Renormalized Periods on $GL(3)$

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Abstract. A theory of renormalization of divergent integrals over torus periods on $GL(3)$ is given, based on a relative truncation. It is shown that the renormalized periods of Eisenstein series have unexpected functional equations.

AMS Classification: 11F12 (Primary), 11F55 (Secondary).

Let F be a global field with adele ring $\mathbb{A}$. Let G be a reductive algebraic group defined over F, and let (π, V) be an automorphic representation of $G_{\mathbb{A}}$. Let H be a subgroup of G and χ a character of $H_{\mathbb{A}}$ which is trivial on H_F. We may consider the period

$$\int_{H_F \backslash H_{\mathbb{A}}} \phi(g)\, \chi(g)\, dg, \qquad \phi \in V. \tag{1}$$

We wish to consider cases where the integral (1) may be divergent, in which case an issue of renormalization arises. If the integral is divergent but has a natural renormalization, we will denote the renormalized period by

$$\mathbf{RN} \int_{H_F \backslash H_{\mathbb{A}}} \phi(g)\, \chi(g)\, dg, \qquad \phi \in V.$$

For example consider the case where G is $PGL(2)$, H is the diagonal torus, and

$$\chi \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} = \left| \frac{y_1}{y_2} \right|^s.$$

If π is cuspidal then the integral (1) is convergent. In this case (1) is an Euler product, which agrees at all unramified places with $L(s+\frac{1}{2}, \pi)$. If on the other hand ϕ is an Eisenstein series with parameter w (i.e. $E_{w,-w}$ in the notation of Section 1), the integral (1) can still be interpreted by means of a renormalization process, which we review in Section 1. Again the renormalized integral gives $L(s + \frac{1}{2}, \pi)$, which in this case equals $\zeta(s+w)\,\zeta(1+s-w)$, where ζ is the Dedekind zeta function of F.

This construction shows the appearance of unexpected symmetries or hidden functional equations. The "expected" functional equations are of course those associated with the functional equation of the Eisenstein series, namely $w \to 1-w$, and those of the Mellin transform of an automorphic form, that is $s \to -s$. Beyond these, there is an *unexpected* functional equation because $\zeta(s+w)\,\zeta(1+s-w)$ is symmetric under $s \to w - \frac{1}{2}$ and $w \to \frac{1}{2} + s$.

It is well known that there is a strong tendency for the period (1) to be a value of an L-function when (G, H, χ) are Gelfand data, by which we mean that the representation of G induced from χ is multiplicity free. Because of their tendency to give Eulerian integrals, most of the period integrals which have been applied in number theory have been over Gelfand data. But we will argue that non-Gelfand periods are still interesting, because they can have unexpected functional equations. In the $PGL(2)$ example, (G, H, χ) are Gelfand data. But cases where (G, H, χ) are not Gelfand data can also show hidden functional equations.

For example, in Bump and Beineke [BB] it is shown that a renormalized integral of four $SL(2, \mathbb{Z})$ Eisenstein series has unexpected functional equations. This can be thought of as an instance of (1), in which $H = GL(2)$ is embedded diagonally in $G = GL(2) \times GL(2) \times GL(2) \times GL(2)$, and χ is trivial. This H is not a Gelfand subgroup, and this integral is not Eulerian. It is a function of four complex variables having $384 = 16 \cdot 24$ "expected" functional equations corresponding to the 16 functional equations of the Eisenstein series and the 24 permutations of them. In addition to these it has "hidden" functional equations for it was proved in [BB] that the actual group of functional equations has order 1152.

The hidden functional equations can be proved by relating this renormalized integral to a torus integral of a $PGL(3)$ Eisenstein series, which is itself a further example of a renormalized period (1). Specifically, we may take $G = PGL(3)$ and $H = A/Z$ where A is the diagonal torus of $GL(3)$ and Z the center. Then if ϕ is an Eisenstein series, we again obtain a function of four complex variables, since there are two complex variables parametrizing the $PGL(3)$ Eisenstein series and two parametrizing the character χ of H. This renormalized integral has as evident symmetries the six functional equations of the Eisenstein series, together with the action of the normalizer of H on χ by conjugation. Thus its group of overt symmetries has order 36.

Once one knows that these two functions of four complex variables are equal, one obtains the full group of functional equations, for the two subgroups of orders 384 and 36 together generate a group of order 1152, which is the group of symmetries of the polar divisor. The coincidence of these two functions of four complex variables is predicted by the "see-saw" formalism, indeed by a variant of the last example in Kudla [K], which was offered to explain the previous example of Bump and Goldfeld [BG]. The relevant see-saw is shown in Figure 1, where the vertical lines are inclusions, and the diagonal lines are theta liftings. The ambient group is $GSp(12)$.

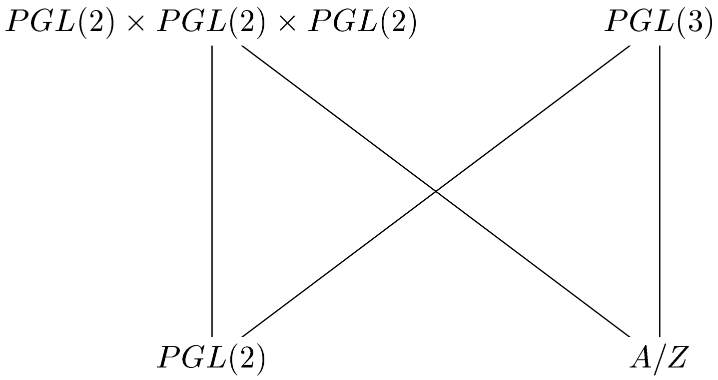

Figure 1. The See-Saw.

Although this strategy of proof underlies [BB] this is in fact *not* what was done there. When [BB] was written there was not available any proper theory of renormalization in this context. Thus [BB] were forced to replaced the integral (1) by a non-invariant one.

A proper theory of renormalization should attach an invariant meaning to (1). By this we mean one in which it is manifest from the definitions that the period inherits functional equations from the conjugations of χ by elements of the Weyl group, as well as the functional equations corresponding to those of the Eisenstein series. (We caution the reader that the term *invariant* could also be used to mean an integral that is unchanged by right translation by an arbitrary element of the group. The renormalized integral is not an invariant functional in the latter sense.) Only after such an invariant definition is given can one ask whether the renormalized integral has further "hidden" functional equations.

In this paper we give such an invariant definition of torus periods of $GL(n)$ Eisenstein series when $n = 3$. The strategy is to define a truncation $\Lambda^T \phi$ such that

$$\int_{A(F)\backslash A(\mathbb{A})} \Lambda^T \phi(g)\, \chi(g)\, dg,$$

is convergent, then to add other terms which make the result independent of T. The truncation Λ^T is not the well-known truncation of Arthur [A] but it is closely related. It is rather a relative truncation similar to the "mixed truncation" of Jacquet, Lapid and Rogawski [JLR]. (Our G and H are different from theirs, but the idea is the same.)

The key feature of the relative truncation is that (quoting Jacquet, Lapid and Rogawski) one *takes constant terms over G yet does the truncation over H*. Strictly speaking, to make sense of this description, in our example H is not A itself, but its normalizer. The truncation involves a summation over the Weyl group, with the subtraction and addition of constant terms along the various parabolic subgroups.

We expect but have not proved that the definition that we give applies to other automorphic forms on $GL(3)$. (The principal fact to be generalized is Proposition 9.) More importantly we expect that it will be clear how to generalize this definition to $GL(n)$. Naturally the combinatorics will be more complicated on $GL(n)$.

The invariant renormalization of the period integral (1) will allow us to determine the polar divisor of this period. It is a 24-cell, a regular polytope in 4 dimensions, whose symmetries include the 36 manifest functional equations, but others as well.

We may now state a theorem about this period. Let μ_1, μ_2 and μ_3 be complex numbers satisfying $\mu_1 + \mu_2 + \mu_3 = 0$, and let s_1, s_2, s_3 satisfy $s_1 + s_2 + s_3 = 0$. We define an Eisenstein series G_{μ_1,μ_2,μ_3} on $GL(3, \mathbb{A})$ in (21) below, and a character of the diagonal torus $A(\mathbb{A})$ by

$$\chi_{s_1,s_2,s_3}(y) = y_1^{s_1}\, y_2^{s_2}\, y_3^{s_3}. \tag{2}$$

We will define a renormalized integral

$$\mathbf{RN} \int_{Z(\mathbb{A})\ A(\mathbb{Q})\backslash A(\mathbb{A})} G_{\mu_1,\mu_2,\mu_3}(y)\, \chi_{s_1,s_2,s_3}(y)\, d^\times y \tag{3}$$

below in (25) by the method we have already described. Since there are relations between the parameters, it is really a function of four complex variables. It has 36 evident functional equations, corresponding to the 6 functional equations of the Eisenstein series, and the 6 evident symmetries coming from conjugations of A by its normalizer. Surprisingly, it has other functional equations.

Theorem 1. *The integral (3) is invariant under*

$$\begin{aligned} \mu_1 &\to \tfrac{1}{3}(2\mu_1 - \mu_3 + s_1), & s_1 &\to \tfrac{1}{3}(-4\,\mu_2 - s_1),\\ \mu_2 &\to \tfrac{1}{3}(\mu_2 - 2\,s_1), & s_2 &\to \tfrac{1}{3}(2\mu_2 + 2s_1 + 3\,s_2),\\ \mu_3 &\to \tfrac{1}{3}\left(-\mu_1 + 2\,\mu_3 + s_1\right), & s_3 &\to \tfrac{1}{3}(2\,\mu_2 - s_1 - 3s_2). \end{aligned}$$

Its full group of symmetries has order 1152.

What is most important in this paper is the "correct" definition of the renormalized $GL(3)$ period. Our principal application is the occurrence of unexpected functional equations. Since the renormalized integral is over a non-Gelfand subgroup, it is not Eulerian and so we are outside the domain of number theory as it is usually understood. It is our view that the hidden functional equations are evidence that this new territory may contain interesting surprises. The work of Woodson [W] gives some indication of what we could expect on $GL(n)$.

This work was supported in part by an AWM-NSF Mentoring Travel Grant and by NSF grants DMS-9970841 and DMS-0203353.

1. Renormalization on $GL(1)$ and Mellin transforms on $GL(2)$. Let Ω be a locally compact abelian group, written multiplicatively. We assume that there is a surjective homomorphism $\omega : \Omega \to \mathbb{R}$ whose kernel is compact. For example if $\Omega = \mathbb{A}^\times/\mathbb{Q}^\times$ we can take $\omega(a) = \log|a|$.

By a *finite function* on Ω we mean a function ρ whose translates f_a defined by $\rho_a(x) = \rho(ax)$ span a finite dimensional vector space. Let $\mathcal{F}(\Omega)$ be the space of finite functions. We would like to be able to integrate functions on Ω which are asymptotic to (possibly different) finite functions of $x \in \Omega$ as $\omega(x)$ tends to $-\infty$ and ∞.

The integration theory we are seeking should define an additive functional $\lambda(\rho)$ on a space $\mathcal{C}$ of functions on Ω. The space $\mathcal{C}$ should contain the Haar integrable functions and the restriction of λ to these should be Haar integrable. The space $\mathcal{C}$ and the functional λ should be invariant under translation. Moreover if $\rho \in \mathcal{C}$ then the truncation $\rho(y)\,\hat\tau\big(\omega(y)\big)$ should also be in $\mathcal{C}$, where $\hat\tau : \mathbb{R} \to \{0,1\}$ is the characteristic function of the positive real numbers.

The space $\mathcal{C}$ cannot contain the constant function. We may see this as follows. If $1 \in \mathcal{C}$ then $\rho(y) = \hat\tau\big(\omega(y)\big)$ and any translate ρ_a would be in $\mathcal{C}$. The difference $\rho - \rho_a$, where a is chosen so that $\omega(a) = 1$ is $\{x \in \Omega | 0 \le \omega(x) \le 1\}$. If $\lambda(\rho) = \lambda(\rho_a)$ then the Haar volume of this set would vanish, which it does not.

The finite functions are finite linear combinations of functions of the form

$$\rho(a) = \chi(a)\,\omega(a)^r \tag{4}$$

where $0 < r \in \mathbb{Z}$, $u \in \mathbb{R}$ and χ is a quasicharacter of Ω. Note that while $\chi(ab) = \chi(a)\,\chi(b)$ we have $\omega(ab) = \omega(a)+\omega(b)$. Let $\mathcal{F}_0(\Omega)$ be the subspace of finite functions whose translates do not contain 1. These are the functions (4) where the quasicharacter χ is nontrivial.

Proposition 2. *Let $\rho \in \mathcal{F}_0(\Omega)$.*
(i) There exists a unique $R \in \mathcal{F}_0(\mathbb{R})$ such that

$$\int_{T<\omega(a)<U} \rho(a)\, da = R(U) - R(T). \tag{5}$$

(ii) If $\int_T^\infty |\rho(a)|\, da < \infty$ then $R(T) = -\int_T^\infty \rho(a)\, da$.
(iii) If $\int_{-\infty}^T |\rho(a)|\, da < \infty$ then $R(T) = \int_{-\infty}^T \rho(a)\, da$.

Proof. We need only define R for $\rho \in \mathcal{F}_0$ as in (4). If χ is nontrivial on $\ker(\omega)$ the integral (5) vanishes since ω is constant on the cosets of $\ker(\omega)$ so we must take $R = 0$.

Assume that χ is trivial on $\ker(\omega)$. Then $\chi(a) = e^{u\,\omega(a)}$ for some complex number $u \neq 0$. The function $\rho(a) = \rho_0(\omega(a))$ where

$$\rho_0(x) = e^{ux}\, x^r, \qquad \rho_0 \in \mathcal{F}_0(\mathbb{R}).$$

Normalizing the Haar integrals appropriately, the left side of (5) equals

$$\int_T^U \rho_0(x)\, dx.$$

Thus we are reduced to the special case where $\Omega = \mathbb{R}$. It is easy to see that the derivative $D : \mathcal{F}_0(\mathbb{R}) \to \mathcal{F}_0(\mathbb{R})$ is bijective, so by the Fundamental Theorem of Calculus we must choose R to be the unique antiderivative of ρ_0 in $\mathcal{F}_0(\mathbb{R})$.

Parts (ii) and (iii) follow from the uniqueness in (i) since it is easily checked that the integrals lie in $\mathcal{F}_0$. ∎

Let $\mathcal{C}$ denote the space of functions f such that there exist functions $\rho_{-\infty}$ and ρ_∞ in $\mathcal{F}_0(\Omega)$ such that $\hat{\tau}\big(\omega(y)\big)\,(\rho - \rho_\infty)(y)$ and such that $\hat{\tau}\big(-\omega(y)\big)\,(\rho - \rho_{-\infty})(y)$ are integrable. Let $R_{-\infty}$ and R_∞ be the functions corresponding to $\rho_{-\infty}$ and ρ_∞ by Proposition 2. Given T and T' such that $T' < T$, define

$$\rho_{T,T'}(y) = \begin{cases} \rho(y) - \rho_\infty(y) & \text{if } \omega(y) > T, \\ \rho(y) & \text{if } T > \omega(y) > T', \\ \rho(y) - \rho_{-\infty}(y) & \text{if } T' > \omega(y). \end{cases}$$

This function is integrable. Define

$$\mathbf{RN} \int_\Omega \rho(y)\, dy = \int_\Omega \rho_{T,T'}(y)\, dy + R_{-\infty}(T') - R_\infty(T). \tag{6}$$

Although ρ_∞ and $\rho_{-\infty}$ are not unique, Proposition 2 (ii) and (iii) imply that (6) does not depend on their choice. Note that by Proposition 2 (i) this is independent of the choice of T, T', and it follows that the renormalized integral is invariant under translation.

In this section let $K = \prod_v K_v$ be a standard maximal compact subgroup of $GL(2,\mathbb{A})$, where

$$K_v = \begin{cases} O(2) & \text{if } v = \infty, \\ GL(2,\mathbb{Z}_p) & \text{if } v = p \text{ is finite.} \end{cases}$$

Let λ be a complex number and let $\mathcal{A}(GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})/K,\lambda)$ be the space of automorphic forms with central character $|\cdot|^\lambda$. Thus an element ϕ of $\mathcal{A}(GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})/K,\lambda)$ is a smooth function of moderate growth which is finite with respect to the Laplace-Beltrami operator in $GL(2,\mathbb{R})$, and which satisfies

$$\phi\left(\begin{pmatrix} z & \\ & z \end{pmatrix} g\right) = |z|^\lambda\,\phi(g).$$

We assume that the constant term

$$\phi_0(g) = \int_{\mathbb{A}/\mathbb{Q}} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = \sum_i c_i f_i(g),$$

where

$$f_i\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} k\right) = |y_1|^{\gamma_i}|y_2|^{\lambda-\gamma_i}$$

when $k \in K$, where c_i and γ_i are suitable constants. Let T and T' be real numbers such that $-T' < T$. We normalize the measure on $\mathbb{A}^\times/\mathbb{Q}^\times$ so that

$$(7) \qquad \int\limits_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ |y|\geq 1}} |y|^s\,d^\times y = -\frac{1}{s} \quad (s<0), \qquad \int\limits_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ |y|\leq 1}} |y|^s\,d^\times y = \frac{1}{s}, \quad (s>0).$$

Define:

$$(8) \quad I(\phi,s,T,T') = \int\limits_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ \log|y|<-T'}} \left(\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} - \phi_0\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix}|y|^\lambda\right)|y|^s\,d^\times y$$

$$+ \int\limits_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ -T'<\log|y|<T}} \phi\begin{pmatrix} y & \\ & 1 \end{pmatrix}|y|^s\,d^\times y + \int\limits_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ T<\log|y|}} \left(\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} - \phi_0\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^s\,d^\times y.$$

The integral is convergent for all s.

Proposition 3. *Let*

$$(9) \qquad I_{\mathrm{r}}(\phi,s) = I(\phi,s,T,T') - \sum c_i\left[\frac{e^{T(\gamma_i+s)}}{\gamma_i+s} - \frac{e^{T'(\gamma_i-\lambda-s)}}{-\gamma_i+\lambda+s}\right].$$

Then $I_{\mathrm{r}}(\phi,s)$ is independent of T and T'.

Proof. Let U, U' be given. Assume that $U > T$ and $U' > T'$. We have

$$I(\phi,s,U,U') - I(\phi,s,T,T') = \sum_i c_i \int\limits_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ T<\log|y|<U}} |y|^{s+\gamma_i}\,d^\times y + \int\limits_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ -U'<\log|y|<-T'}} |y|^{s+\lambda-\gamma_i}\,d^\times y.$$

This equals

$$\sum c_i \left[\frac{e^{U(\gamma_i+s)}}{\gamma_i+s} - \frac{e^{T(\gamma_i+s)}}{\gamma_i+s} - \frac{e^{U'(\gamma_i-\lambda-s)}}{-\gamma_i+\lambda+s} + \frac{e^{T'(\gamma_i-\lambda-s)}}{-\gamma_i+\lambda+s} \right].$$

This implies the independence of $I_{\mathrm{r}}(\phi,s)$ from T and T'. ■

Proposition 3 makes explicit a special case of Proposition 2. So by (6), the renormalized integral

$$\mathbf{RN} \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^s\,d^\times y = I_{\mathrm{r}}(\phi,s).$$

We will denote $\zeta^*(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s)$. Let B_2 be the Borel subgroup of upper triangular matrices. Define a function f_{θ_1,θ_2} on $GL(2,\mathbb{Q})$ by

$$f_{\theta_1,\theta_2}\left(\begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} k\right) = |y_1|^{\theta_1}\,|y_2|^{\theta_2}, \qquad k \in K. \tag{10}$$

Define the normalized spherical Eisenstein series by $E^*_{\theta_1,\theta_2}(g) = \zeta^*(\theta_1-\theta_2)E_{\theta_1,\theta_2}(g)$, where

$$E_{\theta_1,\theta_2}(g) = \sum_{\gamma\in B_2(\mathbb{Q})\backslash GL(2,\mathbb{Q})} f_{\theta_1,\theta_2}(\gamma g).$$

The sum is absolutely convergent if $\mathrm{re}(\theta_1-\theta_2) \geq 1$, and $E^*_{\theta_1,\theta_2}(g)$ has analytic continuation to all $\frac{1}{2}(\theta_1-\theta_2) \neq 0$, 1 with a functional equation

$$E^*_{\theta_1,\theta_2}(g) = E^*_{\theta_2+1,\theta_1-1}(g). \tag{11}$$

We will use (11) frequently and without comment.

Proposition 4. *Let θ_1 and $\theta_2 \in \mathbb{C}$. Then*

$$I_{\mathrm{r}}(E^*_{\theta_1,\theta_2}, s) = \zeta^*(s+\theta_1)\,\zeta^*(s+1+\theta_2).$$

We will give two proofs of this. The zeta functions in the two proofs come out differently, so comparing the two proofs gives a proof of the functional equation of the zeta function.

First Proof. We restrict θ_1, θ_2 to a compact set and choose s so that its real part is large. Then the exponential terms in (9) decay as $T \to -\infty$ and $T' \to \infty$. Taking these limits we see that the integral equals

$$\int_{\mathbb{A}^\times/\mathbb{Q}^\times} \left(\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} - \phi_0\begin{pmatrix} y & \\ & 1 \end{pmatrix} \right) |y|^s \, d^\times y$$

and we recognize it as the Mellin transform of the Eisenstein series minus its constant term. Of course, this is just the L-function of the Eisenstein series evaluated at $s + \frac{1}{2}$. The general case follows by analytic continuation. ∎

Second Proof. This time we take s restricted to a compact set and θ_1, $-\theta_2$ such that $\mathrm{re}(\theta_1 + s) > 1$ and $\mathrm{re}(-\theta_2 - s) > 1$. We will show

$$I_{\mathrm{r}}(E^*_{\theta_1,\theta_2}, s) = \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \left(E^*_{\theta_1,\theta_2}\begin{pmatrix} y & \\ & 1 \end{pmatrix} - \zeta^*(\theta_1 - \theta_2)(|y|^{\theta_1} + |y|^{\theta_2}) \right) |y|^s \, d^\times y. \tag{12}$$

The subtracted terms are *not* the constant term of the Eisenstein series. To prove (12) take $T = T' = 0$ in the definition to see

$$\begin{aligned} I_{\mathrm{r}}(E^*_{\theta_1,\theta_2}, s) = \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \Bigg(& E^*_{\theta_1,\theta_2}\begin{pmatrix} y & \\ & 1 \end{pmatrix} - \hat{\tau}(\log y) \Big[\zeta^*(\theta_1 - \theta_2)|y|^{\theta_1} + \zeta^*(\theta_1 - \theta_2 - 1)|y|^{1+\theta_2} \Big] \\ & - \hat{\tau}(-\log y) \Big[\zeta^*(\theta_1 - \theta_2)|y|^{\theta_2} + \zeta^*(\theta_1 - \theta_2 - 1)|y|^{-1+\theta_1} \Big] \Bigg) |y|^s \, d^\times y \\ & - \frac{\zeta^*(\theta_1 - \theta_2)}{s + \theta_1} + \frac{\zeta^*(\theta_1 - \theta_2)}{s + \theta_2} - \frac{\zeta^*(\theta_1 - \theta_2 - 1)}{s + \theta_2 + 1} + \frac{\zeta^*(\theta_1 - \theta_2 - 1)}{s + \theta_1 - 1}. \end{aligned}$$

Now two applications each of the identities (7) give (12).

We recall a definition of the Eisenstein series from Godement and Jacquet [GJ]. Let Ψ be the Gaussian element of the Schwartz space $S(\mathbb{A}^2)$, $\Psi(x, y) = \psi(x)\psi(y)$ where $\psi(x) = \prod_v \psi_v(x_v)$, the product being over the places of $\mathbb{Q}$, where ψ_v is the characteristic function of $\mathbb{Z}_v$ when v is finite, while $\psi_\infty(x) = e^{-\pi x^2}$. Then $\mathrm{re}(\theta_1 - \theta_2) > 1$ so

$$\zeta^*(\theta_1 - \theta_2)\, f_{\theta_1,\theta_2}(g) = |\det(g)|^{\theta_1} \int_{\mathbb{A}^\times} \Psi((0, t)g)\, |t|^{\theta_1 - \theta_2} \, d^\times t. \tag{13}$$

Substituting this into the definition of the Eisenstein series and parametrizing a coset $\gamma \in B_{\mathbb{Q}} \backslash GL(2, \mathbb{Q})$ by its bottom row $(c, d) \in \mathbb{Q}^\times \backslash (\mathbb{Q}^2 - 0)$ we have

$$E^*\begin{pmatrix} y & \\ & 1 \end{pmatrix} = \sum_{\mathbb{Q}^\times \backslash (\mathbb{Q}^2 - 0)} \int_{\mathbb{A}^\times} |y|^{\theta_1} \Psi(tyc, td) |t|^{\theta_1 - \theta_2} \, d^\times t.$$

The contributions of the terms when $c = 0$ and $d = 0$ are easily computed. They are, respectively $\zeta^*(\theta_1 - \theta_2)|y|^{\theta_1}$ and $\zeta^*(\theta_1 - \theta_2)|y|^{\theta_2}$. Subtracting them,

$$\begin{aligned} E^*_{\theta_1,\theta_2}\begin{pmatrix} y & \\ & 1 \end{pmatrix} - \zeta^*(\theta_1 - \theta_2)(|y|^{\theta_1} + |y|^{\theta_2}) &= \sum_{\mathbb{Q}^\times \backslash (\mathbb{Q}^\times)^2} \int_{\mathbb{A}^\times} |y|^{\theta_1} \Psi(tyc, td) |t|^{\theta_1 - \theta_2} \, d^\times t \\ &= \int_{\mathbb{A}^\times} \sum_{c \in \mathbb{Q}^\times} |y|^{\theta_1} \Psi(tyc, t) |t|^{\theta_1 - \theta_2} \, d^\times t. \end{aligned} \tag{14}$$

Therefore (12) equals

$$\int_{\mathbb{A}^\times} \int_{\mathbb{A}^\times} |y|^{\theta_1} \Psi(ty, t) |t|^{\theta_1 - \theta_2} |y|^s \, d^\times t \, d^\times y.$$

After substituting $y \to y/t$ the variables now separate into a product of two unramified Tate integrals giving $\zeta^*(s + \theta_1)\, \zeta^*(-s - \theta_2)$. ■

2. Renormalization of $GL(3)$ torus integrals. Now we will consider the same problem for $GL(3)$. We take $F = \mathbb{Q}$ and as before $\mathbb{A}$ is its adele ring. Let A be the maximal torus of diagonal matrices in $GL(3)$, and let $\log : A(\mathbb{A}) \to \mathfrak{a} = \mathbb{R}^3$, be the map sending

$$\log(y) = (\log|y_1|, \log|y_2|, \log|y_3|), \qquad y = \begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix}$$

while

$$\log_0 : A(\mathbb{A}) \to \mathfrak{a}_0 = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 \,|\, \sum a_i = 0 \right\}$$

is the composition of log with the Euclidean orthogonal projection onto $\mathfrak{a}_0$.

The *positive Weyl chamber* in $\mathfrak{a}$ is the region

$$\mathfrak{a}^+ = \{(a_1, a_2, a_3) | a_1 \geq a_2 \geq a_3\}.$$

It is a fundamental domain for the Weyl group $W \cong S_3$. Also let $\mathfrak{a}_0^+ = \mathfrak{a}_0 \cap \mathfrak{a}^+$. We will denote the fundamental dominant weights $\omega_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $\omega_2 = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$, and the simple roots $\alpha_1 = 2\omega_1 - \omega_2 = (1, -1, 0)$, $\alpha_2 = 2\omega_2 - \omega_1 = (0, 1, -1)$.

Let P and Q be the standard parabolics with Levi factors $M_P = GL(2) \times GL(1)$ and $M_Q = GL(1) \times GL(2)$, respectively, and let $B = P \cap Q$ be the standard Borel. Its Levi factor $M_B = A$. We will denote by U_P, U_Q and U_B the unipotent radicals of P, Q and B respectively.

Define the functions $\hat{\tau}_{\mathcal{P}} : \mathfrak{a}_0 \to \mathbb{R}$ to be the characteristic functions of the regions:

$$\begin{cases} a_1 + a_2 \geq 0 & \text{if } \mathcal{P} = P, \\ a_1 \geq 0 & \text{if } \mathcal{P} = Q, \\ a_1, a_1 + a_2 \geq 0 & \text{if } \mathcal{P} = B. \end{cases}$$

We extend these functions to $\mathfrak{a}$ by composing them with the orthogonal projection $\mathfrak{a} \to \mathfrak{a}_0$.

Let ϕ be a spherical automorphic form on $PGL(3, \mathbb{A})$. Thus ϕ is a function on

$$Z_{\mathbb{A}} GL(3, F) \backslash GL(3, \mathbb{A}) / K,$$

where now Z is the center of $GL(3)$ and $K = O(3) \prod_p GL(3, \mathbb{Z}_p)$. Define

$$\phi_{\mathcal{P}}(g) = \int_{U_{\mathbb{A}}/U_F} \phi(ug) \, du$$

when $\mathcal{P}$ is a parabolic and U is its unipotent radical. Let $T \in \mathfrak{a}_0^+$. Define, for $y \in A(\mathbb{A})$

(15) $$\Lambda^T\phi(y) = \phi(y) - \sum_{w \in W_P \backslash W} \hat{\tau}_P(\log(wy) - T)\, \phi_P(wy) - \sum_{w \in W_Q \backslash W} \hat{\tau}_Q(\log(wy) - T)\, \phi_Q(wy) + \sum_{w \in W} \hat{\tau}_B(\log(wy) - T)\, \phi_B(wy).$$

Here $W_P = \langle \sigma_1 \rangle \cong S_2$ is the Weyl group of P and $W_Q = \langle \sigma_2 \rangle \cong S_2$ is the Weyl group of Q. Here σ_1 and σ_2 are the two simple reflections. This is well defined because $\hat{\tau}_P(\log(y)-T) = \hat{\tau}_P(\log(\sigma_1 y) - T)$ and $\hat{\tau}_Q(\log(y) - T) = \hat{\tau}_Q(\log(\sigma_2 y) - T)$.

Proposition 5. *$\Lambda^T\phi(y)$ is of rapid decay.*

Proof. What we will actually prove is that the integral of this function against any polynomial in $|y_1|$, $|y_2|$, $|y_3|$ is absolutely convergent on the torus.

To prove this we may assume $\log(y) \in \mathfrak{a}_0^+$, since $\Lambda^T\phi$ is invariant under W. For such y, we have

(16)
$$\Lambda^T\phi(y) = \phi(y) - \hat{\tau}_P(\log\, y - T)\phi_P(y) - \hat{\tau}_Q(\log y - T)\phi_Q(y) + \hat{\tau}_B(\log\, y - T)\phi_B(y) - \hat{\tau}_Q(\log(\sigma_1 y) - T)(\phi_Q(\sigma_1 y) - \phi_B(\sigma_1 y)) - \hat{\tau}_P(\log(\sigma_2 y) - T)(\phi_P(\sigma_2 y) - \phi_B(\sigma_2 y)).$$

Indeed, we have $\hat{\tau}_B(\log(\sigma_1 y) - T) = \hat{\tau}_Q(\log(\sigma_1 y) - T)$ and $\hat{\tau}_B(\log(\sigma_2 y) - T) = \hat{\tau}_P(\log(\sigma_2 y) - T)$ for $y \in \mathfrak{a}_0^+$, so terms in (16) all appear, while the supports of the remaining characteristic functions in (15) all vanish for such y. The Figure 2 shows the regions in (16). Particularly, the support of $\hat{\tau}_P(\log\, y - T)$ restricted to $\mathfrak{a}_0^+$ is the union of (i), (ii), (iii) and (v); the support of $\hat{\tau}_Q(\log\, y - T)$ is the union of (i), (ii), (iv) and (v); the support of $\hat{\tau}_B(\log\, y - T)$ is the union of (i), (ii) and (v), the support of $\hat{\tau}_Q(\log(\sigma_1 y) - T)$ is (i) and the support of $\hat{\tau}_P(\log(\sigma_2 y) - T)$ is (v).

The sum of the first four terms is of rapid decay by Moeglin and Waldspurger [MW] Corollary I.2.12. We will show that

$$\hat{\tau}_Q(\log(\sigma_1 y) - T)(\phi_Q(\sigma_1 y) - \phi_B(\sigma_1 y))$$

and

$$\hat{\tau}_P(\log(\sigma_2 y) - T)(\phi_P(\sigma_2 y) - \phi_B(\sigma_2 y))$$

are of rapid decay on $\log^{-1} \mathfrak{a}_0^+$. These are similar so we will just do the first. We will actually show that $\phi_Q(\sigma_1 y) - \phi_B(\sigma_1 y)$ is of rapid decay on $\log^{-1}(\mathfrak{a}_0^+)$. This may be written

$$\int_{\mathbb{A}/F} \left[\phi_Q\left(\left(\begin{matrix} y_2 & & \\ & y_1 & \\ & & y_3 \end{matrix}\right)\right) - \phi_Q\left(\left(\begin{matrix} 1 & & \\ & 1 & x_3 \\ & & 1 \end{matrix}\right)\left(\begin{matrix} y_2 & & \\ & y_1 & \\ & & y_3 \end{matrix}\right)\right)\right] dx_3.$$

Now ϕ_Q is in the space $\mathcal{A}\big(M_Q(\mathbb{Q})\, U_Q(\mathbb{A})\, Z(\mathbb{A}) \backslash GL(3,\mathbb{A})/K\big)$ of automorphic forms on $GL(3,\mathbb{A})/K$ with respect to Q (see Moeglin and Waldspurger [MW] I.2.17). Subtracting its constant term therefore gives a function of rapid decay as $y_1/y_3 \to \infty$. Within $\log^{-1} \mathfrak{a}_0^+$ the absolute value of y_1/y_3 is large off a compact set, so this is of rapid decay. ■

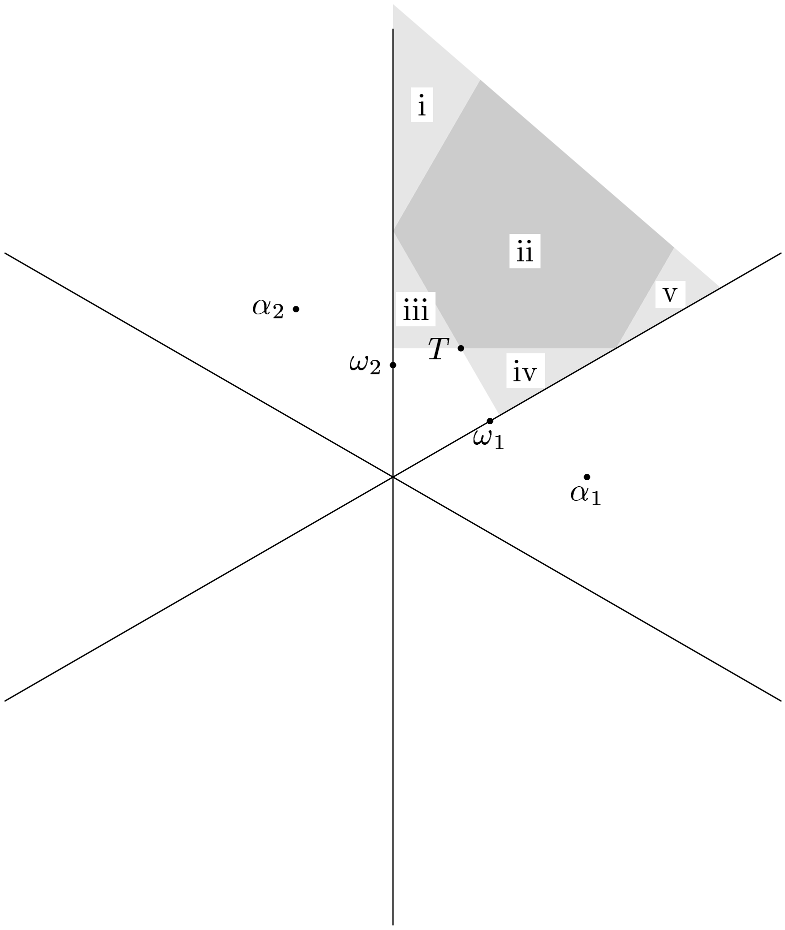

Figure 2. Regions of integration in $\mathfrak{a}_0^+$.

Let Φ be an element of $\mathcal{A}\big(M_P(\mathbb{Q})\,U_P(\mathbb{A})\,Z(\mathbb{A})\backslash GL(3,\mathbb{A})/K\big)$. Let

$$\Phi_0(g) = \int_{\mathbb{A}/F} \Phi\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)\,dx.$$

We define

$$\Lambda_P^T\Phi(y) = \hat{\tau}_P(\log\,y - T)\Phi(y) - \hat{\tau}_B(\log\,y - T)\Phi_0(y) - \hat{\tau}_B(\log\,\sigma_1 y - T)\Phi_0(\sigma_1 y).$$

This type of truncation is closely related to the truncations that appeared in Proposition 3. The integral of $\Lambda_P^T\Phi(y)$ over all of $A(\mathbb{A})$ will be divergent, but the integral over the line $\mathbb{R}\alpha_1$ will be convergent. Similarly, let

$$\Lambda_Q^T\Phi(y) = \hat{\tau}_Q(\log\,y - T)\Phi(y) - \hat{\tau}_B(\log\,y - T)\Phi_0(y) - \hat{\tau}_B(\log\,\sigma_2 y - T)\Phi_0(\sigma_2 y).$$

Proposition 6. *Suppose that $U - T$ is a positive multiple of α_2. Then*

$$(\Lambda^T\phi - \Lambda^U\phi)(y) = -\sum_{w\in W_P\backslash W} \Lambda_P^T\phi_P(wy) + \sum_{w\in W_P\backslash W} \Lambda_P^U\phi_P(wy).$$

Proof. If $U-T$ is a multiple of α_2 then $\hat{\tau}_Q(\log y-T)-\hat{\tau}_Q(\log y-U)=0$, so $\Lambda^T\phi-\Lambda^U\phi$ equals

$$\sum_{w\in W_P\backslash W}\big(\hat{\tau}_P(\log\, wy-U)-\hat{\tau}_P(\log\, wy-T)\big)\phi_P(wy)$$
$$-\sum_{w\in W}\big(\hat{\tau}_B(\log\, wy-U)-\hat{\tau}_B(\log\, wy-T)\big)\phi_B(wy).$$

The proposition follows from grouping the six terms in the second sum in pairs with the three terms of the first sum. ∎

If $U-T$ is a multiple of α_2 then $\hat{\tau}_P(\log y-T)-\hat{\tau}_P(\log y-U)$ and $\hat{\tau}_B(\log y-T)-\hat{\tau}_B(\log y-U)$ are also the characteristic functions of uncomplicated sets. Referring to Figure 3, $\hat{\tau}_P(\log y-T)-\hat{\tau}_P(\log y-U)$ is the characteristic function of the entire shaded strip, while $\hat{\tau}_B(\log y-T)-\hat{\tau}_B(\log y-U)$ is the characteristic function of the rightmost lighter-shaded piece.

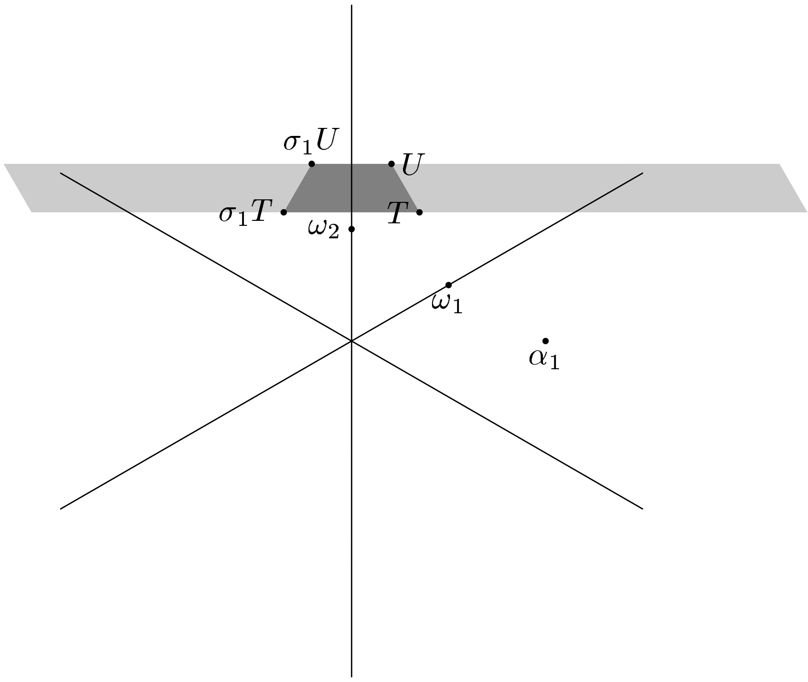

Figure 3. Domain of integration in Proposition 7 (i).

Let $\Phi\in\mathcal{A}\big(M_P(\mathbb{Q})\,U_P(\mathbb{A})\,Z(\mathbb{A})\backslash GL(3,\mathbb{A})/K\big)$. Define a character $\chi(y)=\chi_{s_1,s_2,s_3}(y)$ of $A(\mathbb{A})$ by (2), where $\sum s_i=0$. Associate with these data a value $B_P(\Phi,\chi,T)$ which is to be linear in Φ. If Φ is cuspidal, then $B_P(\Phi,\chi,T)=0$. On the other hand suppose that $\Phi=\Phi_P^{\theta_1,\theta_2,\theta_3}$ where

$$\Phi_P^{\theta_1,\theta_2,\theta_3}\left(\begin{pmatrix} a & b & * \\ c & d & * \\ & & y_3\end{pmatrix}k\right)=E^*_{\theta_1,\theta_2}\begin{pmatrix} a & b \\ c & d\end{pmatrix}y_3^{\theta_3},\qquad (k\in K)\tag{17}$$

where $\sum \theta_i = 0$. Denoting $\gamma_{T_1,T_3}(u,v) = \frac{e^{T_1 u + T_3 v}}{uv}$, we define

$$B_P(\Phi,\chi,T) = B_P^0(\Phi,\chi,T) + B_P^1(\Phi,\chi,T),$$

where

$$B_P^1(\Phi,\chi,T) = \frac{2e^{3(\theta_3+s_3)T_3/2}}{\theta_3+s_3}\zeta^*\big(\tfrac{1}{2}(\theta_1-\theta_2+s_1-s_2)\big)\,\zeta^*\big(\tfrac{1}{2}(\theta_1-\theta_2-s_1+s_2)\big)$$

and

$$\begin{aligned}B_P^0(\Phi,\chi,T) = 6\zeta^*(\theta_1-\theta_2)\,\gamma_{T_1,T_3}(\theta_1-\theta_2+s_1-s_2,-\theta_2+\theta_3+s_3-s_2)\\ + 6\zeta^*(\theta_1-\theta_2)\,\gamma_{T_1,T_3}(\theta_1-\theta_2-s_1+s_2,-\theta_2+\theta_3-s_1+s_3)\\ + 6\zeta^*(\theta_1-\theta_2-1)\,\gamma_{T_1,T_3}(2-\theta_1+\theta_2+s_1-s_2,1-\theta_1+\theta_3-s_2+s_3)\\ + 6\zeta^*(\theta_1-\theta_2-1)\,\gamma_{T_1,T_3}(2-\theta_1+\theta_2-s_1+s_2,1-\theta_1+\theta_3-s_1+s_3).\end{aligned}$$

Similarly if $\Phi \in \mathcal{A}\big(M_Q(\mathbb{Q})\,U_Q(\mathbb{A})\,Z(\mathbb{A})\backslash GL(3,\mathbb{A})/K\big)$ we define an analogous factor B_Q. Particularly if $\Phi = \Phi_Q^{\theta_1,\theta_2,\theta_3}$ where

$$\Phi_Q^{\theta_1,\theta_2,\theta_3}\left(\begin{pmatrix} y_1 & * & * \\ & a & b \\ & c & d\end{pmatrix} k\right) = E^*_{\theta_2,\theta_3}\begin{pmatrix} a & b \\ c & d\end{pmatrix} y_1^{\theta_1}, \qquad (k \in K), \tag{18}$$

let

$$B_Q(\Phi,\chi,T) = B_Q^0(\Phi,\chi,T) + B_Q^1(\Phi,\chi,T),$$

$$B_Q^1(\Phi,\chi,T) = -\frac{2e^{3T_1(\theta_1+s_1)/2}}{\theta_1+s_1}\,\zeta^*\big(\tfrac{1}{2}(\theta_2-\theta_3+s_2-s_3)\big)\,\zeta^*\big(\tfrac{1}{2}(\theta_2-\theta_3-s_2+s_3)\big),$$

$$\begin{aligned}B_Q^0(\Phi,\chi,T) = 6\zeta^*(\theta_2-\theta_3)\,\gamma_{T_1,T_3}(\theta_1-\theta_2+s_1-s_3,-\theta_2+\theta_3+s_2-s_3)\\ + 6\zeta^*(\theta_2-\theta_3)\,\gamma_{T_1,T_3}(\theta_1-\theta_2+s_1-s_2,-\theta_2+\theta_3-s_2+s_3)\\ + 6\zeta^*(\theta_2-\theta_3-1)\,\gamma_{T_1,T_3}(-1+\theta_1-\theta_3+s_1-s_3,-2+\theta_2-\theta_3+s_2-s_3)\\ + 6\zeta^*(\theta_2-\theta_3-1)\,\gamma_{T_1,T_3}(-1+\theta_1-\theta_3+s_1-s_2,-2+\theta_2-\theta_3-s_2+s_3).\end{aligned}$$

Proposition 7. *(i) If $\Phi \in \mathcal{A}\big(M_P(\mathbb{Q})\,U_P(\mathbb{A})\,Z(\mathbb{A})\backslash GL(3,\mathbb{A})/K\big)$ and $U-T$ is a multiple of α_2 then*

$$\int_{Z(\mathbb{A})\;A(\mathbb{Q})\backslash A(\mathbb{A})} (\Lambda_P^T\Phi - \Lambda_P^U\Phi)(y)\,\chi(y)\,dy = B_P(\Phi,\chi,T) - B_P(\Phi,\chi,U). \tag{19}$$

(ii) If $\Phi \in \mathcal{A}\big(M_Q(\mathbb{Q})\,U_Q(\mathbb{A})\,Z(\mathbb{A})\backslash GL(3,\mathbb{A})/K\big)$ and $U-T$ is a multiple of α_1 then

$$\int_{Z(\mathbb{A})\;A(\mathbb{Q})\backslash A(\mathbb{A})} (\Lambda_Q^T\Phi - \Lambda_Q^U\Phi)(y)\,\chi(y)\,dy = B_Q(\Phi,\chi,T) - B_Q(\Phi,\chi,U).$$

We have not yet specified the Haar measure on $Z(\mathbb{A})\,A(\mathbb{Q})\backslash A(\mathbb{A})$. We do that now, for there is a particular normalization implicit in the Proposition.

Proof. To prove (i), we may assume that $U - T$ is a positive multiple of α_2. Write $2\log_0(y) = u\omega_2 + v\alpha_1$, so that the domain of integration is $-\infty < v < \infty$ and u lies in an interval to be described. See Figure 3.

If $T = (T_1, T_2, T_3)$ then our assumption that $U - T$ is a multiple of α_2 implies that $U = (T_1, U_2, U_3)$ where $U_2 + U_3 = T_2 + T_3 = -T_1$. We have

$$2\log_0(y) = (\tfrac{1}{3}u + v, \tfrac{1}{3}u - v, -\tfrac{2}{3}u), \qquad u = \log|y_1 y_2 y_3^{-2}|, \qquad v = \log|y_1/y_2|.$$

The constraint on u is

$$-U_3 \geq \tfrac{1}{3}u \geq -T_3 > 0.$$

The term $\hat{\tau}_R(\log(y) - T)$ is nonzero if and only if $v \geq 2T_1 - \frac{1}{3}u$. Using this, we may fix u and integrate with respect to v with $\log|y_1 y_2 y_3^{-2}| = u$ fixed. The integrand is invariant under $Z(\mathbb{A})$ so we may fix $y_2 = 1$ and integrate with respect to y_1 and y_3. Without loss of generality we may take Φ_P in the form (17). The integrand is

$$e^{-u(\theta_3+s_3)/2}\, E^*_{\theta_1,\theta_2}\begin{pmatrix} y_1 & \\ & 1 \end{pmatrix} |y_1|^{\frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1}.$$

With notation as in (8) the inner integral is

$$e^{-(u/2)(\theta_3+s_3)}\, I(E^*_{\theta_1,\theta_2}, \tfrac{1}{2}\theta_3 + \tfrac{1}{2}s_3 + s_1, 2T_1 - \tfrac{u}{3}, 2T_1 - \tfrac{u}{3}).$$

Thus the integral is

$$\int_{-3T_3}^{-3U_3} e^{-(u/2)(\theta_3+s_3)}\, I(E^*_{\theta_1,\theta_2}, \tfrac{1}{2}\theta_3 + \tfrac{1}{2}s_3 + s_1, 2T_1 - \tfrac{u}{3}, 2T_1 - \tfrac{u}{3})\, du$$

We apply Proposition 3 with:

$$c_1 = \zeta^*(\theta_1 - \theta_2)\, e^{-u(\theta_3+s_3)/2}, \qquad \gamma_1 = \theta_1,$$
$$c_2 = \zeta^*(\theta_1 - \theta_2 - 1)\, e^{-u(\theta_3+s_3)/2}, \qquad \gamma_2 = 1 + \theta_2,$$

and $\lambda = \theta_1 + \theta_2$. We get

$$\int_{-3T_3}^{-3U_3} \Bigg[e^{-u(\theta_3+s_3)/2}\, I_{\mathrm{r}}(E^*_{\theta_1,\theta_2}, \tfrac{1}{2}\theta_3 + \tfrac{1}{2}s_3 + s_1)$$
$$+ \zeta^*(\theta_1 - \theta_2)\, e^{-u(\theta_3+s_3)/2} \frac{e^{(2T_1 - \frac{u}{3})(\theta_1 + \frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1)}}{\theta_1 + \frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1}$$
$$- \zeta^*(\theta_1 - \theta_2)\, e^{-u(\theta_3+s_3)/2} \frac{e^{-(2T_1 - \frac{u}{3})(\theta_2 + \frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1)}}{\theta_2 + \frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1}$$
$$+ \zeta^*(\theta_1 - \theta_2 - 1)\, e^{-u(\theta_3+s_3)/2} \frac{e^{(2T_1 - \frac{u}{3})(1 + \theta_2 + \frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1)}}{1 + \theta_2 + \frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1}$$
$$- \zeta^*(\theta_1 - \theta_2 - 1)\, e^{-u(\theta_3+s_3)/2} \frac{e^{-(2T_1 - \frac{u}{3})(-1 + \theta_1 + \frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1)}}{-1 + \theta_1 + \frac{1}{2}\theta_3 + \frac{1}{2}s_3 + s_1} \Bigg]\, du.$$

Using the value of $I_{\mathrm{r}}(E^*_{\theta_1,\theta_2}, s)$ from Proposition 4, together with $\theta_1 + \theta_2 + \theta_3 = 0$ and $s_1 + s_2 + s_3 = 0$, this equals $B_P(\Phi, \chi, T) - B_P(\Phi, \chi, U)$. The proof of (ii) is similar. ∎

Proposition 8.

$$\int_{Z_{\mathbb{A}}\, A(\mathbb{Q})\backslash A(\mathbb{A})} \Lambda^T \phi(y)\, \chi(y)\, dy + \sum_{w\in W_P\backslash W} B_P(\phi_P, {}^w\chi, T)$$

is convergent for any χ and is unchanged if a real multiple of α_2 is added to T. Moreover

$$\int_{Z_{\mathbb{A}}\, A(\mathbb{Q})\backslash A(\mathbb{A})} \Lambda^T \phi(y)\, \chi(y)\, dy + \sum_{w\in W_Q\backslash W} B_Q(\phi_Q, {}^w\chi, T)$$

is convergent for any χ and is unchanged if a real multiple of α_1 is added to T.

Proof. The convergence of the integral follows from Proposition 5. The invariance follows from Proposition 6 and Proposition 7. ■

If $\sum \mu_i = 0$, define

$$f_{\mu_1,\mu_2,\mu_3}\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k\right) = |y_1|^{\mu_1}|y_2|^{\mu_2}|y_3|^{\mu_3}, \qquad k\in K, \tag{20}$$

and

$$G(g) = G_{\mu_1,\mu_2,\mu_3}(g) = \zeta^*(\mu_1-\mu_2)\,\zeta^*(\mu_2-\mu_3)\,\zeta^*(\mu_1-\mu_3-1)\sum_{B_F\backslash GL(3,F)} f_{\mu_1,\mu_2,\mu_3}(\gamma g) \tag{21}$$

where K is the standard maximal compact subgroup and B the Borel subgroup of $GL(3,\mathbb{A})$. This is convergent if $\mathrm{re}(\mu_1-\mu_2), \mathrm{re}(\mu_2-\mu_3) > 1$ and has meromorphic continuation to all μ_i. With this notation the functional equations of $G_{\mu_1,\mu_2,\mu_3}(g)$ consist of the six permutations of μ_1-1, μ_2 and μ_3+1.

The constant terms:

$$\begin{aligned} G_P\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3\end{pmatrix} &= \zeta^*(\mu_2-\mu_3)\,\zeta^*(\mu_1-\mu_3-1)\,|y_3|^{\mu_3}\, E^*_{\mu_1,\mu_2}\begin{pmatrix} y_1 & \\ & y_2\end{pmatrix} \\ &+ \zeta^*(\mu_1-\mu_2-1)\,\zeta^*(\mu_1-\mu_3-2)\,|y_3|^{\mu_1-2}\, E^*_{\mu_2+1,\mu_3+1}\begin{pmatrix} y_1 & \\ & y_2\end{pmatrix} \\ &+ \zeta^*(\mu_1-\mu_2)\,\zeta^*(\mu_2-\mu_3-1)\,|y_3|^{\mu_2-1}\, E^*_{\mu_3+2,\mu_1-1}\begin{pmatrix} y_1 & \\ & y_2\end{pmatrix}, \end{aligned} \tag{22}$$

$$\begin{aligned} G_Q\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3\end{pmatrix} &= \zeta^*(\mu_1-\mu_2)\,\zeta^*(\mu_1-\mu_3-1)\,|y_1|^{\mu_1}\, E^*_{\mu_2,\mu_3}\begin{pmatrix} y_2 & \\ & y_3\end{pmatrix} \\ &+ \zeta^*(\mu_2-\mu_3)\,\zeta^*(\mu_1-\mu_2-1)\,|y_1|^{\mu_2+1}\, E^*_{\mu_3+1,\mu_1-2}\begin{pmatrix} y_2 & \\ & y_3\end{pmatrix} \\ &+ \zeta^*(\mu_1-\mu_3-2)\,\zeta^*(\mu_2-\mu_3-1)\,|y_1|^{\mu_3+2}\, E^*_{\mu_1-1,\mu_2-1}\begin{pmatrix} y_2 & \\ & y_3\end{pmatrix}, \end{aligned} \tag{23}$$

and

$$G_B\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} =$$
$$\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)\zeta^*(\mu_2-\mu_3)|y_1|^{\mu_1}|y_2|^{\mu_2}|y_3|^{\mu_3}$$
$$+\,\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_1}|y_2|^{\mu_3+1}|y_3|^{\mu_2-1}$$
$$+\,\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3)|y_1|^{\mu_2+1}|y_2|^{\mu_3+1}|y_3|^{\mu_1-2}$$
$$+\,\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-1)\zeta^*(\mu_2-\mu_3)|y_1|^{\mu_2+1}|y_2|^{\mu_1-1}|y_3|^{\mu_3}$$
$$+\,\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_3+2}|y_2|^{\mu_1-1}|y_3|^{\mu_2-1}$$
$$+\,\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_3+2}|y_2|^{\mu_2}|y_3|^{\mu_1-2}$$

Proposition 9.

$$\sum_{w\in W_P\backslash W} B_P^0(G_P, {}^w\chi, T) = \sum_{w\in W_Q\backslash W} B_Q^0(G_Q, {}^w\chi, T). \tag{24}$$

Proof. This follows from the definitions of B_P^0 and B_Q^0 together with (22) and (23). ■

In view of Proposition 9 we will denote (24) unbiasedly as $B^0(G,\chi,T)$. It clearly satisfies

$$B^0(G,\chi,T) = B^0(G,{}^w\chi,T)$$

for any $w \in W$.

Theorem 10. *The expression:*

$$\int\limits_{Z(\mathbb{A})\ A(\mathbb{Q})\backslash A(\mathbb{A})} \Lambda^T G(y)\,\chi(y)\,d^\times y + B^0(G,\chi,T) \tag{25}$$
$$+\sum_{w\in W_P\backslash W} B_P^1(G,{}^w\chi,T) + \sum_{w\in W_Q\backslash W} B_Q^1(G,{}^w\chi,T)$$

is independent of T*.*

Proof. It is sufficient to show that (25) is unchanged when T is shifted by an element of either α_1 or α_2. In view of Proposition 9, we may express $B^0(G,\chi,T)$ in terms of either the B_P^0 or B_Q^0. In either case, the invariance follows from Proposition 8, together with the obvious fact that $B_P^1(G_P,\chi,T)$ is unchanged if T is changed by a multiple of α_1, and that $B_Q^1(G_Q,\chi,T)$ is unchanged if T is changed by a multiple of α_2. ■

Define the renormalized integral $\mathbf{RN}\int_{Z(\mathbb{A})\ A(\mathbb{Q})\backslash A(\mathbb{A})} G(y)\,\chi(y)\,d^\times y$ to equal (25).

Theorem 11. *The poles of* $\mathbf{RN}\int_{Z(\mathbb{A})\,A(\mathbb{Q})\backslash A(\mathbb{A})} G(y)\,\chi(y)\,d^\times y$ *are the* 24 *hyperplanes:*

$$\mu_1-\mu_2=0,\,2 \qquad \mu_2-\mu_3=0,\,2, \qquad \mu_1-\mu_3=1,\,3$$

$$\mu_1+s_i=0,\,2, \qquad \mu_2+s_i=-1,\,1, \qquad \mu_3+s_i=-2,\,2.$$

Proof. There appear to be other poles but these cancel. For example, let us show that there is no pole along the hyperplane

$$\theta_1-\theta_2+s_1-s_2=0.$$

Let

$$A=\tfrac{1}{2}(\theta_1-\theta_2+s_1-s_2),$$

$$B=\tfrac{1}{2}(\theta_1-\theta_2-s_1+s_2),$$

$$C=s_3+\theta_3$$

Four terms are polar when $A=0$. Two of these have sum

$$-2C^{-1}e^{3CT_3/2}\zeta^*(A)\zeta^*(B)+2A^{-1}(C+2A/3)^{-1}e^{3CT_3/2}\zeta^*(A+B)e^{A(2T_1+T_3)}.$$

Since $\zeta^*(A)-A^{-1}$ is holomorphic at $A=0$ this differs by an analytic function from

$$\frac{1}{A}\left[-2C^{-1}e^{3CT_3/2}\zeta^*(B)+2(C+2A/3)^{-1}e^{3CT_3/2}\zeta^*(A+B)e^{A(2T_1+T_3)}\right].$$

The expression in brackets vanishes when $A=0$, so there is no pole along this line. The other two terms cancel similarly so there is no pole along this line.

We leave it to the reader to show that all poles except the ones described cancel like this. ■

It is clear from the definitions that this renormalized integral has as functional equations the 6 functional equations of the Eisenstein series, which transform the μ_i and leave the s_i unchanged, as well as the 6 permutations of the s_i, corresponding to $\chi\to{}^w\chi$ for $w\in W$. Thus it has at least 36 symmetries or functional equations. However the polytope spanned by these 24 hyperplanes has other symmetries not among these 36, for example that in Theorem 1.

3. A generalization. We now generalize Theorem 10. In the generalization we specify for each $w\in W$ a T^w. The special case where the T^w are all equal to a fixed T coincides with our previous truncation. The purpose of this generalization is that the parameters T^w can be moved around independently, as for example in the first proof of Proposition 4 we specialized T and T' differently. We will not make use of this result in this paper.

We ask that $\sigma_1 T^w - T^{\sigma_1 w} \in \mathbb{R}\alpha_1$ and $\sigma_2 T^w - T^{\sigma_2 w} \in \mathbb{R}\alpha_2$. This implies that if $w^{-1}T^w$ and $w'^{-1}T^{w'}$ are in adjacent Weyl chambers then their difference is a root. The six $w^{-1}T^w$ are thus the vertices of a hexagon with parallel opposite sides. Let

$$\begin{aligned}
&\Lambda^{\{T^w\}}\phi(y) = \\
&\quad \phi(y) - \sum_{w \in W_P\backslash W} \hat\tau_P(\log(wy) - T^w)\, \phi_P(wy) - \sum_{w \in W_Q\backslash W} \hat\tau_Q(\log(wy) - T^w)\, \phi_Q(wy) \\
&\qquad\qquad + \sum_{w \in W} \hat\tau_B(\log(wy) - T^w)\, \phi_B(wy).
\end{aligned}$$

The analog of Proposition 5 is true, namely this function is of rapid decay.

If T and T' are arbitrary and $\Phi \in \mathcal{A}\big(M_P(\mathbb{Q})\, U_P(\mathbb{A})\, Z(\mathbb{A})\backslash GL(3,\mathbb{A})/K\big)$, let

$$\begin{aligned}
B_P^0(\Phi,\chi,T,T') = {}& 6\zeta^*(\theta_1-\theta_2)\, \gamma_{T_1,T_3}(\theta_1-\theta_2+s_1-s_2, -\theta_2+\theta_3+s_3-s_2) \\
&+ 6\zeta^*(\theta_1-\theta_2)\, \gamma_{T_1',T_3'}(\theta_1-\theta_2-s_1+s_2, -\theta_2+\theta_3-s_1+s_3) \\
&+ 6\zeta^*(\theta_1-\theta_2-1)\, \gamma_{T_1,T_3}(2-\theta_1+\theta_2+s_1-s_2, 1-\theta_1+\theta_3-s_2+s_3) \\
&+ 6\zeta^*(\theta_1-\theta_2-1)\, \gamma_{T_1',T_3'}(2-\theta_1+\theta_2-s_1+s_2, 1-\theta_1+\theta_3-s_1+s_3),
\end{aligned}$$

while if $\Phi \in \mathcal{A}\big(M_Q(\mathbb{Q})\, U_Q(\mathbb{A})\, Z(\mathbb{A})\backslash GL(3,\mathbb{A})/K\big)$, let

$$\begin{aligned}
B_Q^0(\Phi,\chi,T,T') = {}& 6\zeta^*(\theta_2-\theta_3)\, \gamma_{T_1,T_3}(\theta_1-\theta_2+s_1-s_3, -\theta_2+\theta_3-s_3+s_2) \\
&+ 6\zeta^*(\theta_2-\theta_3)\, \gamma_{T_1',T_3'}(\theta_1-\theta_2+s_1-s_2, -\theta_2+\theta_3+s_3-s_2) + \\
&+ 6\zeta^*(\theta_2-\theta_3-1)\, \gamma_{T_1,T_3}(-1+\theta_1-\theta_3+s_1-s_3, -2+\theta_2-\theta_3-s_3+s_2) \\
&+ 6\zeta^*(\theta_2-\theta_3-1)\, \gamma_{T_1',T_3'}(-1+\theta_1-\theta_3+s_1-s_2, -2+\theta_2-\theta_3+s_3-s_2).
\end{aligned}$$

The analog of Proposition 9 is true for the Eisenstein series

$$\sum_{w \in W_P\backslash W} B_P^0(G_P, {}^w\chi, T^w, T^{\sigma_1 w}) = \sum_{w \in W_Q\backslash W} B_Q^0(G_Q, {}^w\chi, T^w, T^{\sigma_2 w}). \tag{26}$$

We will denote (26) as $B^0(G,\chi,\{T^w\})$.

Theorem 12. *With these notations,* $\mathbf{RN}\int_{Z(\mathbb{A})\ A(\mathbb{Q})\backslash A(\mathbb{A})} G(y)\,\chi(y)\,dy$ *equals*

$$\begin{aligned}
&\int_{Z(\mathbb{A})\ A(\mathbb{Q})\backslash A(\mathbb{A})} \Lambda^{\{T^w\}}G(y)\,\chi(y)\,d^\times y + B^0(G,\chi,\{T^w\}) \\
&\qquad + \sum_{w \in W_P\backslash W} B_P^1(G_P, {}^w\chi, T^w) + \sum_{w \in W_Q\backslash W} B_Q^1(G_Q, {}^w\chi, T^w).
\end{aligned} \tag{27}$$

Proof. Let Φ be an element of $\mathcal{A}\big(M_P(\mathbb{Q})\, U_P(\mathbb{A})\, Z(\mathbb{A})\backslash GL(3,\mathbb{A})/K\big)$, and let Φ_0 be as before. Let T' be such that $T - T'$ is a multiple of α_1 and define

$$\Lambda_P^{T,T'}\Phi(y) = \hat\tau_P(\log y - T)\Phi(y) - \hat\tau_B(\log y - T)\Phi_0(y) - \hat\tau_B(\log \sigma_1 y - T')\Phi_0(\sigma_1 y).$$

We let T and U be such that $T^w - U^w$ is a multiple of α_2 for all w. Then as in Proposition 6 we have

$$(\Lambda^{\{T^w\}}\phi - \Lambda^{\{U^w\}}\phi)(y) = -\sum_{w\in W_P\backslash W} \Lambda_P^{T^w,T^{\sigma_1 w}}\phi_P(wy) + \sum_{w\in W_P\backslash W} \Lambda_P^{U^w,U^{\sigma_1 w}}\phi_P(wy).$$

Note that this is well defined modulo W_P because

$$\Lambda_P^{T,T'}\Phi(y) = \Lambda_P^{T',T}\Phi(\sigma_1 y).$$

Assuming that $T - T'$ is a multiple of α_1 so that $T_3' = T_3$, define

$$B_P(\Phi,\chi,T,T') = B_P^0(\Phi,\chi,T,T') + B_P^1(\Phi,\chi,T),$$

where $B_P^1(\Phi,\chi,T) = B_P^1(\Phi,\chi,T')$ is as before. We have

$$B_P^0(\Phi,\chi,T,T') = B_P^0(\Phi,{}^{\sigma_1}\chi,T',T).$$

Similarly, if $T - T'$ is a multiple of α_2 then $T_1' = T_1$ and in this case we let

$$B_Q(\Phi,\chi,T,T') = B_Q^0(\Phi,\chi,T,T') + B_Q^1(\Phi,\chi,T),$$

Assume that $T - U = \sigma_1(T' - U')$ is a multiple of α_2, and that $U - U'$ and $T - T'$ are (different) multiples of α_1. Then the analog of Proposition 7 is the formula

$$\int_{Z(\mathbb{A})\;A(\mathbb{Q})\backslash A(\mathbb{A})} (\Lambda_P^{T,T'}\Phi - \Lambda_P^{U,U'}\Phi)(y)\,\chi(y)\,dy = B_P(\Phi,\chi,T,T') - B_P(\Phi,\chi,U,U').$$

We see that (27) is unchanged if $\{T^w\}$ are all changed by multiples of α_2, and it is similarly unchanged if they are all changed by multiples of α_1. Combining both cases, it is independent of the choices of $\{T^w\}$. In particular, taking the T^w all equal to T, it is equal to the expression in Theorem 10. ■

4. Renormalization on $GL(2)$. The theory in this section is modeled on the results of Zagier [Z]. The principal difference is that we work on the adele group.

In this section K will denote the maximal compact subgroup $\prod_v K_v$ where $K_\infty = O(2)$ and $K_v = GL(2,\mathbb{Z}_p)$ when $v = p$ is a finite place. We will also let $K_{\mathrm{f}} = \prod_{v<\infty} K_v$ denote the maximal compact subgroup of the finite adeles. Also Z_2 will denote the center of $GL(2)$ consisting of scalar matrices.

Let ϕ be a function on $GL(2,\mathbb{A})$ left invariant by $GL(2,\mathbb{Q})$ and $Z_2(\mathbb{A})$. We do not assume that ϕ is an automorphic form since we wish to include products of Eisenstein series. However we assume that there exist finite functions $\rho_1,\cdots,\rho_r$ on $\mathbb{A}^\times/\mathbb{Q}^\times$ and smooth functions $\xi_1,\cdots,\xi_r$ on K such that

$$|y|^{-1}\,\phi\left(\begin{pmatrix} y & x \\ & 1\end{pmatrix} k\right) - \rho(y,k)$$

is of rapid decay as $|y| \to \infty$, uniformly in x and k, where

$$\rho(y,k) = \sum_{i=1}^{r} \rho_i(y) \left| \frac{y_1}{y_2} \right|^{-1} \xi_i(k), \qquad y_i \in \mathbb{A}^\times,\ x \in \mathbb{A},\ k \in K. \tag{28}$$

We assume that the ρ_i are (in the notation of Section 1) in $\mathcal{F}_0(\mathbb{A}^\times/\mathbb{Q}^\times)$. This restriction prohibits ϕ from being the constant function. If $\epsilon \in \prod_p \mathbb{Z}_p^\times$ then

$$\sum_i \rho_i(y)\,\xi_i(k) = \sum_i \rho_i(y\epsilon)\,\xi_i\left(\begin{pmatrix} \epsilon^{-1} & \\ & 1 \end{pmatrix} k\right),$$

so

$$\sum_i \rho_i(y) \int_K \xi_i(k)\,dk = \sum_i \rho_i(y\epsilon) \int_K \xi_k(k)\,dk. \tag{29}$$

Let $R_i : \mathbb{R} \to \mathbb{C}$ be related to ρ_i as in Proposition 2, so that

$$\int\limits_{\substack{y \in \mathbb{A}^\times/\mathbb{Q}^\times \\ T < \log|y| < U}} \rho_i(y)\,d^\times y = R_i(U) - R_i(T).$$

Since the narrow class number of $\mathbb{Q}$ is one, we may identify $\mathbb{A}^\times/\mathbb{Q}^\times = \mathbb{R}_+^\times \prod_p \mathbb{Z}_p^\times$, so

$$\int_{e^T}^{e^U} \int\limits_{\prod_p \mathbb{Z}_p^\times} \rho_i(y\epsilon)\,d^\times\epsilon\,\frac{dy}{y} = R_i(U) - R_i(T), \tag{30}$$

where y is taken from $\mathbb{R}_+^\times$ embedded into $\mathbb{A}^\times$ at the infinite place. Define

$$R(T) = \sum_{i=1}^{r} R_i(T) \int_K \xi_i(k)\,dk.$$

Using (29) and (30) we have

$$\int_{e^T}^{e^U} \rho_i(y) \int_K \xi_i(k)\,dk\,\frac{dy}{y} = R(U) - R(T). \tag{31}$$

Let B_2 be the Borel subgroup of $GL(2)$. Then $GL(2,\mathbb{A}) = B_2(\mathbb{A})\,K$. We define a *height function* $h : GL(2,\mathbb{A}) \to \mathbb{R}_+$ by $h(g) = |y_1/y_2|$ when we write

$$g = \begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} k,$$

with $k \in K$. This is well-defined. The function $h(g)$ plays an analogous role in the reduction theory to the imaginary part y of a point $z = x + iy$ in the upper half plane.

Lemma 13. *Let $g \in GL(2,\mathbb{A})$, $\gamma \in GL(2,\mathbb{Q})$. If $h(g), h(\gamma g) > 1$ then $\gamma \in B_2(\mathbb{Q})$.*

Proof. Let

$$g = \begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} k, \qquad \gamma g; = \begin{pmatrix} y_1' & * \\ & y_2' \end{pmatrix} k',$$

with $k, k' \in K$. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad k'k^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then $cy_1/y_2 = Cy_2'/y_1'$. Thus $|c|h(g)h(\gamma g) = |C|$. Since $k'k^{-1} \in K$, $|C| \leq 1$ so $|c| < 1$. Since $c \in \mathbb{Q}$ this implies that $c = 0$ so $\gamma \in B_2(\mathbb{Q})$. ∎

We also define

$$H(g) = \sup \{h(\gamma g) \,|\, \gamma \in GL(2,\mathbb{Q})\}.$$

The lemma implies that this supremum exists.

Let $T > 0$. Let $GL(2,\mathbb{A})_T = \{g \in GL(2,\mathbb{A}) \,|\, H(g) < T\}$. Now we define

$$\phi_T(g) = \begin{cases} \phi(g) & \text{if } g \in GL(2,\mathbb{A})_T; \\ \phi(g) - \rho(g') & \text{if } g = \gamma g', \text{ where } \log\, h(g') > T, \end{cases}$$

$$\phi^T(g) = \begin{cases} \phi(g) & \text{if } g \in GL(2,\mathbb{A})_T; \\ 0 & \text{if } g = \gamma g', \text{ where } \log\, h(g') > T. \end{cases}$$

It follows from the Lemma that if $g \notin GL(2,\mathbb{A})_T$ then g' is unique modulo $B_2(\mathbb{Q})$, so $h(g')$ is uniquely determined. We define

$$\mathbf{RN}\int_{GL(2,\mathbb{Q})Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})} \phi(g)\,dg = \int_{GL(2,\mathbb{Q})Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})} \phi_T(g)\,dg - R(T). \tag{32}$$

Proposition 14. *The expression (32) is independent of T.*

Proof. Suppose that $U > T$. We have

$$\int_{GL(2,\mathbb{Q})Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})} (\phi_U - \phi_T)(g)\,dg = \int_{GL(2,\mathbb{Q})Z_2(\mathbb{A})\backslash (GL(2,\mathbb{A})_U - GL(2,\mathbb{A})_T)} \rho(g)\,dg =$$
$$\int_{\mathbb{A}/\mathbb{Q}} \int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ T < \log|y| < U}} \int_K \rho(y,k)|y|^{-1}\,dk\,d^\times y\,dx$$

Using (28) and (31) this is $R(U) - R(T)$. ∎

Let $\mathcal{L}_T = \{g \in GL(2,\mathbb{A}) \,|\, h(g) < T\}$, and let $\mathcal{U}_T = \{g \in GL(2,\mathbb{A}) \,|\, h(g) \geq T\}$ be its complement. Let $\mathcal{S}_T$ be a fundamental domain for the action of $B_2(F)$ on $\mathcal{U}_T$. It is a "Siegel set" in the sense of reduction theory. The Lemma implies that the inclusion of $\mathcal{U}_T$ into $GL(2,\mathbb{A})$ induces a homeomorphism

$$\mathcal{S}_T \cong B_2(\mathbb{Q})\, Z_2(\mathbb{A})\backslash \mathcal{U}_T \cong GL(2,\mathbb{Q})\, Z_2(\mathbb{A})\backslash (GL(2,\mathbb{A}) - GL(2,\mathbb{A})_T). \tag{33}$$

Moreover, if $\mathcal{F}_T$ is a fundamental domain for $Z_2(\mathbb{A})GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})_T$, then the Lemma also implies that

$$\mathcal{L}_T = GL(2,\mathbb{A})_T \cup \bigcup_{\gamma\in GL(2,\mathbb{Q})-B_2(\mathbb{Q})} \gamma^{-1}\mathcal{S}_T \qquad \text{(disjoint)}. \tag{34}$$

is a fundamental domain for $Z_2(\mathbb{A})\,GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})$.

Let $E^*(g,s) = E^*_{s,-s}(g)$ in the notation of Section 1. Also let $f_s(g) = f_{s,-s}(g)$ in the notation (10). The constant term

$$E_0^*(g,s) = \int_{\mathbb{A}/\mathbb{Q}} E^*\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g, s\right) = \zeta^*(2s)\, f(g,s) + \zeta^*(2-2s)\, f(g,1-s).$$

Let $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}$ be the additive character whose conductor is $\mathbb{Z}_p$ for every finite place p, and whose infinite component is $\psi_\infty(x) = e^{2\pi i x}$. The Whittaker function

$$\int_{\mathbb{A}/\mathbb{Q}} E^*\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g, s\right)\psi(x)\,dx = W(g,s) = \prod_v W_v(g,s)$$

where

$$W_\infty\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}, s\right) = \sqrt{y}\, K_{s-1/2}(2\pi y)$$

and

$$W_v\begin{pmatrix} y & \\ & 1\end{pmatrix} = \begin{cases} p^{-n/2}\,\dfrac{p^{(s-1/2)(n+1)/2} - p^{-(s-1/2)(n+1)/2}}{p^{s-1/2} - p^{-(s-1/2)}} & \text{if } n = \mathrm{ord}_p(y) \geq 0, \\ 0 & \text{otherwise,}\end{cases} \tag{35}$$

when $v = p$ is a finite place. See Bump [B], (7.33) on p. 358. We have the Fourier expansion

$$E^*(g,s) = E_0^*(g,s) + \sum_{\alpha\in\mathbb{Q}^\times} W\left(\begin{pmatrix} \alpha & \\ & 1\end{pmatrix} g, s\right). \tag{36}$$

Up to this point we have not assumed that ϕ is right K-invariant. Now however we assume this. Let

$$a_0(y) = \phi_0\begin{pmatrix} y & \\ & 1\end{pmatrix}, \qquad \phi_0(g) = \int_{\mathbb{A}/\mathbb{Q}} \phi\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right)\,dx$$

be the constant term of ϕ. Since $\rho(y,k)$ is now independent of $k \in K$ we denote it as simply ρ.

Theorem 15. *We have*

$$\mathbf{RN}\int_{GL(2,\mathbb{Q})\,Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})} E^*(g,s)\,\phi(g)\,dg = \zeta^*(2s)\,\mathbf{RN}\int_{\mathbb{A}^\times/\mathbb{Q}^\times} a_0(y)\,|y|^{s-1}\,d^\times y. \tag{37}$$

This is an adelic version of the Theorem in Zagier [Z].

Proof. Both sides are meromorphic in s, so it is sufficient to prove this when $\mathrm{re}(s)$ is large.

Following Zagier, the first step is to prove that if $\mathrm{re}(s)$ is sufficiently large

$$\begin{aligned}\int_{GL(2,\mathbb{Q})\,Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})_T} & E^*(g,s)\,\phi(g)\,dg \\ & + \int_{\mathcal{S}_T} [E^*(g,s)-\zeta^*(2s)f_s(g)-\zeta^*(2-2s)f_{1-s}(g)]\,\phi(g)\,dg = \\ & \zeta^*(2s)\int_{\substack{A^\times/\mathbb{Q}^\times \\ \log|y|<T}} a_0(y)\,|y|^{s-1}\,d^\times y - \zeta^*(2-2s)\int_{\substack{A^\times/\mathbb{Q}^\times \\ \log|y|>T}} a_0(y)\,|y|^{-s}\,d^\times y.\end{aligned} \tag{38}$$

We can unfold

$$\begin{aligned}\int_{GL(2,\mathbb{Q})\,Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})_T} & E^*(g,s)\,\phi(g)\,dg = \\ & \int_{GL(2,\mathbb{Q})\,Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})} E^*(g,s)\,\phi^T(g)\,dg = \\ & \zeta^*(2s)\int_{B_2(\mathbb{Q})\,Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})} f_s(g)\,\phi^T(g)\,dg.\end{aligned}$$

Using (34) this equals

$$\begin{aligned}\zeta^*(2s)\int_{B_2(\mathbb{Q})\,Z_2(\mathbb{A})\backslash\mathcal{L}_T} & f_s(g)\,\phi(g)\,dg \\ & - \zeta^*(2s)\int_{B_2(\mathbb{Q})\,Z_2(\mathbb{A})\backslash\bigcup_{\gamma\in GL(2,\mathbb{Q})-B_2(\mathbb{Q})}\gamma^{-1}\mathcal{S}_T} f_s(g)\,\phi(g)\,dg\end{aligned}$$

The first term is evaluated using the Iwasawa decomposition and equals

$$\int_{\log|y|<T} a_0(y)\,y^{s-1}\,d^\times y.$$

The second is evaluated by interchanging the summation and the integration and changing $g\to\gamma g$. It equals

$$\zeta^*(2s)\sum_{\gamma\in B_2(\mathbb{Q})\backslash(GL(2,\mathbb{Q})-B_2(\mathbb{Q}))}\int_{\mathcal{S}_T} f_s(\gamma g)\,\phi(g)\,dg = \int_{\mathcal{S}_T}[E^*(g,s)-\zeta^*(2s)\,f_s(g)]\,\phi(g)\,dg.$$

Noting that

$$\int_{\mathcal{S}_T} \zeta^*(2-2s)\, f_{1-s}(g)\, \phi(g)\, dg = \int_{\log|y|>T} a_0(y)\, |y|^{-s}\, d^\times y$$

we obtain (38).

Note that in the integration over $\mathcal{S}_T$ in the second term on the left hand side of (38) we may replace $f_s(g)\,\phi(g)$ and $f_{1-s}(g)\,\phi(g)$ by $f_s(g)\,\phi_0(g)$ and $f_{1-s}(g)\,\phi_0(g)$ respectively. Thus if we add

$$\int_T^\infty \big(a_0(y)-\rho(y)\big)\big(\zeta^*(2s)|y|^{s-1}+\zeta^*(2-2s)|y|^{-s}\big)\, d^\times y$$
$$-\,\zeta^*(2s)\int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times\\ \log|y|<T}} \rho(y)|y|^{s-1}\, d^\times y + \zeta^*(2-2s)\int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times\\ \log|y|<T}} \rho(y)|y|^{-s}\, d^\times y$$

to both sides of (38) we get

$$\int_{GL(2,\mathbb{Q})\,Z_2(\mathbb{A})\backslash GL(2,\mathbb{A})_T} E^*(g,s)\,\phi(g)\, dg$$
$$+\int_{\mathcal{S}_T} \big[E^*(g,s)\,\phi(g)-\zeta^*(2s)|y|^{s-1}\rho(y)-\zeta^*(2-2s)|y|^{-s}\rho(y)\big]\, dg$$
$$-\,\zeta^*(2s)\int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times\\ \log|y|<T}} \rho(y)|y|^{s-1}\, d^\times y+\zeta^*(2-2s)\int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times\\ \log|y|<T}} \rho(y)|y|^{-s}\, d^\times y =$$
$$\zeta^*(2s)\int_{\mathbb{A}^\times/\mathbb{Q}^\times} \big(a_0(y)-\rho(y)\big)\, |y|^{s-1}\, d^\times y.$$

Assuming that $\mathrm{re}(s)$ is sufficiently large, then using Proposition 2 (ii) and (iii), the left hand side is the renormalized Rankin-Selberg integral on the left side of (37). The right hand side is the renormalized Mellin transform on the right side of (37), and we are done. ■

The theorem implies the functional equation under $s \to 1-s$ of the right hand side. It also allows computation of its polar divisor as in Zagier [Z]. We will omit this, however.

5. The comparison. In this section we prove the identity of two renormalized integrals, one on $GL(2)$ and one on $GL(3)$, leading to the proof of Theorem 1.

We apply Theorem 15 now in the case of the product of three Eisenstein series. Let

$$\phi(g) = E^*(g,\nu_1)\, E^*(g,\nu_2)\, E^*(g,\nu_3). \tag{39}$$

We choose the parameters s and ν_i so that

$$s = \tfrac{1}{2}(\mu_1-\mu_3-1)$$

and

$$\nu_i = \tfrac{1}{2}(s_i+\mu_2+1), \qquad (1 \le i \le 3).$$

Define

$$(40)\quad H\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} =$$

$$G\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} - \zeta^*(\mu_1-\mu_2)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)\Bigg[|y_3|^{\mu_3}E_{\mu_1,\mu_2}\begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix}$$

$$+|y_2|^{\mu_3}E_{\mu_1,\mu_2}\begin{pmatrix} y_1 & \\ & y_3 \end{pmatrix} + |y_1|^{\mu_3}E_{\mu_1,\mu_2}\begin{pmatrix} y_2 & \\ & y_3 \end{pmatrix} + |y_3|^{\mu_1}E_{\mu_2,\mu_3}\begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix}$$

$$+|y_2|^{\mu_1}E_{\mu_2,\mu_3}\begin{pmatrix} y_1 & \\ & y_3 \end{pmatrix} + |y_1|^{\mu_1}E_{\mu_2,\mu_3}\begin{pmatrix} y_2 & \\ & y_3 \end{pmatrix}$$

$$-|y_1|^{\mu_2}|y_2|^{\mu_1}|y_3|^{\mu_3} - |y_1|^{\mu_2}|y_2|^{\mu_3}|y_3|^{\mu_1} - |y_1|^{\mu_1}|y_2|^{\mu_2}|y_3|^{\mu_3}$$

$$-|y_1|^{\mu_3}|y_2|^{\mu_2}|y_3|^{\mu_1} - |y_1|^{\mu_1}|y_2|^{\mu_3}|y_3|^{\mu_2} - |y_1|^{\mu_3}|y_2|^{\mu_1}|y_3|^{\mu_2}\Bigg]$$

$$-\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)\Bigg[|y_3|^{\mu_2+1}E_{\mu_1-1,\mu_3}\begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix}$$

$$+|y_1|^{\mu_2+1}E_{\mu_1-1,\mu_3}\begin{pmatrix} y_2 & \\ & y_3 \end{pmatrix} + |y_2|^{\mu_2+1}E_{\mu_1-1,\mu_3}\begin{pmatrix} y_1 & \\ & y_3 \end{pmatrix}$$

$$-|y_1|^{\mu_2+1}|y_2|^{\mu_1-1}|y_3|^{\mu_3} - |y_1|^{\mu_2+1}|y_2|^{\mu_3}|y_3|^{\mu_1-1} - |y_1|^{\mu_1-1}|y_2|^{\mu_2+1}|y_3|^{\mu_3}$$

$$-|y_1|^{\mu_3}|y_2|^{\mu_2+1}|y_3|^{\mu_1-1} - |y_1|^{\mu_1-1}|y_2|^{\mu_3}|y_3|^{\mu_2+1} - |y_1|^{\mu_3}|y_2|^{\mu_1-1}|y_3|^{\mu_2+1}\Bigg].$$

Proposition 16. *With* $\mathrm{re}(\nu_i) > \frac{1}{2}$ *and* $\mathrm{re}(s)$ *sufficiently large (depending on the* ν_i*)*

$$(41)\qquad \mathbf{RN}\int_{Z_2(\mathbb{A})\,GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})} \phi(g)\,E(g,s)\,dg = \int_{Z_{\mathbb{A}}\,A(\mathbb{Q})\backslash A(\mathbb{A})} H(y)\,\chi(y)\,dy.$$

The integral on the right hand side is convergent.

Compare Proposition 1 in [BB].

Proof. If $\mathrm{re}(s)$ is sufficiently large then $\mathrm{re}(\mu_1-\mu_2), \mathrm{re}(\mu_2-\mu_3) > 1$, and this implies that the $GL(2)$ Eisenstein series appearing in (40) are given by convergent series.

Using (36), we have

$$f(y) = E_0^*(g,\nu_1)\,E_0^*(g,\nu_2)\,E_0^*(g,\nu_3), \qquad g = \begin{pmatrix} y & \\ & 1 \end{pmatrix}$$

and the constant term $a_0(y)$ satisfies

$$(42)\quad a_0(y) - f(y) =$$

$$\sum_{\alpha_1+\alpha_2+\alpha_3=0} W\left(\begin{pmatrix} \alpha_1 y & \\ & 1 \end{pmatrix}, \nu_1\right) W\left(\begin{pmatrix} \alpha_2 y & \\ & 1 \end{pmatrix}, \nu_2\right) W\left(\begin{pmatrix} \alpha_3 y & \\ & 1 \end{pmatrix}, \nu_3\right)$$

$$+ E_0^*(g,\nu_1) \sum_\alpha W\left(\begin{pmatrix} \alpha y & \\ & 1 \end{pmatrix}, \nu_2\right) W\left(\begin{pmatrix} \alpha y & \\ & 1 \end{pmatrix}, \nu_3\right)$$

$$+ E_0^*(g,\nu_2) \sum_\alpha W\left(\begin{pmatrix} \alpha y & \\ & 1 \end{pmatrix}, \nu_3\right) W\left(\begin{pmatrix} \alpha y & \\ & 1 \end{pmatrix}, \nu_1\right)$$

$$+ E_0^*(g,\nu_3) \sum_\alpha W\left(\begin{pmatrix} \alpha y & \\ & 1 \end{pmatrix}, \nu_1\right) W\left(\begin{pmatrix} \alpha y & \\ & 1 \end{pmatrix}, \nu_2\right)$$

We have chosen $\mathrm{re}(s)$ large, so in (8) we may let $T, -T' \to -\infty$ in the renormalized Mellin transform of a_0 in Theorem 15, so

$$(43)\qquad \mathbf{RN} \int_{Z_2(\mathbb{A})\, GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})} \phi(g)\, E(g,s)\, dg = \int_{\mathbb{A}^\times/\mathbb{Q}^\times} (a_0(y) - f(y))\, |y|^{s-1}\, d^\times y.$$

We will show that the right hand side of (41) produces the same terms as substituting (42) into this Mellin transform.

The convergence of the right hand side of (41) will emerge from the proof. Using the Plücker parametrization of $B\backslash GL(3)$, we will break G into pieces, some of which will be paired with some of the remaining terms in H. The sum of all terms agrees with the result of substituting (42) into (43).

Define an involutory automorphism of $GL(3)$ by:

$$^\iota g = J\, {}^t g^{-1}\, J, \qquad J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Let Ψ_1 and Ψ_2 be Schwartz functions on $\mathbb{A}^3$. Define

$$(44)\quad \zeta^*(\mu_1-\mu_2)\,\zeta^*(\mu_2-\mu_3)\, f(g) =$$

$$|\det(g)|^{\mu_2} \int_{\mathbb{A}^\times} \int_{\mathbb{A}^\times} \Psi_1((0,0,t)g)\, \Psi_2((0,0,u)^\iota g)\, |t|^{\mu_2-\mu_3}\, |u|^{\mu_1-\mu_2}\, d^\times t\, d^\times u.$$

This integral is convergent and satisfies

$$f\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} g\right) = |y_1|^{\mu_1}\, |y_2|^{\mu_2}\, |y_3|^{\mu_3}\, f(g).$$

Let us take $\Psi_1 = \Psi_2 = \Psi$ to be the Gaussian element of the Schwartz space, $\Psi(x_1,x_2,x_3) = \psi(x_1)\,\psi(x_2)\,\psi(x_3)$ where ψ is defined in Section 1. In this case f is K-invariant and $f(1) = 1$, so $f = f_{\mu_1,\mu_2,\mu_3}$ defined in (20).

We now recall the Plücker parametrization of $B(\mathbb{Q})\backslash GL(3,\mathbb{Q})$. Let $\gamma \in GL(3,\mathbb{Q})$. Let $(\beta_1,\beta_2,\beta_3)$ and $(\alpha_3,\alpha_2,\alpha_1)$ be the bottom rows of γ and ${}^\iota\gamma$. Then $\beta_1\alpha_1+\beta_2\alpha_2+\beta_3\alpha_3=0$. The vectors $(\beta_1,\beta_2,\beta_3)$ and $(\alpha_3,\alpha_2,\alpha_1)$ only change by a constant multiple if γ is changed on the left by an element of $B(\mathbb{Q})$. Thus the coset $B(\mathbb{Q})\backslash GL(3,\mathbb{Q})$ is parametrized uniquely by a pair of triples $(\beta_1,\beta_2,\beta_3)$ and $(\alpha_3,\alpha_2,\alpha_1)$ in $\mathbb{Q}^\times\backslash(\mathbb{Q}^3-\{0\})$ satisfying the Plücker relation $\sum\beta_i\alpha_i=0$.

We now let $G_{\{\beta_i,\alpha_i\neq 0\}}(g)$ denote the contribution of the terms with Plücker invariants β_i, α_i all nonzero in the Eisenstein series (21). We prove

$$\int_{Z(\mathbb{A})\,A(\mathbb{Q})\backslash A(\mathbb{A})} G_{\{\beta_i,\alpha_i\neq 0\}}(y)\,\chi(y)\,dy = \zeta^*(2s)\int_{\mathbb{A}^\times/\mathbb{Q}^\times}\sum_{\substack{\alpha_i\in\mathbb{Q}^\times\\ \alpha_1+\alpha_2+\alpha_3=0}}|u|^{s-1} \times W\left(\begin{pmatrix}\alpha_1 u & \\ & 1\end{pmatrix},\nu_1\right) W\left(\begin{pmatrix}\alpha_2 u & \\ & 1\end{pmatrix},\nu_2\right) W\left(\begin{pmatrix}\alpha_3 u & \\ & 1\end{pmatrix},\nu_3\right) d^\times u \tag{45}$$

with χ as in Section 2. This is

$$\zeta^*(\mu_1-\mu_3-1)\int_{Z(\mathbb{A})A(\mathbb{Q})\backslash A(\mathbb{A})}\int_{A^\times/\mathbb{Q}^\times}\int_{A^\times/\mathbb{Q}^\times} \sum_{\substack{0\neq\beta_i\in\mathbb{Q}\\ 0\neq\alpha_i\in\mathbb{Q}\\ \beta_1\alpha_1+\beta_2\alpha_2+\beta_3\alpha_3=0}} \psi(\beta_1y_1t)\,\psi(\alpha_1y_1^{-1}u)\,\psi(\beta_2y_2t)\,\psi(\alpha_2y_2^{-1}u)\,\psi(\beta_3y_3t)\,\psi(\alpha_3y_3^{-1}u)$$
$$\times |t|^{\mu_2-\mu_3}\,|u|^{\mu_1-\mu_2}\,|y_1|^{s_1+\mu_2}\,|y_2|^{s_2+\mu_2}\,|y_3|^{s_3+\mu_2}\,d^\times t\,d^\times u\,dy.$$

In (44) we integrated t and u over $\mathbb{A}^\times$, and here we are integrating over $\mathbb{A}^\times/\mathbb{Q}^\times$. This is because we have summed over all nonzero rows $(\alpha_1,\alpha_2,\alpha_3)$ and $(\beta_1,\beta_2,\beta_3)$ subject to the Plücker relation, instead of dividing by the action of $\mathbb{Q}^\times$ in the Plücker parametrization. Now we may drop the integration over t and the summation over the β_i if we integrate over $A(\mathbb{A})$ instead of the quotient. The integral thus becomes

$$\zeta^*(\mu_1-\mu_3-1)\int_{A^\times}\int_{A^\times}\int_{A^\times}\int_{A^\times/\mathbb{Q}^\times} \sum_{\substack{\alpha_i\in\mathbb{Q}^\times\\ \alpha_1+\alpha_2+\alpha_3=0}} \psi(y_1)\,\psi(\alpha_1y_1^{-1}u)\,\psi(y_2)\,\psi(\alpha_2y_2^{-1}u)\,\psi(y_3)\,\psi(\alpha_3y_3^{-1}u)\,|u|^{\mu_1-\mu_2}$$
$$\times |y_1|^{s_1+\mu_2}\,|y_2|^{s_2+\mu_2}\,|y_3|^{s_3+\mu_2}\,d^\times u\,d^\times y_1\,d^\times y_2\,d^\times y_3.$$

We have

$$\int_{\mathbb{A}^\times}\psi(y)\,\psi(uy^{-1})\,|y|^{2\nu-1}\,d^\times y = |u|^{\nu-1}\,W\left(\begin{pmatrix}u & \\ & 1\end{pmatrix},\nu\right). \tag{46}$$

Indeed, the integral factorizes. At the archimedean place this equals

$$\int_0^\infty e^{-\pi(y^2+u^2y^{-2})}\,|y|^{2\nu-1}\,\frac{dy}{y} = |u|^{\nu-1/2}\,K_{\nu-1/2}(2\pi u),$$

and the nonarchimedean integrals are also easily compared with (35). Recalling that $s_1+s_2+s_3=0$, we obtain (45).

Next we have

$$\int_{Z(\mathbb{A})\,A(\mathbb{Q})\backslash A(\mathbb{A})} G_{\{\alpha_1=0,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\neq 0\}}(y)\,\chi(y)\,dy = \zeta^*(2s)\int_{\mathbb{A}^\times/\mathbb{Q}^\times} |u|^{s-1}\times \tag{47}$$

$$\zeta^*(2\nu_1-1)\,|u|^{1-\nu_1}\sum_{\alpha\in\mathbb{Q}^\times} W\left(\begin{pmatrix}\alpha u & \\ & 1\end{pmatrix},\nu_2\right) W\left(\begin{pmatrix}\alpha u & \\ & 1\end{pmatrix},\nu_3\right)\,du.$$

Indeed, this is proved like (45), making use of

$$\int_{\mathbb{A}^\times} \psi(y)\,|y|^\nu\,d^\times y = \zeta^*(\nu). \tag{48}$$

Next we prove that

$$\int_{Z(\mathbb{A})\,A(\mathbb{Q})\backslash A(\mathbb{A})} \Bigg[G_{\{\beta_1=0,\alpha_2,\alpha_3,\beta_2,\beta_3\neq 0\}}(y) \tag{49}$$

$$-\,\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_2+1}E_{\mu_1-1,\mu_3}\begin{pmatrix} y_2 & \\ & y_3\end{pmatrix}$$

$$+\,|y_1|^{\mu_2+1}\,|y_2|^{\mu_1-1}\,|y_3|^{\mu_3}+|y_1|^{\mu_2+1}\,|y_2|^{\mu_3}\,|y_3|^{\mu_1-1}\Bigg]\chi(y)\,dy=$$

$$\zeta^*(2s)\int_{\mathbb{A}^\times/\mathbb{Q}^\times}\sum_{\alpha\in\mathbb{Q}}|u|^{s-1}\,\zeta^*(2\nu_1)\,|u|^{\nu_1}\,W\left(\begin{pmatrix}\alpha u & \\ & 1\end{pmatrix},\nu_2\right) W\left(\begin{pmatrix}\alpha u & \\ & 1\end{pmatrix},\nu_3\right)\,du.$$

We note that by Poisson summation

$$\sum_{\alpha_1\in\mathbb{Q}}\psi(\alpha_1 y_1^{-1}u) = |y_1u^{-1}|\sum_{\alpha_1\in\mathbb{Q}}\psi(\alpha_1 y_1 u^{-1}).$$

So we get

$$\zeta^*(\mu_1-\mu_3-1)\int_{A^\times/\mathbb{Q}^\times}\int_{A^\times/\mathbb{Q}^\times} \tag{50}$$

$$\sum_{\alpha_1\in\mathbb{Q}}\ \sum_{\substack{0\neq\beta_2,\beta_3\in\mathbb{Q}\\ 0\neq\alpha_2,\alpha_3\in\mathbb{Q}\\ \beta_2\alpha_2+\beta_3\alpha_3=0}} \psi(\alpha_1y_1u^{-1})\,\psi(\beta_2y_2t)\,\psi(\alpha_2y_2^{-1}u)\,\psi(\beta_3y_3t)\,\psi(\alpha_3y_3^{-1}u)\,|y_1u^{-1}|$$

$$\times\,|t|^{\mu_2-\mu_3}\,|u|^{\mu_1-\mu_2}\,|y_1|^{s_1+\mu_2}\,|y_2|^{s_2+\mu_2}\,|y_3|^{s_3+\mu_2}\,d^\times t\,d^\times u.$$

We consider first the contribution when $\alpha_1=0$. Collapse the integrations with the sum over α_3, β_2. Thus $\alpha_2+\beta_3=0$. Denoting $\alpha=\alpha_2=-\beta_3$, making the variable change

$u \to y_2 y_3 u$ and noting that ψ is even we get

$$\zeta^*(\mu_1 - \mu_3 - 1) \int_{A^\times} \int_{A^\times} \sum_{\alpha \in \mathbb{Q}^\times} \psi(y_2 t)\, \psi(\alpha y_3 u)\, \psi(\alpha y_3 t)\, \psi(y_2 u) \times |t|^{\mu_2 - \mu_3}\, |u|^{\mu_1 - \mu_2 - 1}\, |y_1|^{s_1 + \mu_2 + 1} |y_2|^{s_2 + \mu_1 - 1}\, |y_3|^{s_3 + \mu_1 - 1}\, d^\times u\, d^\times t.$$

Using (13) and noting

$$f_{\theta_1 + \theta_1', \theta_2 + \theta_2'}(g) = f_{\theta_1, \theta_2}(g)\, f_{\theta_1', \theta_2'}(g),$$

we argue in the proof of (14) to obtain

$$\zeta^*(\theta_1 - \theta_2)\, \zeta^*(\theta_1' - \theta_2') \left[E_{\theta_1 + \theta_1', \theta_2 + \theta_2'} \begin{pmatrix} y_2 & \\ & y_3 \end{pmatrix} - |y_2|^{\theta_1 + \theta_1'} |y_3|^{\theta_2 + \theta_2'} - |y_3|^{\theta_1 + \theta_1'} |y_2|^{\theta_2 + \theta_2'} \right] =$$
$$|y_2 y_3|^{\theta_1 + \theta_1'} \sum_{\mathbb{Q}^\times} \int_{\mathbb{A}^\times} \int_{\mathbb{A}^\times} \psi(t y_2)\, \psi(t y_3 \alpha)\, \psi(u y_2)\, \psi(u y_3 \alpha) |t|^{\theta_1 - \theta_2}\, |u|^{\theta_1' - \theta_2'}\, d^\times t\, d^\times u.$$

Applying this with $\theta_1 = -\mu_3$, $\theta_2 = -\mu_2$, $\theta_1' = -\mu_2 - 1$ and $\theta_2' = -\mu_1$ shows that the contribution with $\alpha_1 = 0$ exactly cancels the last three terms in brackets on the left side of (49).

If $\alpha_1 \neq 0$ then, integrating y over $Z(\mathbb{A})\, A(\mathbb{Q}) \backslash A(\mathbb{A})$, we may collapse the summation over $\alpha_1, \beta_2, \beta_3$ and the integration over t, replacing the integral over $Z(\mathbb{A})\, A(\mathbb{Q}) \backslash A(\mathbb{A})$ by an integral over all of $\mathbb{A}$. Using (46) and (48) and letting $\alpha = \alpha_2 = -\alpha_3$ we get the right hand side of (49).

Taking (45), (47), (49) and four other identities obtained by permuting the indices cyclicly in the (47), (49) accounts for most of G in (40), and all of the terms in the second set of square brackets. The remaining terms of G in (40) correspond to the terms in the first set of square brackets. Indeed, the terms in G still not accounted for consist of the contributions where all $\alpha_1 \beta_1 = \alpha_2 \beta_2 = \alpha_3 \beta_3 = 0$. Recall that at least one of the α_i and at least one of the β_i must be nonzero. Grouping these terms as $\alpha_1 = \alpha_2 = \beta_3 = 0$ etc. produces six Eisenstein series. However there will be overlap in the six terms such as $\alpha_1 = \alpha_2 = \beta_2 = \beta_3 = 0$, and subtracting these overcounted terms produces the first expression in brackets. This completes the proof of Proposition 16. ∎

Proposition 17. *Assume that* $\mathrm{re}(\mu_2) > 0$ *and that* $\mathrm{re}(\mu_1 - \mu_2)$ *is sufficiently large (depending on* μ_2*). Then*

$$\mathbf{RN} \int_{Z(\mathbb{A})\ A(\mathbb{Q}) \backslash A(\mathbb{A})} G(y)\, \chi(y)\, dy = \int_{Z(\mathbb{A})\ A(\mathbb{Q}) \backslash A(\mathbb{A})} H(y)\, \chi(y)\, dy.$$

Proof. We take $T = 0$ in the definition of the renormalized integral. We find some cancellation in $H - \Lambda^0 G$. Of course the Eisenstein series G cancels and for each remaining term where there is cancellation. However the cancellation is only partial because the term appears in Λ^0 with a coefficient of $\hat{\tau}_P(\sigma y)$, $\hat{\tau}_Q(\sigma y)$ or $\hat{\tau}_B(\sigma y)$ with $\sigma \in W$, while in H it appears with no such coefficient. We therefore use the identities $1 - \hat{\tau}_P(y) = \hat{\tau}_Q(\sigma_1\sigma_2 y)$, $1 - \hat{\tau}_Q(y) = \hat{\tau}_P(\sigma_2\sigma_1 y)$, $1 - \hat{\tau}_B(y) = \hat{\tau}_B(\sigma_1\sigma_2 y) + \hat{\tau}_B(\sigma_1\sigma_2 y)$. This results in an expression containing 18 Eisenstein series, half with $+$ and half with $-$, and 48 monomials, half with $+$ and half with $-$. We may group these as

$$
\begin{aligned}
(51)\quad & \Big[\hat{\tau}_P(y)\zeta^*(\mu_1-\mu_2-1)\,\zeta^*(\mu_1-\mu_3-2)\,|y_3|^{\mu_1-2}\,E^*_{\mu_2+1,\mu_3+1}\begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} \\
& - \hat{\tau}_B(y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3)|y_1|^{\mu_2+1}|y_2|^{\mu_3+1}|y_3|^{\mu_1-2} \\
& - \hat{\tau}_B(\sigma_1 y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3)|y_1|^{\mu_3+1}|y_2|^{\mu_2+1}|y_3|^{\mu_1-2}\Big] \\
& + \Big[\hat{\tau}_P(y)\zeta^*(\mu_1-\mu_2)\,\zeta^*(\mu_2-\mu_3-1)\,|y_3|^{\mu_2-1}\,E^*_{\mu_1,\mu_3+1}\begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} \\
& - \hat{\tau}_B(y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_1}|y_2|^{\mu_3+1}|y_3|^{\mu_2-1} \\
& - \hat{\tau}_B(\sigma_1 y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_3+1}|y_2|^{\mu_1}|y_3|^{\mu_2-1}\Big] \\
& + \Big[\hat{\tau}_Q(y)\zeta^*(\mu_1-\mu_3-2)\,\zeta^*(\mu_2-\mu_3-1)\,|y_1|^{\mu_3+2}\,E^*_{\mu_1-1,\mu_2-1}\begin{pmatrix} y_2 & \\ & y_3 \end{pmatrix} \\
& - \hat{\tau}_B(y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_3+2}|y_2|^{\mu_1-1}|y_3|^{\mu_2-1} \\
& - \hat{\tau}_B(y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_3+2}|y_2|^{\mu_2}|y_3|^{\mu_1-2} \\
& - \hat{\tau}_B(\sigma_2 y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_3+2}|y_2|^{\mu_2-1}|y_3|^{\mu_1-1} \\
& - \hat{\tau}_B(\sigma_2 y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3-1)|y_1|^{\mu_3+2}|y_2|^{\mu_1-2}|y_3|^{\mu_2}\Big],
\end{aligned}
$$

together with the results of cyclicly permuting the y_i, as well as

$$
\begin{aligned}
(52)\quad & -\Big[\hat{\tau}_P(y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)|y_3|^{\mu_1}E^*_{\mu_2,\mu_3}\begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} \\
& - \hat{\tau}_B(y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_2}|y_2|^{\mu_3}|y_3|^{\mu_1} \\
& - \hat{\tau}_B(\sigma_1 y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_3}|y_2|^{\mu_2}|y_3|^{\mu_1}\Big] \\
& - \Big[\hat{\tau}_P(y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_2-\mu_3)|y_3|^{\mu_2+1}E^*_{\mu_1-1,\mu_3}\begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} \\
& - \hat{\tau}_B(y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_1-1}|y_2|^{\mu_3}|y_3|^{\mu_2+1} \\
& - \hat{\tau}_B(\sigma_1 y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_3}|y_2|^{\mu_1-1}|y_3|^{\mu_2+1}\Big] \\
& - \Big[\hat{\tau}_Q(y)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_3}E^*_{\mu_1,\mu_2}\begin{pmatrix} y_2 & \\ & y_3 \end{pmatrix} \\
& - \hat{\tau}_B(\sigma_2 y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_3}|y_2|^{\mu_2}|y_3|^{\mu_1} \\
& - \hat{\tau}_B(\sigma_2 y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_3}|y_2|^{\mu_1-1}|y_3|^{\mu_2+1} \\
& - \hat{\tau}_B(y)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_3}|y_2|^{\mu_1}|y_3|^{\mu_2} \\
& - \hat{\tau}_B(y)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)|y_1|^{\mu_3}|y_2|^{\mu_2+1}|y_3|^{\mu_1-1}\Big]
\end{aligned}
$$

and the terms resulting from cyclicly permuting the y_i. Grouped this way, the integral of each term in brackets in (51) or (52) against χ is convergent, and in fact may be evaluated by adapting Proposition 7 and moving U to infinity. Thus (51) produces

$$
\begin{aligned}
(53)\quad & \zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)B_P^1(\Phi_P^{\mu_2+1,\mu_3+1,\mu_1-2},\chi,0) \\
&+\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)}{(\mu_2-\mu_3+s_1-s_2)(\mu_1-\mu_3-3+s_3-s_2)}+\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)}{(\mu_2-\mu_3-s_1+s_2)(\mu_1-\mu_3-3-s_1+s_3)} \\
&+\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_2-\mu_3-1)B_P^1(\Phi_P^{\mu_1,\mu_3+1,\mu_2-1},\chi,0) \\
&+\frac{6\zeta^*(\mu_2-\mu_3-1)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)}{(\mu_1-\mu_3-1+s_1-s_2)(\mu_2-\mu_3-2+s_3-s_2)}+\frac{6\zeta^*(\mu_2-\mu_3-1)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)}{(\mu_1-\mu_3-1-s_1+s_2)(\mu_2-\mu_3-2-s_1+s_3)} \\
&+\zeta^*(\mu_1-\mu_3-2)\zeta^*(\mu_2-\mu_3-1)B_Q^1(\Phi_Q^{\mu_3+2,\mu_1-1,\mu_2-1},\chi,0) \\
&+\frac{6\zeta^*(\mu_2-\mu_3-1)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-2)}{(\mu_3-\mu_1+3+s_1-s_2)(\mu_2-\mu_1-s_2+s_3)}+\frac{6\zeta^*(\mu_2-\mu_3-1)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-2)}{(\mu_3-\mu_1+3+s_1-s_3)(\mu_2-\mu_1+s_2-s_3)} \\
&+\frac{6\zeta^*(\mu_2-\mu_3-1)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)}{(\mu_3-\mu_2+2+s_1-s_2)(\mu_1-\mu_2-2-s_2+s_3)}+\frac{6\zeta^*(\mu_2-\mu_3-1)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-2)}{(\mu_3-\mu_2+2+s_1-s_3)(\mu_1-\mu_2-2+s_2-s_3)}
\end{aligned}
$$

while (52) yields

$$
\begin{aligned}
(54)\quad & -\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)B_P^1(\Phi_P^{\mu_2,\mu_3,\mu_1},\chi,0) \\
&-\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)}{(\mu_2-\mu_3+s_1-s_2)(\mu_1-\mu_3+s_3-s_2)}-\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)}{(\mu_2-\mu_3-s_1+s_2)(\mu_1-\mu_3-s_1+s_3)} \\
&-\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_2-\mu_3)B_P^1(\Phi_P^{\mu_1-1,\mu_3,\mu_2+1},\chi,0) \\
&-\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-1)}{(\mu_1-\mu_3-1+s_1-s_2)(\mu_2-\mu_3+1+s_3-s_2)}-\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-1)}{(\mu_1-\mu_3-1-s_1+s_2)(\mu_2-\mu_3+1-s_1+s_3)} \\
&-\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_3-1)B_Q^1(\Phi_Q^{\mu_3,\mu_1,\mu_2},\chi,0) \\
&-\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)}{(\mu_3-\mu_1+s_1-s_2)(\mu_2-\mu_1-s_2+s_3)}-\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2)\zeta^*(\mu_1-\mu_3-1)}{(\mu_3-\mu_1+s_1-s_3)(\mu_2-\mu_1+s_2-s_3)} \\
&-\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-1)}{(\mu_3-\mu_2-1+s_1-s_2)(\mu_1-\mu_2-2-s_2+s_3)}-\frac{6\zeta^*(\mu_2-\mu_3)\zeta^*(\mu_1-\mu_2-1)\zeta^*(\mu_1-\mu_3-1)}{(\mu_3-\mu_2-1+s_1-s_3)(\mu_1-\mu_2-2+s_2-s_3)}
\end{aligned}
$$

In addition to (53) and (54) we have the terms gotten by cyclicly permuting the s_i, as one may see from a change of variables.

These remaining terms must equal the terms

$$B^0(G,\chi,T)+\sum_{w\in W_P\backslash W} B_P^1(G,{}^w\chi,T)+\sum_{w\in W_Q\backslash W} B_Q^1(G,{}^w\chi,T)$$

to obtain cancellation when these are taken into account in the definition (25). To see that the B^1 terms match, it is necessary to make use of the identity

$$B_P^1(\Phi_P^{\theta_1,\theta_2,\theta_3},\chi,0)=-B_Q^1(\Phi_Q^{\theta_3,\theta_1,\theta_2},{}^{\sigma_1\sigma_2}\chi,0).$$

To see that the B_0 terms match, we note that these *mostly* match but we have 48 such terms where we should only have 36. The discrepancy is accounted for by identities such as

$$\frac{1}{(\mu_2-\mu_3+s_1-s_2)(\mu_1-\mu_3+s_3-s_2)}-\frac{1}{(\mu_1-\mu_3+s_3-s_2)(\mu_2-\mu_1-s_3+s_1)}$$
$$=\frac{1}{(\mu_2-\mu_1-s_3+s_1)(\mu_2-\mu_3+s_1-s_2)}.$$

This completes the proof of Proposition 17. ■

Theorem 18. *With ϕ as in (39), choose ν_i and s so that $\mathrm{re}(\nu_i) > \frac{1}{2}$ and $\mathrm{re}(s)$ is sufficiently large. Then*

$$\mathbf{RN}\int_{Z_2(\mathbb{A})\,GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})}\phi(g)\,E(g,s)\,dg=\mathbf{RN}\int_{Z_{\mathbb{A}}\,A(\mathbb{Q})\backslash A(\mathbb{A})}G(y)\,\chi(y)\,dy,$$

the integral on the right side being absolutely convergent.

Proof. This follows from Proposition 16 and Proposition 17. ■

This implies Theorem 1. Indeed, using the functional equation $E^*(g,s)=E^*(g,1-s)$, a special case of (11), the functional equation in (Theorem 1) corresponds to $\nu_1\to 1-\nu_1$, with s, ν_2 and ν_3 unchanged. Other functional equations may be obtained by permuting s, ν_1, ν_2 and ν_3 and combining these with the symmetries of the $GL(3)$ integral gives the full group of symmetries of the polar divisor, a group of order 1152.

References

[A] J. Arthur, A trace formula for reductive groups, I. Terms associated to classes in $G(Q)$, *Duke Math. J.* **45** (1978), 911–952.

[B] D. Bump, *Automorphic Forms and Representations*, Cambridge (1997).

[BB] D. Bump and J. Beineke, Hidden symmetries for a renormalized integral of Eisenstein series, *Forum Math.* 14 (2002), 457–475.

[BG] D. Bump and D. Goldfeld, A Kronecker limit formula for cubic fields, in *Modular forms*, R. Rankin ed. (1984), 43–49.

[GJ] R. Godement and H. Jacquet, *Zeta Functions of simple algebras*, Springer Lecture Notes in Mathematics **260** (1972).

[JLR] H. Jacquet, E. Lapid and J. Rogawski, Periods of automorphic forms, *J. Amer. Math. Soc.* **12** (1999), 173–240.

[K] S. Kudla, Seesaw dual reductive pairs in *Automorphic forms of several variables*, I. Satake and Y. Morita ed., Birkhäuser Progr. Math. **46** (1984), 244–268.

[MW] C. Moeglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Une paraphrase de l'Écriture, Cambridge (1995).

[W] J. Woodson, *The dual pair $GL \times GL$*, Stanford dissertation (1994).

[Z] D. Zagier, The Rankin-Selberg method for automorphic functions which are not of rapid decay. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981), 415–437 (1982).

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