplus_{\pi \in \text{Irr}(G)} \pi \otimes \pi
\]
Polynomial version

In this decomposition

\[ L^2(G') = \bigoplus_{\pi \in \text{Irr}(G)} \pi \otimes \pi \quad (\text{Hilbert space } \oplus .) \]

we may restrict ourselves to the subspace of \( U(n) \)-finite vectors, which form the affine ring

\[ A = \mathbb{C}[X_{ij}, \det^{-1}], \quad X_{ij} = \text{coordinate functions.} \]

(\( G \) is compact so continuous functions are \( L^2 \).) Then

\[ A = \bigoplus_{\pi \in \text{Irr}(G)} \pi \otimes \pi \quad (\text{Algebraic } \oplus .) \]

- \( \pi \) runs through \( \pi_{\lambda} \) where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a weight, i.e. \( \lambda_1 \geq \cdots \geq \lambda_n, \lambda_i \in \mathbb{Z} \)
- The weights \( \lambda \) with \( \lambda_N \geq 0 \) are partitions.
- Affine ring \( \mathbb{C}[X_{ij}] \) of Mat\( _n \) is

\[ \mathbb{C}[X_{ij}] = \bigoplus_{\text{partitions } \lambda} \pi_{\lambda} \otimes \pi_{\lambda} \quad (\text{Algebraic } \oplus .) \]

- This is because only the matrix coefficients of \( \pi_{\lambda} \) where \( \lambda \) is a partition are regular on the determinant locus.
Proof of the Cauchy Identity

We have

\[ \mathbb{C}[X_{ij}] = \bigoplus_{\text{partitions } \lambda} \pi_\lambda \otimes \pi_\lambda \quad (\text{Algebraic } \bigoplus) \quad (1) \]

as \( U(n) \times U(n) \) modules, where \((g, h)\) act by

\[ (g, h) f(x) = f(tgxh), \quad f: \text{Mat}_n(\mathbb{C}) \rightarrow \mathbb{C}. \]

Let \( g, h \) have eigenvalues \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \).

Taking the trace in this identity gives

\[ \sum_\lambda s_\lambda(\alpha_1, \ldots, \alpha_n)s_\lambda(\beta_1, \ldots, \beta_n) = \prod_{i, j} (1 - \alpha_i\beta_j)^{-1}. \]

- The series is only convergent if \( |\alpha_i|, |\beta_j| < 1 \). But (1) extends to \( \text{GL}(n, \mathbb{C}) \times \text{GL}(n, \mathbb{C}) \) of which \( U(n) \times U(n) \) is a maximal compact, and \( \pi_\lambda \)
  extend to analytic reps of \( \text{GL}(n, \mathbb{C}) \).

- The case \( n > m \) can be deduced by specializing
  \[ b_{m+1}, \ldots, \beta_n \rightarrow 0. \]

- At the heart of the proof is a correspondence with \( G, H = U(n) \) and

  \[ \omega = \bigoplus_{\text{partitions } \lambda} \pi_\lambda \otimes \pi_\lambda = \text{polynomials on Mat}_n(\mathbb{C}). \]

- This a case of the Howe correspondence.
**Dual Cauchy Identity**

If $\lambda$ is a partition represent it by a **Young diagram**. Transposing the diagram of $\lambda$ gives the diagram of the **conjugate partition** $\lambda'$.

**Example.** $N = 4$, $\lambda = (2, 2, 1, 0) = (2, 2, 1) = (2^21)$.

Thus $\lambda$ is a partition of 5 of length 3 or into 3 parts.

$\lambda'$ is a partition of 5 of length 2 or into 2 parts.

The ring $\Lambda$ has a basis $s_\lambda$ which specialize to characters of irreducible rep's of $S_k$ ($k = \sum \lambda_i$) or $U(N)$ $N \geq \text{length}(\lambda)$. The map $s_\lambda \mapsto s_{\lambda'}$ is an **automorphism** of $\Lambda$ which we will call the **involution**.

Applying the involution to one set of variables in the Cauchy identity

$$\sum_\lambda s_\lambda(\alpha_1, \ldots, \alpha_n)s_\lambda(\beta_1, \ldots, \beta_n) = \prod_{i,j} (1 - \alpha_i \beta_j)^{-1}.$$

produces the **dual Cauchy identity**

$$\sum_\lambda s_\lambda(\alpha_1, \ldots, \alpha_n)s_{\lambda'}(\beta_1, \ldots, \beta_n) = \prod_{i,j} (1 + \alpha_i \beta_j).$$
Keating and Snaith

The following theorem was very influential in the application of RMT to $\zeta$.

**Theorem.** We have

$$\int_{U(N)} |\det(g - I)|^{2k} dg = \prod_{j=0}^{N-1} \frac{j!(j + 2k)!}{(j + k)!^2}.$$

The same constants appear in the (conjectural) $2k$-th moment of $\zeta$.

- The original proof of Keating and Snaith used the Selberg integral.
- We will give another proof (due to Gamburd) that uses $\text{GL}(N) \times \text{GL}(2k)$ duality.
- The two proofs have different generalizations. The proof of Keating and Snaith allows interpolation of $k$ to real numbers, while Gamburd’s proof allows more general evaluations such as

$$\int_{U(N)} |\det(g - I)|^{2k} \chi_\lambda(g) dg$$

where $\chi_\lambda$ is the character of $\pi_\lambda$. 
Proof of Keating-Snaith formula

If $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$ are complex numbers, we will show

$$
\int_{U(N)} \prod_{i=1}^k \left\{ \det(I + \alpha_i g) \det(I + \beta_i^{-1} g^{-1}) \right\} d g = s_{(N^k)}(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k). \tag{2}
$$

The left-hand side equals

$$
\prod \beta_i^{-1} \int_{U(N)} \prod_{i=1}^k \{ \det(I + \alpha_i g) \det(g \beta_i + I) \} \overline{\det(g)^k} d g.
$$

By dual Cauchy id, if $t_1, \ldots, t_N$ are eigenvalues of $g,$

$$
\prod_{i=1}^k \{ \det(\cdots) \det(\cdots) \} = \sum_{\lambda} s_{\lambda}(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k) s_{\lambda'}(t_1, \ldots, t_N),
$$

Since $\det(g)^k = s_{\lambda'}(t_1, \ldots, t_N),$ integrating over $g$ picks off just one term, with $\lambda' = (k^N)$ and so $\lambda = (N^k)$ so. This proves (2). Taking $\alpha_i = \beta_i = 1$

$$
\int_{U(N)} |\det(g - I)|^{2k} d g = s_{(N^k)}(1, \ldots, 1),
$$

the dimension of the rep’n $\pi_{(N^k)}$ of $U(2k).$ This dimension is computed using Weyl’s dimension formula, proving the theorem of Keating and Snaith.
**Analysis of the proof**

- Underlying this computation is the (dual) Cauchy identity.
- The Cauchy identity for $\text{GL}(N) \times \text{GL}(2k)$ amounts to the use of the Howe correspondence for $\text{GL}(N) \times \text{GL}(2k)$.
- In this correspondence, if $\lambda$ is a dominant weight, $\pi_{\lambda}^{\text{GL}(N)}$ corresponds to $\pi_{\lambda}^{\text{GL}(2k)}$.
- But in the Cauchy identity $\pi_{\lambda}^{\text{GL}(N)}$ corresponds to $\pi_{\lambda'}^{\text{GL}(2k)}$.
- When $\lambda = (N^k)$, $\lambda' = (k^N)$ and the answer turns out to be the dimension of this $\pi_{\lambda'}^{\text{GL}(2k)}$.
- **We used the correspondence to transfer the computation from $\text{GL}(N)$ to $\text{GL}(2k)$**.
- Similarly, moments for classical groups can be expressed in terms of characters of other groups parametrized by rectangular partitions. Thus a result of Keating and Snaith can be written:

\[
\int_{\text{Sp}(2N)} \prod_{j=1}^{k} \det(I - x_j g) \, dg = \\
(\mathfrak{x}_1 \ldots \mathfrak{x}_k)^N \chi_{\langle N^k \rangle}^{\text{Sp}(2k)}(\mathfrak{x}_1^{\pm 1}, \ldots, \mathfrak{x}_k^{\pm 1}) = \\
\sum_{\varepsilon \in \{\pm 1\}} \prod_{j=1}^{k} \mathfrak{x}_j^{N(1 - \varepsilon_j)} \prod_{i \leq j} (1 - \mathfrak{x}_i^{\varepsilon_i} \mathfrak{x}_j^{\varepsilon_j})^{-1}.
\]
Ratios

Let $\Xi_{L,K}$ consist of permutations $\sigma \in S_{K+L}$ such that

$$\sigma(1) < \cdots < \sigma(L), \quad \sigma(L+1) < \cdots < \sigma(L+K).$$

**Theorem.** (Conrey, Farmer and Zirnbauer) If $N \geq Q, R$ and $|\gamma_q|, |\delta_r| < 1$ we have

$$
\int_{U(N)} \frac{\prod_{i=1}^{L} \det (I + \alpha_i^{-1} \cdot g^{-1}) \cdot \prod_{k=1}^{K} \det (I + \alpha_{L+k} \cdot g)}{\prod_{q=1}^{Q} \det (I - \gamma_q \cdot g) \cdot \prod_{r=1}^{R} \det (I - \delta_r \cdot g^{-1})} \, dg = \\
\sum_{\sigma \in \Xi_{L,K}} \prod_{k=1}^{K} \left( \alpha_{\sigma(L+k)}^{-1} \alpha_{L+k} \right)^N \times \\
\frac{\prod_{q=1}^{Q} \prod_{i=1}^{L} \left( 1 + \gamma_q \alpha_{\sigma(i)}^{-1} \right) \prod_{r=1}^{R} \prod_{k=1}^{K} \left( 1 + \delta_r \alpha_{\sigma(L+k)} \right)}{\prod_{k=1}^{K} \prod_{l=1}^{L} \left( 1 - \alpha_{\sigma(l)}^{-1} \alpha_{\sigma(K+k)} \right) \prod_{r=1}^{R} \prod_{q=1}^{Q} \left( 1 - \gamma_q \delta_r \right)}.
$$

- According to CFZ the assumption that $N$ is large can be eliminated.
- Proof will depend on the **generalized Cauchy identity** involving **Littlewood-Schur functions**.
- These were studied by Littlewood and also by Berele and Regev, who apparently rediscovered them.
Generalized Cauchy identity

Define the Littlewood-Schur polynomial

\[ \text{LS}_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_l) = \sum_{\mu, \nu} c^\lambda_{\mu\nu} s_\mu(x_1, \ldots, x_k) s_\nu(y_1, \ldots, y_l). \]

The \( c^\lambda_{\mu\nu} \) are the Littlewood-Richardson coefficients.

Theorem. (Berele and Regev)

\[
\sum_\lambda \text{LS}_\lambda(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n) \text{LS}_\lambda(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots, \delta_t) = \prod_{i,k} (1 - \alpha_i \gamma_k)^{-1} \prod_{i,l} (1 + \alpha_i \delta_l) \prod_{j,k} (1 + \beta_j \gamma_k) \prod_{j,l} (1 - \beta_j \delta_l)^{-1}.
\]

We will assume it now and discuss the proof later.

Laplace expansion

Let \((a_{ij})\) be \((L + K) \times (L + K)\). Then \(\det(a_{ij}) = \)

\[
\sum_{\sigma \in \Xi_{L,K}} \text{sgn}(\sigma) \begin{vmatrix} a_{1,\sigma(1)} & \cdots & a_{1,\sigma(L)} \\ \vdots & & \vdots \\ a_{L,\sigma(1)} & \cdots & a_{L,\sigma(L)} \\ a_{L+1,\sigma(L+1)} & \cdots & a_{L+K,\sigma(L+K)} \\ \vdots & & \vdots \\ a_{L+K,\sigma(L+1)} & \cdots & a_{L+K,\sigma(L+K)} \end{vmatrix} \times \text{Proof easy.}
\]

20
Laplace expansion for $\text{LS}_\lambda$

**Proposition.** Suppose $\lambda$ of length $\leq K$ such that $\lambda_L \geq \lambda_{L+1} + Q$, let $\lambda = \tau \cup \rho$ with

$$\tau = (\lambda_1, \ldots, \lambda_L), \quad \rho = (\lambda_{L+1}, \ldots, \lambda_{L+K}).$$

Then

$$\text{LS}_\lambda(\alpha_1, \ldots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) = \sum_{\sigma \in \Xi_{L,K}} \prod_{1 \leq l \leq L} (\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)})^{-1}$$

$$\text{LS}_\tau\langle KL \rangle(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(L)}; \gamma_1, \ldots, \gamma_Q) \text{LS}_\rho(\alpha_{\sigma(L+1)}, \ldots, \alpha_{\sigma(L+K)}; \gamma_1, \ldots, \gamma_Q)$$

**Proof.** Induction on $Q$. If $Q = 0$, this says

$$\sum_{\sigma \in \Xi_{L,K}} \prod_{1 \leq l \leq L} (\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)})^{-1}$$

$$s_\lambda(\alpha_1, \ldots, \alpha_{L+K}) = s_{\tau+\langle KL \rangle}(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(L)})s_{\rho}(\alpha_{\sigma(L+1)}, \ldots, \alpha_{\sigma(L+K)}).$$

This is proved by applying the Laplace expansion to the determinant definition of the Schur function. For $Q > 0$ one adds the $\gamma_i$ one at a time using Pieri’s formula (i.e. the degenerate Littlewood-Richardson rule). \qed
Proof (sketch) of unitary CFZ

By the dual Cauchy identity,

\[
\prod_{l=1}^{L} \det (I + \alpha_l^{-1} \cdot g^{-1}) \cdot \prod_{k=1}^{K} \det (I + \alpha_{L+k} \cdot g) = \det (g)^L \prod_{l=1}^{L} \alpha_l^{-N} \prod_{k=1}^{K+L} \det (I + \alpha_k g)
\]

\[
= \det (g)^L \prod_{l=1}^{L} \alpha_l^{-N} \sum_{\lambda} s_{\lambda}(\alpha_1, \ldots, \alpha_{K+L}) \chi_{\lambda'}(g)
\]

On the other hand by the Cauchy identity

\[
\prod_{q=1}^{Q} \det (I - \gamma_q g)^{-1} = \sum_{\mu} s_{\mu}(\gamma_1, \ldots, \gamma_Q) \chi_{\mu}(g)
\]

and

\[
\prod_{r=1}^{R} \det (I - \delta_r \cdot g^{-1})^{-1} = \sum_{\nu} s_{\nu}(\delta_1, \ldots, \delta_R) \chi_{\nu}(g).
\]

By Schur orthogonality

\[
\int_{U(N)} \prod_{l=1}^{L} \det (I + \alpha_l^{-1} \cdot g^{-1}) \cdot \prod_{k=1}^{K} \det (I + \alpha_{L+k} \cdot g) \det (I - \gamma_q \cdot g) \prod_{r=1}^{R} \det (I - \delta_r \cdot g^{-1})
\]

\[
= \sum_{\lambda, \mu, \nu} \langle \chi_{\lambda'} \chi_{\mu}, \det^L \chi_{\nu} \rangle
\]

\[
\prod_{l=1}^{L} \alpha_l^{-N} s_{\lambda}(\alpha_1, \ldots, \alpha_{L+K}) s_{\mu}(\gamma_1, \ldots, \gamma_Q) s_{\nu}(\delta_1, \ldots, \delta_R).
\]
We rewrite this as
\[
\prod_{l=1}^{L} \alpha_l^{-N} \sum_{\lambda,\mu,\nu} c_{\lambda,\mu}^\nu s_{\lambda}(\alpha_1, \ldots, \alpha_{L+K}) s_{\mu}(\gamma_1, \ldots, \gamma_Q) s_{\nu}(\delta_1, \ldots, \delta_R) =
\prod_{l=1}^{L} \alpha_l^{-N} \sum_{\nu} LS_{\tilde{\nu}}(\gamma_1, \ldots, \gamma_Q; \alpha_1, \ldots, \alpha_{L+K}) s_{\nu}(\delta_1, \ldots, \delta_R) =
\prod_{l=1}^{L} \alpha_l^{-N} \sum_{\nu} LS_{\tilde{\nu}}(\alpha_1, \ldots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) s_{\nu}(\delta_1, \ldots, \delta_R),
\]
where \( \tilde{\nu} = \nu + \langle L^N \rangle \) and \( \hat{\nu} = \tilde{\nu}' = N^L \cup \nu' \). Using the Laplace expansion for \( LS_{\tilde{\nu}} \):
\[
LS_{\tilde{\nu}}(\alpha_1, \ldots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) = \sum_{\sigma \in \Xi_{L,K}} \prod_{1 \leq l \leq L \atop 1 \leq k \leq K} (\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)})^{-1} \times LS_{\langle (N+K)^L \rangle}(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(L)}; \gamma_1, \ldots, \gamma_Q) \times LS_{\nu'}(\alpha_{\sigma(L+1)}, \ldots, \alpha_{\sigma(L+K)}; \gamma_1, \ldots, \gamma_Q).
\]
Substituting this, using generalized Cauchy identity to evaluate the sum over \( \nu \), and Littlewood’s formula
\[
LS_{\langle (l+m)^k \rangle}(x_1, \ldots, x_k; y_1, \ldots, y_l) = \left( \prod_{i=1}^{k} x_i \right)^m \prod_{1 \leq i \leq k} (x_i + y_j)
\]
gives
\[
\sum_{\sigma \in \Xi_{L,K}} \prod_{1 \leq k \leq K} (\alpha_{\sigma(L+k)}^{-1} \alpha_{L+k})^N \times
\frac{\prod_{q=1}^{Q} \prod_{l=1}^{L} (1 + \gamma_q \alpha_{\sigma(l)}^{-1}) \prod_{r=1}^{R} \prod_{k=1}^{K} (1 + \delta_r \alpha_{\sigma(L+k)})}{\prod_{k=1}^{K} \prod_{l=1}^{L} (1 - \alpha_{\sigma(l)}^{-1} \alpha_{\sigma(K+k)}) \prod_{r=1}^{R} \prod_{q=1}^{Q} (1 - \gamma_q \delta_r)}.
\]

Remarks on the proof

- There may be more than one way to proceed once we have an adequate set of tools.
- The tools, mainly the generalized Cauchy identity, Laplace expansion and Littlewood’s identity are themselves of considerable interest.

We will concentrate on ideas around the generalized Cauchy identity and Laplace expansion.

\[ U(p + q) \longrightarrow U(p) \times U(q) \] branching

As before, \( \chi_\lambda = \) character of \( \pi_\lambda \), \( \lambda \) a dominant weight. We assume \( \lambda \) is a partition, so \( \pi_\lambda \) is a polynomial rep’n.

**Theorem.** (i) We have

\[
\chi_\nu^{(p+q)}|_{U(p) \times U(q)} \begin{pmatrix} g_1 & g_2 \end{pmatrix} = \sum_{\lambda, \mu} \sum_{\nu} c_{\lambda \mu}^\nu \chi_\lambda(g_1) \chi_\mu(g_2),
\]

\[
\pi_\nu^{(p+q)} \begin{pmatrix} g_1 & g_2 \end{pmatrix} = \bigoplus_{\lambda, \mu} \sum_{\nu} c_{\lambda \mu}^\nu \pi_\lambda(g_1) \otimes \pi_\mu(g_2).
\]

(ii) Let \( x_1, \ldots, x_p \) and \( y_1, \ldots, y_q \) be two sets of variables.

\[
s_\lambda(x_1, \ldots, x_p, y_1, \ldots, y_q) = \sum_{\mu, \nu} c_{\mu \nu}^\lambda s_\mu(x_1, \ldots, x_p) s_\nu(y_1, \ldots, y_q).
\]

The two statements are equivalent. (Take \( x_i, y_i \) to be eigenvalues of \( g_1 \) and \( g_2 \).)
Proof of unitary branching

In the context of the unitary groups the Littlewood-Richardson rules occur in 2 distinct ways:

- **Clebsch-Gordan coef**: \(c^\lambda_{\mu\nu}\) is the multiplicity of \(\pi_\lambda\) in \(\pi_\mu \otimes \pi_\nu\) reps of \(U(N)\) or \(GL(N, \mathbb{C})\).

- **Unitary branching rule**: \(c^\lambda_{\mu\nu}\) is the multiplicity of \(\pi^\text{GL}(p)\) \(\otimes \pi^\text{GL}(q)\) in the restriction of \(\pi^\text{GL}(p+q)\).

The See-Saw:

\[
\begin{array}{ccc}
U(p+q) & U(n) \times U(n) \\
\uparrow & \downarrow & \uparrow \\
U(p) \times U(q) & U(n) \\
\end{array}
\]

- Vertical lines are inclusions
- Diagonal lines are correspondences
- Let \(\omega = \) action of \(U(p+q), U(n)\) on symmetric algebra of \(\text{Mat}_{(p+q)n}(\mathbb{C})\) (left, right translation)

  \(\omega = \bigoplus_\lambda \pi_\lambda^{U(p+q)} \otimes \pi_\lambda^{U(N)}\).

- Alternatively we have action \(\omega\) of \(U(p) \times U(q)\) and \(U(n) \times U(n)\) on same symmetric algebra.

  \(\omega = \bigoplus_{\mu,\nu} (\pi_\mu^{U(p)} \otimes \pi_\nu^{U(q)}) \otimes (\pi_\mu^{U(n)} \otimes \pi_\nu^{U(n)})\)
Unitary branching, continued

The representation $\omega$ is the action of $U((p+q)n)$ on the symmetric algebra on $\text{Mat}_{(p+q)\times n}(\mathbb{C})$. Both dual pairs can be embedded

$$U(p+q) \times U(n) \rightarrow U((p+q)n)$$

$$(U(p) \times U(q)) \times (U(n) \times U(n))$$

The actions are as follows. Let

$$X = \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right) \in \text{Mat}_{(p+q)\times n}(\mathbb{C}), \quad X_1 \in \text{Mat}_{p\times n}(\mathbb{C}), \quad X_2 \in \text{Mat}_{q\times n}(\mathbb{C}).$$

- Action of $U(p+q)$ is by left multiplication.
- $U(n) \times U(n)$ is by right multiplication on $X_1$ and $X_2$ individually.

$\quad$ $\quad$ $\quad$

- The unitary branching rule now follows ...
See-Saw

Let \( \omega \) be a representation of \( \Omega \). Let \( G_1 \subseteq \Omega \), and \( H_2 \) be its centralizer. Assume

\[
\omega|_{G_1 \times H_2} = \bigoplus_{i \in I} \pi_i^{(1)} \otimes \sigma_i^{(2)}
\]

where \( \pi_i^{(1)} \) and \( \sigma_i^{(2)} \) are irreducible rep’s of \( G_1 \) and \( H_2 \), and \( \pi_i^{(1)} \leftrightarrow \sigma_j^{(2)} \) is the graph of a correspondence.

Let \( H_1 \subseteq G_1 \). The centralizer \( G_2 \) of \( H_1 \) contains \( H_2 \).

Assume \( \omega|_{H_1 \times G_2} \) is also a correspondence.

\[
\omega|_{H_1 \times G_2} = \bigoplus_{j \in J} \sigma_j^{(1)} \otimes \pi_j^{(2)}.
\]

Lemma 2. Assume the branching rules

\[
\pi_i^{(1)} = \sum_{j \in J} c_{ij} \sigma_j^{(1)}, \quad \pi_j^{(2)} = \sum_{i \in I} d_{ji} \sigma_i^{(2)} \quad (3)
\]

Then the \( c_{ij} = d_{ij} \).

Proof. Both \( c_{ij} \) and \( d_{ij} = \) multiplicity of \( \sigma_j^{(1)} \times \sigma_i^{(2)} \) in \( \omega \) as \( H_1 \times H_2 \) modules.
Proof of generalized Cauchy

Recall that the involution $\iota$ “roughly” interchanges the two generating functions:

$$\sum_{t=0}^{\infty} t^k h_k(\alpha_1, \ldots, \alpha_N) = \prod_{j=1}^{N} (1 - t \alpha_j)^{-1}.$$ 

$$\sum_{t=0}^{\infty} t^k e_k(\alpha_1, \ldots, \alpha_N) = \prod_{j=1}^{N} (1 + t \alpha_j).$$

Start with Cauchy identity, apply unitary branching:

$$\prod_{i,k} (1 - \alpha_i \gamma_k)^{-1} \prod_{i,l} (1 - \alpha_i \delta_l)^{-1} \prod_{j,k} (1 - \beta_j \gamma_k)^{-1} \prod_{j,l} (1 - \beta_j \delta_l)^{-1} =$$

$$= \sum_{\lambda} s_\lambda(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) s_\lambda(\gamma_1, \ldots, \gamma_s, \delta_1, \ldots, \delta_t) =$$

$$= \sum_{\lambda} \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\alpha_1, \ldots, \alpha_m) s_\nu(\beta_1, \ldots, \beta_n) \sum_{\sigma,\tau} c^\lambda_{\sigma\tau} s_\sigma(\gamma_1, \ldots, \gamma_s) s_\tau(\delta_1, \ldots, \delta_t).$$

Now apply $\iota$ in variables $\beta$ and $\delta$:

$$\prod_{i,k} (1 - \alpha_i \gamma_k)^{-1} \prod_{i,l} (1 + \alpha_i \delta_l) \prod_{j,k} (1 + \beta_j \gamma_k) \prod_{j,l} (1 - \beta_j \delta_l)^{-1} =$$

$$= \sum_{\lambda} LS_\lambda(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n) LS_\lambda(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots, \delta_t).$$

Thus we obtain the Generalized Cauchy identity.
Hopf algebra structure for \( \Lambda \)

The Generalized Cauchy identity is equivalent to an important fact. The ring \( \Lambda \) is a graded algebra.

\[
\Lambda = \bigoplus_{k} \Lambda_k, \quad \Lambda_k = \text{gen. characters of } S_k.
\]

The multiplication (\textit{induction}) is a bilinear map \( \Lambda \times \Lambda \to \Lambda \) that induces a homomorphism \( \Lambda \otimes \Lambda \to \Lambda \). In degree \( k \) this is a map

\[
\bigoplus_{p+q=k} \Lambda_p \otimes \Lambda_q \mapsto \Lambda_k.
\]

On \( \Lambda_p \times \Lambda_q \) this is \textit{induction} of chars \( S_p \times S_q \to S_{p+q} \). There is a dual operation, namely \textit{restriction} of chars \( S_{p+q} \to S_p \times S_q \). This gives a homomorphism of graded rings \( \Lambda \to \Lambda \otimes \Lambda \) called \textit{comultiplication}.

\textbf{Theorem. (Geissinger)} The two operations of multiplication and comultiplication make \( \Lambda \) a Hopf algebra.

This means that comultiplication is a homomorphism of graded algebras, or (equivalently) that multiplication is a homomorphism of graded coalgebras.

- The Hopf algebra structure was popularized by Zelevinsky.
- We will show that the theorem is \textit{equivalent} to the Generalized Cauchy identity!
The Hopf axiom

Geissinger’s theorem boils down to the commutativity of the following diagram:

\[
\begin{array}{ccc}
\Lambda \otimes \Lambda & \xrightarrow{m^* \otimes m^*} & \Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda \\
\downarrow{m} & & \downarrow{m \otimes m} \\
\Lambda & \xrightarrow{m^*} & \Lambda \otimes \Lambda
\end{array}
\]

\[1 \otimes \tau \otimes 1\]

\[\Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda \]

\[m = \text{multiplication}, \quad m^* = \text{comult.}, \quad \tau(x \otimes y) = y \otimes x.\]

- Start with a character of in \(\Lambda_p \otimes \Lambda_q\) and push it forward to \(\Lambda_r \otimes \Lambda_s\), where \(p + q = r + s = k\).
- Thus we are inducing a character from \(S_p \times S_q\) to \(S_k\), then restricting to \(S_r \times S_s\).

Mackey theory

If \(G \supset H_1, H_2\) (finite groups) there are two ways we can get from characters \(\chi\) of \(H_1\) to characters of \(H_2\). We can

**Induce then restrict** or **restrict then induce**

And these are the same. More exactly

\[
\text{Res}_{H_2} \text{Ind}_{H_1}^G (\chi) = \bigoplus_{\gamma \in H_2 \setminus G/H_1} \text{Ind}_{H_2}^{H_{\gamma}} \text{Res}_{H_{\gamma}} (\gamma \chi)
\]

where \(H_{\gamma} = H_2 \cap \gamma H_1 \gamma^{-1}\) and \(\gamma \chi(h) = \chi(\gamma^{-1}h \gamma)\). For symmetric groups this gives the **Hopf axiom**.
Hopf Axiom = Generalized Cauchy

The Hopf axiom reduces to the formula

$$\sum_{\lambda} c_{\mu\nu}^{\lambda} c_{\sigma\tau}^{\lambda} = \sum_{\varphi, \eta} c_{\varphi\eta}^{\tau} c_{\psi\xi}^{\mu} c_{\varphi\xi}^{\nu} c_{\psi\eta}^{\nu}, \quad (4)$$

since if we apply $m^* \circ m$ to $s_{\mu} \otimes s_{\nu}$, the coefficient of $s_{\sigma} \otimes s_{\tau}$ is the left side, $(m \otimes m) \circ (1 \otimes \tau \otimes 1) \circ (m^* \otimes m^*)$ gives the right side.

To deduce (4) from the generalized Cauchy identity we note that (in obvious notation) the right side of

$$\prod_{i,k} (1 - \alpha_i \gamma_k)^{-1} \prod_{i,l} (1 + \alpha_i \delta_l) \prod_{j,k} (1 + \beta_j \gamma_k) \prod_{j,l} (1 - \beta_j \delta_l)^{-1} =$$

$$\sum_{\lambda} \text{LS}_{\lambda}(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n) \text{LS}_{\lambda}(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots, \delta_t).$$

is

$$\sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\mu}(\alpha)s_{\nu'}(\beta)c_{\sigma\tau}^{\lambda} s_{\sigma}(\gamma)s_{\tau'}(\delta)$$

while the left side is

$$\sum \text{LS}_{\lambda}(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n) \text{LS}_{\lambda}(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots, \delta_t).$$

Comparing, we obtain the result.