Unitary Correlations and the Fejér Kernel

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Abstract. *Let M be a unitary matrix with eigenvalues t_j, and let f be a function on the unit circle. Define $X_f(M) = \sum f(t_j)$. We derive exact and asymptotic formulae for the covariance of X_f and X_g with respect to the measures $|\chi(M)|^2\,dM$ where dM is Haar measure and χ an irreducible character. The asymptotic results include an analysis of the Fejér kernel which may be of independent interest.*

1. Introduction. Random matrix theory uses the eigenvalues of typical large matrices as models of a wide variety of natural phenomena from nuclear spectra to the zeros of the Riemann zeta function. Surveys of this area may be found in Mehta [M], Tracy and Widom [TW1] or Hejhal, Friedman, Gutzwiller and Odlyzko [HFGO].

We study the eigenvalues of typical unitary matrices M. These are n points t_j on $\mathbb{T}$, the unit circle. Sometimes we will write $t_j = e^{i\theta_j}$ with $0 \le \theta_j < 2\pi$. The distribution of the traces of powers of M is studied in Diaconis and Shahshahani [DS], the number of eigenvalues in an interval is studied in Wieand [W], and the log characteristic polynomial is studied in Keating and Snaith [KS1], [KS2] and in Hughes, Keating and O'Connell [HKO]. These are all examples of additive functions of M, that is, functions of the form

$$X_f(M) = \sum_{j=1}^{n} f(t_j),$$

where $f : \mathbb{T} \to \mathbb{C}$ is a function. The limiting behavior of such functionals is studied in the papers cited above as well as in Diaconis and Evans [DE] and Soshnikov [So1], [So2], where a variety of central limit theorems are proved.

In this paper we study the variance and covariance of X_f and X_g. We are interested not merely in the limiting behavior, but in the behavior for n of moderate size.

We will normalize Haar measures $\int_{\mathbb{T}} dt$ and $\int_{U(n)} dM$ so that the compact groups $\mathbb{T}$ and $U(n)$ have volume 1. The convolution of two functions on $\mathbb{T}$ is defined by

$$(f * g)(x) = \int_{\mathbb{T}} f(t)\, g(xt^{-1})\, dt.$$

The *Fejér kernel*

$$K_n(t) = \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) t^k = \frac{\sin\left(\frac{n\theta}{2}\right)^2}{n \sin\left(\frac{\theta}{2}\right)^2}, \qquad t = e^{i\theta}. \tag{1}$$

Fejér introduced this kernel to prove that Fourier coefficients determine the function at a point under suitable conditions. See Körner [Ko] for background, and Zygmund [Z] pp. 88–90 for the classical theory.

If Φ and Ψ are functions on $U(n)$, define the covariance

$$\mathrm{Cov}(\Phi,\Psi) = \int_{U(n)} \Phi(M)\,\Psi(M)\,dM - \overline{\Phi}\,\overline{\Psi},$$

where the mean value

$$\overline{\Phi} = \int_{U(n)} \Phi(M)\,dM.$$

Let $\widetilde{g}(t) = g(t^{-1})$.

Theorem 1. *For $f, g \in L^2(\mathbb{T})$,*

(2) $$\mathrm{Cov}(X_f, X_g) = n(f * \tilde{g})(1) - n(f * \tilde{g} * K_n)(1).$$

It is striking that this covariance only depends on $f * \widetilde{g}$.

We will give two proofs of Theorem 1, in sections 2 and 3. We will see in Section 2 that Theorem 1 is equivalent to an icon of random matrix theory, a formula of Dyson for the pair correlation function. We will generalize Theorem 1 in Theorem 4 below to obtain covariances with respect to the measure $|\chi(M)|^2\,dM$, where χ is an irreducible character of $U(n)$ by a very similar formula.

The Fejér kernel is *a Dirac sequence*. This means that $K_n \geq 0$, $\int_{\mathbb{T}} K_n(t)\,dt = 1$ and $K_n \to 0$ uniformly on any compact subset of $\mathbb{T}$ excluding the point 1. Consequently the sequence of trigonometric polynomials $\phi * K_n \to \phi$ uniformly for continuous functions ϕ on $\mathbb{T}$. Taking $\phi = f * \widetilde{g}$ we see that the covariance (2) may be interpreted as the error in the approximation of ϕ by $\phi * K_n$.

Suppose that c_k and d_k are the Fourier coefficients of f and g, so that $f(t) = \sum c_k t^k$ and $g(t) = \sum d_k t^k$. Recalling that convolution of functions corresponds to multiplication of the Fourier coefficients, we may write (2) in the equivalent form:

(3) $$\mathrm{Cov}(X_f, X_g) = n \sum_{k=-\infty}^{\infty} c_k\, d_{-k} - \sum_{k=-(n-1)}^{n-1} (n - |k|)\, c_k\, d_{-k}.$$

As an example, we consider the case where f and g are the characteristic functions of two intervals. Thus X_f and X_g count the number of eigenvalues in each interval. Wieand [W] showed that in the limit as $n \to \infty$, these functions are uncorrelated unless the intervals share an endpoint. She found that if they share a left or right endpoint, then there is a positive limiting correlation, but if the left endpoint of one interval is a right endpoint of the other, then there is a negative limiting correlation.

Meanwhile Rains [R] also considered the number of eigenvalues in an interval; he found the complete asymptotic expansion for the variance. It is possible to go from Rains' asymptotic results on the variance of the number of eigenvalues in an interval I to an asymptotic result on the covariance of the number of eigenvalues in two different intervals I and J. The work involved in this process is not entirely trivial, since the results of [R] would need to be applied to several intervals: if K is a connected component of the symmetric difference $I \triangle J$ of I and J, then one needs to know the variances in the number of eigenvalues in the six intervals I, J, K, $I \triangle K$, $J \triangle K$ and $I \triangle (J \triangle K) = (I \triangle J) \triangle K$. By an inclusion exclusion process, one may infer the covariance of the number of eigenvalues in the intervals I and J.

We can consider this matter from our point of view using Theorem 1, recovering and extending these results of Weiand and Rains. As an example, let $0 \le \omega \le 2\pi$, and let f and g be the characteristic functions of the images under $\theta \to e^{i\theta}$ of the intervals $[0, \omega]$ and $[\theta, \omega + \theta]$, respectively. We have

$$c_k = e^{ik\theta} d_k = \begin{cases} \frac{1}{2\pi i k}(1 - e^{-ik\omega}) & \text{if } k \neq 0 \\ \frac{\omega}{2\pi} & \text{if } k = 0. \end{cases}$$

Let $\phi = f * \widetilde{f}$. Thus

$$\phi(e^{i\theta}) = \frac{\max(\omega - \theta, 0) + \max(\omega + \theta - 2\pi, 0)}{2\pi}. \tag{4}$$

Because $\big(f * \widetilde{g}\big)(1) = \phi(e^{i\theta})$ and $\big(f * \widetilde{g} * K_n\big)(1) = \big(\phi * K_n\big)(e^{i\theta})$, Theorem 1 shows that $\mathrm{Cov}(X_f, X_g) = n(\phi - \phi * K_n)(e^{i\theta})$, and it is this that we study. The graph of $\phi(e^{i\theta})$ in the case $\omega = \frac{\pi}{2}$ is shown in Figure 1.

Figure 1. $\phi(e^{i\theta})$ as a function of θ for $\omega = \frac{\pi}{2}$.

The series (3) simplifies to:

$$\mathrm{Cov}(X_f, X_g) = n\,(\phi - \phi * K_n)(e^{i\theta}) = n\,\phi(e^{i\theta}) - n\left(\frac{\omega}{2\pi}\right)^2 - \sum_{k=1}^{n-1} \frac{(n-k)}{\pi^2 k^2}\big(1 - \cos(k\omega)\big)\cos(k\theta). \tag{5}$$

This formula is suitable for numerical computation. For $n = 42$ and $\omega = \frac{\pi}{2}$, Figure 2 shows the graph of this quantity as a function of θ.

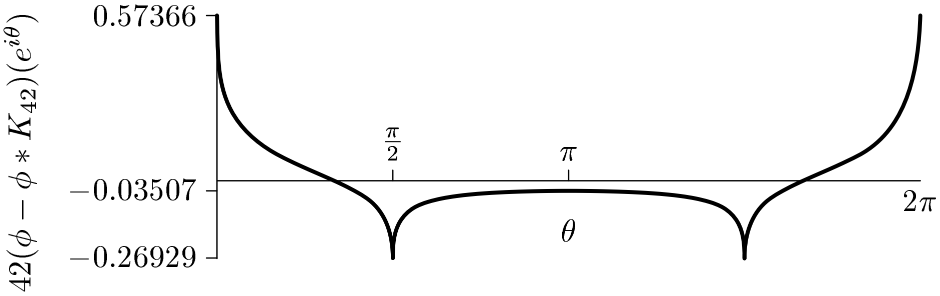

Figure 2. The function $n(\phi - \phi * K_n)$ when $n = 42$.

Figure 2 displays the covariance between the number of eigenvalues of a random unitary matrix in two given intervals of length $\frac{\pi}{2}$ as one of the intervals slides around the circle. We see that the places where (5) has the largest magnitude are the locations of the discontinuities in the derivative ϕ'. We will justify this qualitative observation in Section 4 by an asymptotic analysis of $\phi - K_n * \phi$ for a function ϕ which (like this one) is continuous, but whose derivative has jump discontinuities.

Let Ci and Si be the cosine and sine integrals,

$$\mathrm{Ci}(x) = \gamma + \log(x) + \int_0^x \frac{\cos(t) - 1}{t}\, dt = -\int_x^\infty \frac{\cos(t)}{t}\, dt, \qquad |\arg(x)| < \pi, \tag{6}$$

where $\gamma = 0.57721\cdots$ is Euler's constant, and

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt = \frac{\pi}{2} - \int_x^\infty \frac{\sin(t)}{t}\, dt, \qquad |\arg(x)| < \pi. \tag{7}$$

The asymptotic expansion of Ci as $x \to \infty$, obtained from the last expression in (6) by integration by parts, is

$$\mathrm{Ci}(x) \sim \frac{\sin(x)}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \dots\right) - \frac{\cos(x)}{x^2}\left(1 - \frac{3!}{x^2} + \frac{5!}{x^4} - \dots\right). \tag{8}$$

Similarly

$$\mathrm{Si}(x) \sim \frac{\pi}{2} - \frac{\sin(x)}{x^2}\left(1 - \frac{3!}{x^2} + \frac{5!}{x^4} - \dots\right) - \frac{\cos(x)}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \dots\right). \tag{9}$$

If $|\theta| \le \pi$, let

$$\Phi_n(\theta) = \begin{cases} 2\big(1 + \gamma + \log(2n)\big) & \text{if } \theta = 0, \\ 2\,\mathrm{Ci}(n|\theta|) - 2\,\log\sin\!\left(\frac{|\theta|}{2}\right) + 2\,\cos(n\theta) - n\pi|\theta| + 2\,n\theta\,\mathrm{Si}(n|\theta|) & \text{otherwise.} \end{cases} \tag{10}$$

By (8) and (9) this function is continuous at $\theta = 0$. We make Φ_n into a 2π periodic function. The function Φ_n is "spiky," increasingly so as n increases. Indeed, (8) and (9) show that

$$\lim_{n\to\infty} \Phi_n(\theta) = -2\log\sin(|\theta|/2) \tag{11}$$

$n \to \infty$. The graph of Φ_5 is shown in Figure 3.

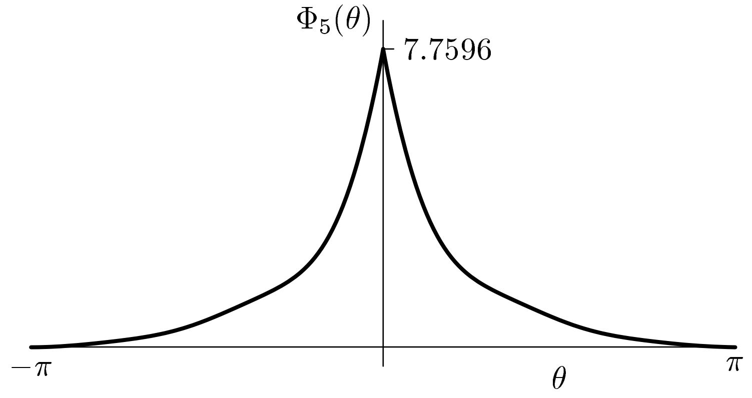

Figure 3. The function Φ_5.

We can now describe the results of our asymptotic analysis of $\phi - K_n * \phi$, which are given in Theorems 9 and 10 below. The most important and interesting contributions to these come from the jump discontinuities in the derivative of ϕ. Indeed, we will show that if $\theta \to \phi(e^{i\theta})$ has a jump discontinuity at $\theta = \theta_0$, then the asymptotic form of $n(\phi - K_n * \phi)(e^{i\theta})$ contains a constant multiple of $\Phi_n(\theta - \theta_0)$.

When ϕ is the function (4), the derivative jumps at $\theta = 0$, ω and $2\pi - \omega$, and Theorem 10 shows that

$$\mathrm{Cov}(X_f, X_g) = n(\phi - K_n * \phi)(\theta) = \\ \frac{1}{2\pi^2}\Phi_n(\theta) - \frac{1}{4\pi^2}\Phi_n(\theta - \omega) - \frac{1}{4\pi^2}\Phi_n(\theta - 2\pi + \omega) + O(n^{-1})$$

uniformly in θ. In practice this approximation is quite good. For $n = 42$, the graph of the approximation is indistinguishable from Figure 2, for its values agree with those of the original function to about five decimal places. The largest error is less than 1.4×10^{-5}.

Theorem 1, together with (11) and Theorem 9 show that as $n \to \infty$ the covariance $\mathrm{Cov}(X_f, X_g)$ tends to a limiting distribution which is singular at the discontinuities of the derivative of ϕ. In the example of Figure 1, (11) shows that the limiting covariance equals

$$\frac{1}{2\pi^2}\log\left|\frac{\sin\left(\frac{\theta-\pi/2}{2}\right)\sin\left(\frac{\theta-3\pi/2}{2}\right)}{\sin^2(\theta/2)}\right|.$$

The graph of this function is shown in Figure 4.

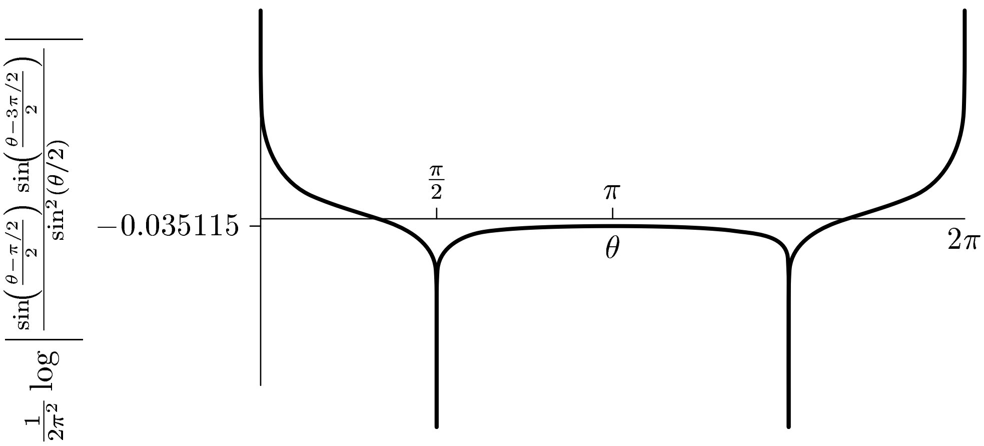

Figure 4. The limiting covariance as $n \to \infty$ for Figure 1.

We will generalize Theorem 1 by obtaining covariances with respect to the probability measure $|\chi_\lambda(M)|^2\, dM$, where λ is a partition of length $\leq n$, and χ_λ is the character of the irreducible representation indexed by λ, defined in (16) below. Denoting this covariance as Cov_λ, we will show in Theorem 4 that if λ is fixed, there is a Dirac sequence $K_{n,\lambda}$ such that

$$\mathrm{Cov}_\lambda(X_f, X_g) = n(f * \tilde{g} - f * \tilde{g} * K_{n,\lambda})(1).$$

For example if $\lambda = (1)$, that is, the partition $(1, 0, 0, \cdots)$, then $\chi_\lambda(M) = \mathrm{tr}(M)$ is just the character of the standard representation of $U(n)$. When $n = 42$ and ϕ is the function (4) with $\omega = \frac{\pi}{2}$, the graph of this covariance is shown in Figure 5.

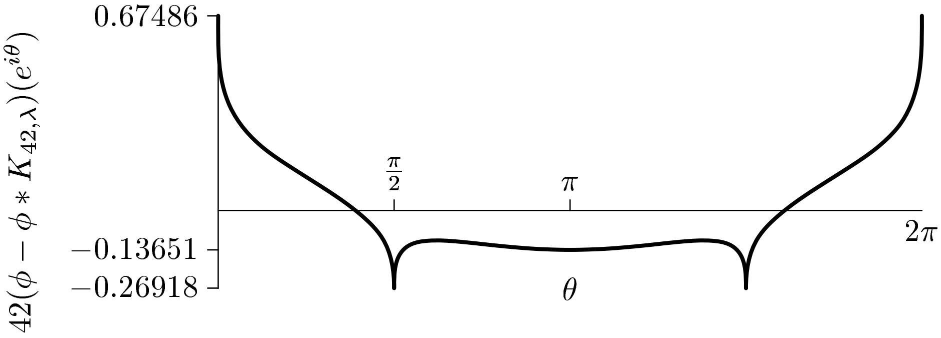

Figure 5. The function $n(\phi - \phi * K_{n,\lambda})$ when $n = 42$, $\lambda = (1)$.

We will show in Theorem 11 that the functions Φ_n can be generalized to functions $\Phi_{n,\lambda}$ giving in a similar way the asymptotic forms of $\mathrm{Cov}_\lambda(X_f, X_g)$, at least when f and g are piecewise linear and continuous. We do not work out the exact form of $\Phi_{n,\lambda}$, though we give the interested reader enough information to do so in the proofs of Theorems 3 and 11. We find that $\Phi_{n,\lambda}(\theta) - \Phi_n(\theta)$ is a trigonometric polynomial independent of n. For example, in the case where $\lambda = (1)$, we find that

$$\Phi_{n,\lambda}(\theta) - \Phi_n(\theta) = 2\cos(\theta).$$

This result shows that if λ is fixed and $n \to \infty$ the covariance $\mathrm{Cov}_\lambda(X_f, X_g)$ tends to a limiting distribution, which is finite except at the discontinuities of the derivative of ϕ.

There has been some related work since the circulation of a preliminary version of this paper. Hughes [Hu] derives related approximations in comparing the variance of the number of eigenvalues and zeta zeros in matching intervals. In the matter of Fejér asymptotics, Pinsky [P] recognized the correction term in Theorem 7 as the Hilbert transform of f'. He derives L^2 convergence results for a variety of summability kernels and extends his results from the circle to the line. Taylor [T] gives a version of Pinsky's results for a function on an n-dimensional Riemannian manifold expanded in eigenfunctions of the Laplacian. He further extends our results to uniform asymptotics for piecewise smooth f with a simple jump across a smooth hypersurface. We thank these authors for keeping us informed.

2. Pair Correlation. The formula (2) is equivalent to a basic formula of unitary statistics, Dyson's formula for the correlation function for unitary eigenvalues. The correlation functions were found by Dyson in part III of [Dy1]; Dyson gave a proof and a generalization to other ensembles in [Dy2]; see also Mehta [M], and Tracy and Widom [TW1] for different proofs and extensions.

The m-level correlation $R_m(t_1, \cdots, t_m)$ measures the density that $t_1, \cdots, t_m$ are the eigenvalues of a Haar random unitary matrix. Concretely, if $f(t_1, \cdots, t_m)$ is a test function on $\mathbb{T}^m$, then

$$\int_{\mathbb{T}^m} R_m(u_1, \cdots, u_m)\, f(u_1, \cdots, u_m)\, du_1 \cdots du_m = \int_{U(n)} \sum\nolimits^* f(t_{i_1}, \cdots, t_{i_m})\, dM, \tag{12}$$

where the sum on the right is over the $n!/(n-m)!$ different m-tuples $(i_1, \cdots, i_m)$ where the i_j are pairwise distinct integers between 1 and n and where $t_1, \cdots, t_n$ are the eigenvalues of M.

Dyson found that

$$R_m(t_1,\cdots,t_m)=\det\big(s_n(\theta_j-\theta_k)\big)_{j,k},\qquad t_j=e^{i\theta_j}, \tag{13}$$

where

$$s_n(\theta)=\begin{cases}\frac{\sin(n\theta/2)}{\sin(\theta/2)} & \theta\neq 0;\\ n & \theta=0.\end{cases}$$

In the case $m=2$ (the only case we need), this amounts to

$$R_2(t_1,t_2)=n^2-n\,K_n\big(t_1t_2^{-1}\big). \tag{14}$$

Proof of Theorem 1. If t_i are the eigenvalues of M, denote

$$\Omega(M)=\sum_{j=1}^{n}f(t_j)\,g(t_j).$$

Evidently

$$X_f(M)\,X_g(M)-\Omega(M)=\sum_{j\neq k}f(t_j)\,g(t_k).$$

Using (12) we have

$$\int_{U(n)}\big(X_f(M)\,X_g(M)-\Omega(M)\big)\,dM=\int_{\mathbb{T}^2}R(u_1,u_2)\,f(u_1)\,g(u_2)\,du_1\,du_2=$$
$$\int_{\mathbb{T}^2}\big(n^2-nK_n(u_1u_2^{-1})\big)\,f(u_1)\,g(u_2)\,du_1\,du_2.$$

The left hand side here equals

$$\int_{U(n)}X_f(M)\,X_g(M)\,dM-n\,\big(f*\widetilde{g}\big)(1),$$

and the right side equals

$$\overline{X_f}\,\overline{X_g}-n\,(f*\widetilde{g}*K_n)(1).$$

Comparing these gives (2). ■

Let λ be a *partition*, that is, a decreasing sequence $\lambda_1\geq\lambda_2\geq\lambda_3\geq\ldots$ of integers such that $\lambda_j=0$ for j sufficiently large. The largest l such that $\lambda_l\neq 0$ is called the *length* $l(\lambda)$ of λ. If the length of λ is $\leq n$, let

$$s_\lambda(t_1,\cdots,t_n)=\frac{\begin{vmatrix}t_1^{\lambda_1+n-1} & t_2^{\lambda_1+n-1} & \cdots & t_n^{\lambda_1+n-1}\\ t_1^{\lambda_2+n-2} & t_2^{\lambda_2+n-2} & \cdots & t_n^{\lambda_2+n-2}\\ \vdots & & & \vdots\\ t_1^{\lambda_n} & t_2^{\lambda_n} & \cdots & t_n^{\lambda_n}\end{vmatrix}}{\begin{vmatrix}t_1^{n-1} & t_2^{n-1} & \cdots & t_n^{n-1}\\ t_1^{n-2} & t_2^{n-2} & \cdots & t_n^{n-2}\\ \vdots & & & \vdots\\ 1 & 1 & \cdots & 1\end{vmatrix}}. \tag{15}$$

This is the *Schur function* as defined in Macdonald [M], (3.1) on p. 30. See Macdonald [M], volume 2 of Stanley [St] and Bump and Diaconis [BD] for background on the Schur functions and related representation theory.

If $t_1, \cdots, t_n$ are the eigenvalues of $M \in U(n)$, then

$$\chi_\lambda(M) = s_\lambda(t_1, \cdots, t_n) \tag{16}$$

is the character of an irreducible representation of $U(n)$, denoted $\mathbf{s}_\lambda(M)$ in [BD]. (This is essentially the Weyl character formula.) With dM Haar measure, $|\chi_\lambda(M)|^2\, dM$ defines a probability measure on $U(n)$. We now investigate the m-level correlation function $R_{m,\lambda}(t_1, \cdots, t_m)$ with respect to the measure $|\chi_\lambda(M)|^2\, dM$. We define this by analogy with (12) by asking that

(17)
$$\int_{\mathbb{T}^m} R_{m,\lambda}(t_1, \cdots, t_m)\, f(t_1, \cdots, t_m)\, dt_1 \cdots dt_m = \int_{U(n)} \sum{}^{*} f(t_{i_1}, \cdots, t_{i_m})\, |\chi_\lambda(M)|^2\, dM.$$

We will prove a formula generalizing Dyson's (13) for $R_{m,\lambda}$. To motivate this, rewrite (13) this way:

$$R_m(t_1, \cdots, t_m) = \det(AA^*),$$

where A is the (nonsquare) matrix

$$\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ \vdots & \vdots & & & \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^{n-1} \end{pmatrix}$$

and A^* is its conjugate transpose. The Hermitian matrix AA^* is not equal to the square matrix on the right side of (13), but is conjugate to it by a diagonal matrix.

Theorem 2. *We have*

$$R_{m,\lambda}(t_1, \cdots, t_n) = \det(A_\lambda A_\lambda^*) \tag{18}$$

with

$$A_\lambda = \begin{pmatrix} t_1^{-\lambda_1} & t_1^{1-\lambda_2} & \cdots & t_1^{-\lambda_n+n-1} \\ \vdots & \vdots & & \vdots \\ t_m^{-\lambda_1} & t_m^{1-\lambda_2} & \cdots & t_m^{-\lambda_n+n-1} \end{pmatrix}$$

Proof. We may prove this formula as follows. First suppose that $m = n$. Then the matrix A_λ is square, and

$$\det(A_\lambda A_\lambda^*) = |\det(A_\lambda^*)|^2.$$

We must show that the function on the right side of (18) satisfies (17). By the Weyl integration formula, if we substitute $|\det(A^*_\lambda)|^2$ for $R_{m,\lambda}$ on the left side of (17) we obtain

$$\int_{\mathbb{T}^n} \begin{vmatrix} t_1^{\lambda_1} & t_2^{\lambda_1} & \cdots & t_n^{\lambda_1} \\ t_1^{\lambda_2-1} & t_2^{\lambda_2-1} & \cdots & t_n^{\lambda_2-1} \\ \vdots & & & \vdots \\ t_1^{\lambda_n-n+1} & t_2^{\lambda_n-n+1} & \cdots & t_n^{\lambda_2-n+1} \end{vmatrix}^2 f(t_1,\cdots,t_n)\,dt_1\cdots dt_n.$$

Let $F(t_1,\cdots,t_n)=\sum_{\sigma\in S_n} f(t_{\sigma(1)},\cdots,t_{\sigma(n)})$ be the symmetrization of f. The Vandermonde identity and the symmetry of the Schur function show that the last expression equals

$$\frac{1}{n!}\int_{\mathbb{T}^n} \frac{\begin{vmatrix} t_1^{\lambda_1+n-1} & t_2^{\lambda_1+n-1} & \cdots & t_n^{\lambda_1+n-1} \\ t_1^{\lambda_2+n-2} & t_2^{\lambda_2+n-2} & \cdots & t_n^{\lambda_2+n-2} \\ \vdots & & & \vdots \\ t_1^{\lambda_n} & t_2^{\lambda_n} & \cdots & t_n^{\lambda_n} \end{vmatrix}^2}{\begin{vmatrix} t_1^{n-1} & t_2^{n-1} & \cdots & t_n^{n-1} \\ t_1^{n-2} & t_2^{n-2} & \cdots & t_n^{n-2} \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}^2} F(t_1,\cdots,t_n)\prod_{i<j}|t_i-t_j|^2\,dt_1\cdots dt_n.$$

Now using (15) and the Weyl integration formula (Goodman and Wallach [GW], p. 343), we obtain the right side of (17).

Therefore (18) is true when $m=n$. The case $m<n$ follows by the same argument as in Mehta [M], pp. 195-196. The downward induction is based on Theorem 5.2.1 on p. 89 of [M], where in the case at hand the function $f(x,y)$ on $\mathbb{T}$ is

$$f(x,y)=\sum_{j=1}^{n}(xy^{-1})^{j-1-\lambda_j}.$$

The other ingredient of the proof is the analog of eq. (5.1.2) of [M], which for us is the formula

$$R_{m,\lambda}(x_1,\cdots,x_m)=\frac{1}{(n-m)!}\int_{\mathbb{T}^{n-m}} R_{n,\lambda}(x_1,\cdots,x_n)\,dx_{m+1}\cdots dx_n.$$

It is straightforward to deduce this from (17) by taking a test function $f(t_1,\cdots,t_n)$ which only depends on $t_1,\cdots,t_m$. ■

If $n\geq l(\lambda)$, define

$$K_{n,\lambda}(t)=\frac{1}{n}\left|\sum_{j=1}^{n}t^{\lambda_j-j}\right|^2.$$

Theorem 3. *(i) The sequence of functions $K_{n,\lambda}$ is a Dirac sequence.*
(ii) There exists polynomials $f_\lambda(t)$ and $g_\lambda(t)$ in t and t^{-1} such that $f_\lambda(t) = f_\lambda(t^{-1})$ and

$$n\big(K_{n,\lambda}(t) - K_n(t)\big) = f_\lambda(t) + t^n\, g_\lambda(t) + t^{-n}\, g_\lambda(t^{-1}). \tag{19}$$

Proof. It is evident that $K_{n,\lambda}$ is positive on $\mathbb{T}$.

Writing

$$K_{n,\lambda}(t) = \frac{1}{n} \sum_{j,k} t^{\lambda_j - \lambda_k - j + k},$$

its mean value on $\mathbb{T}$ is $1/n$ times the number of pairs j, k with $\lambda_j - j = \lambda_k - k$. Since $\lambda_j - j$ is a decreasing sequence, these pairs occur precisely when $j = k$, so $\int_{\mathbb{T}} K_{n,\lambda}(t)\, dt = 1$.

Next we show that if $t \neq 1$, then $K_{n,\lambda}(t) \to 0$; the convergence is uniform on compact subsets of $\mathbb{T} - \{1\}$. If $l = l(\lambda)$ is the length of λ, then

$$K_{n,\lambda}(t) = \frac{1}{n} \left| \sum_{j=1}^{l} (t^{\lambda_j} - 1) t^{-j} + \frac{1 - t^{-(n+1)}}{1 - t^{-1}} \right|^2. \tag{20}$$

Now $\sum_{j=1}^{l} (t^{\lambda_j} - 1) t^{-j}$ is independent of n, and $(1 - t^{-(n+1)})/(1 - t^{-1})$ is bounded independently of n if t is bounded away from 1. Hence (20) is $O(n^{-1})$.

We have established that $K_{n,\lambda}$ is a Fejér sequence. It follows from (20) that

$$n\big(K_{n,\lambda}(t) - K_n(t)\big) = \\ \left| \sum_{j=1}^{l} t^{\lambda_j - j} - t^{-j} \right|^2 + \sum_{j=1}^{l} \frac{t^{\lambda_j - j} - t^{-j}}{1 - t} (1 - t^{n+1}) + \sum_{j=1}^{l} \frac{t^{-\lambda_j + j} - t^{j}}{1 - t^{-1}} (1 - t^{-n-1}).$$

Noting that $(t^{\lambda_j - j} - t^{-j})/(1 - t)$ is a polynomial in t and t^{-1}, simplifying this gives (19). ∎

If Φ and Ψ are functions on $U(n)$ we will define $\mathrm{Cov}_\lambda(\Phi, \Psi)$ to be the covariance with respect to the measure $|\chi_\lambda(M)|^2\, dM$.

Theorem 4. *We have*

$$\mathrm{Cov}_\lambda(X_f, X_g) = n(f * \tilde{g})(1) - n(f * \tilde{g} * K_{n,\lambda})(1). \tag{21}$$

Proof. If $m = 2$, we may rewrite Theorem 2 in the form

$$R_{2,\lambda}(t_1, t_2) = n^2 - n K_{n,\lambda}(t_1 t_2^{-1}).$$

The proof of Theorem 4 is now the same as that of Theorem 1. ∎

3. Principal Toeplitz minors. In this section we reprove Theorems 1 and 4 by an entirely different method. A classical identity of Heine and Szegö expresses Toeplitz determinants as integrals over the unitary group. See Bump and Diaconis [BD] for generalizations and applications of this identity. Here we will derive another generalization of this identity which contains information equivalent to the m-level correlation function of unitary statistics.

Let $1 \leq m \leq n$. If M is any matrix, let $E_m(M)$ denote the sum of the $\binom{n}{m}$ principal $m \times m$ minors of M. Since this is just the trace of M in the m-th exterior power representation of $U(n)$, it is invariant under conjugation, so to compute it we may assume M is diagonal. Thus $E_m(M) = e_m(t_1, \cdots, t_n)$, where t_j are the eigenvalues of M, and e_m is the m-th elementary symmetric polynomial in n variables.

If f is a continuous function on $\mathbb{T}$, we may associate with f a continuous function $\mathcal{U}_f : U(n) \to U(n)$, namely

$$\mathcal{U}_f\left(h \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} h^{-1}\right) = h \begin{pmatrix} f(t_1) & & \\ & \ddots & \\ & & f(t_n) \end{pmatrix} h^{-1},$$

with $h \in U(n)$. Note that this is well defined. If d_j are the Fourier coefficients of f, so that, $f(t) = \sum d_j t^j$, let T_{n-1} be the $n \times n$ Toeplitz matrix

$$T_{n-1}(f) = \begin{pmatrix} d_0 & d_1 & \cdots & d_{n-1} \\ d_{-1} & d_0 & \cdots & d_{n-2} \\ \vdots & & & \vdots \\ d_{-(n-1)} & d_{-(n-2)} & \cdots & d_0 \end{pmatrix}.$$

Theorem 5. *With these notations,*

$$\int_{U(n)} E_m\big(\mathcal{U}_f(M)\big)\, dM = E_m\big(T_{n-1}(f)\big). \tag{22}$$

Proof. If $m = n$, this identity may be written

$$\int_{U(n)} \det\big(\mathcal{U}_f(M)\big)\, dM = \det\big(T_{n-1}(f)\big), \tag{23}$$

and this is the Heine-Szegö identity. See Bump and Diaconis [BD] for a proof and generalizations. The general case of (22) follows by replacing f by $1 + \lambda f$ in (23), then expanding and comparing the coefficients of λ^m. ■

One may deduce Theorem 1 from Theorem 5 by an argument we will give below in the second proof of Theorem 4. To obtain Theorem 4, we need to generalize Theorem 5.

If λ and μ are partitions of length $\leq n$, let $T_{n-1}^{\lambda,\mu}(f)$ be the *Toeplitz minor* which is the $n \times n$ matrix whose (j,k)-th entry is $d_{\lambda_j - \lambda_k - j + k}$. It is a minor in a larger Toeplitz matrix.

The asymptotics of large Toeplitz minors were studied by Bump and Diaconis [BD]. If $\lambda = \mu$ this is a principal minor. These are the ones which we will need.

Theorem 6. *We have*

$$\int_{U(n)} E_m\big(\mathcal{U}_f(M)\big)\,\overline{\chi_\lambda(M)}\,\chi_\mu(M)\,dM = E_m\big(T^{\lambda,\mu}_{n-1}(f)\big). \tag{24}$$

Proof. We take $\lambda = \mu$ in Theorem 3 of Bump and Diaconis [BD]. The quoted theorem asserts that

$$\int_{U(n)} \det\big(\mathcal{U}_f(M)\big)\,\overline{\chi_\lambda(M)}\,\chi_\mu(M)\,dM = \det\big(T^{\lambda,\mu}_{n-1}(f)\big).$$

Proceeding as in the proof of Theorem 5 we obtain (24). ∎

In our applications of Theorem 6 we will take $\lambda = \mu$. The special case $m = 1$ of (24) is worth noting:

$$\int_{U(n)} X_f(M)\,|\chi_\lambda(M)|^2\,dM = n\int_{\mathbb{T}} f(t)\,dt \tag{25}$$

Second proof of Theorem 4. Let $f(t) = \sum c_k t^k$ and $g(t) = \sum d_k t^k$. Let $M \in U(n)$ have eigenvalues t_j. Then

$$X_f(M)\,X_g(M) = \sum_{j,k} f(t_j)\,g(t_k).$$

We integrate this with respect to the measure $|\chi_\lambda(M)|^2\,dM$, separating the diagonal terms ($j = k$) from the terms with $j \neq k$. The diagonal contribution equals the integral of $X_{fg}(M)$, which by (25) equals

$$n\int_{\mathbb{T}} f(t)\,g(t)\,dt = n\sum_k c_k d_{-k} \tag{26}$$

We are left with the integral of the off-diagonal terms

$$\sum_{j\neq k} f(t_j)\,g(t_k) = E_2\big(\mathcal{U}_{f+g}(M)\big) - E_2\big(\mathcal{U}_f(M)\big) - E_2\big(\mathcal{U}_g(M)\big).$$

We evaluate this by means of (24). It equals

$$E_2\big(T^{\lambda,\lambda}_{n-1}(f+g)\big) - E_2\big(T^{\lambda,\lambda}_{n-1}(f)\big) - E_2\big(T^{\lambda,\lambda}_{n-1}(g)\big).$$

Let $1 \le k \le n-1$. If $1 \le j \le k \le n$, then $T^{\lambda,\lambda}_{n-1}(f+g)$ has a principal minor of the form

$$\begin{vmatrix} c_0 + d_0 & c_{\lambda_j-\lambda_k-j+k} + d_{\lambda_j-\lambda_k-j+k} \\ c_{-(\lambda_j-\lambda_k-j+k)} + d_{-(\lambda_j-\lambda_k-j+k)} & c_0 + d_0 \end{vmatrix}.$$

From this, we subtract the two corresponding minors

$$\begin{vmatrix} c_0 & c_{\lambda_j-\lambda_k-j+k} \\ c_{-(\lambda_j-\lambda_k-j+k)} & c_0 \end{vmatrix} + \begin{vmatrix} d_0 & d_{\lambda_j-\lambda_k-j+k} \\ d_{-(\lambda_j-\lambda_k-j+k)} & d_0 \end{vmatrix}$$

of $T_{n-1}^{\lambda,\lambda}(f)$ and $T_{n-1}^{\lambda,\lambda}(g)$ to obtain

$$2c_0d_0 - c_{\lambda_j-\lambda_k-j+k}\, d_{-(\lambda_j-\lambda_k-j+k)} - c_{-(\lambda_j-\lambda_k-j+k)}\, d_{\lambda_j-\lambda_k-j+k}.$$

Summing these terms gives

$$(n^2-n)c_0d_0 - \sum_{1\le j<k\le n} \left(c_{\lambda_j-\lambda_k-j+k}\, d_{-(\lambda_j-\lambda_k-j+k)} + c_{-(\lambda_j-\lambda_k-j+k)}\, d_{\lambda_j-\lambda_k-j+k}\right) = \\ (n^2-n)c_0d_0 - \sum_{1\le j\ne k\le n} c_{\lambda_j-\lambda_k-j+k}\, d_{-(\lambda_j-\lambda_k-j+k)}.$$

Adding back the diagonal terms (26) gives

$$\int_{U(n)} X_f(M)\, X_g(M)\, |\chi_\lambda(M)|^2\, dM = n\sum_{k=-\infty}^{\infty} c_k d_{-k} + n(n-1)c_0d_0 \\ - \sum_{1\le j\ne k\le n} c_{\lambda_j-\lambda_k-j+k}\, d_{-(\lambda_j-\lambda_k-j+k)}.$$

By (25) we have $\overline{X_f} = \int_{U(n)} X_f(M)\, |\chi_\lambda(M)|^2\, dM = nc_0$ and $\overline{X_g} = nd_0$. Thus

$$\tfrac{1}{n}\mathrm{Cov}_\lambda(X_f, X_g) = \sum_{k=-\infty}^{\infty} c_k d_{-k} - \sum_{k=-\infty}^{\infty} \rho_l c_l d_{-l},$$

where $\rho_l = 1$ if $l=0$; more generally, it is $1/n$ times the number of pairs (j,k) with $\lambda_j - \lambda_k - j + k = l$. Evidently $\sum \rho_l t^l = K_{n,\lambda}(t)$, whence (21). ■

4. Fejér asymptotics. As we have noted, the convolution of a function f with the Fejér kernel gives a sequence of approximations to f by trigonometric polynomials. We have expressed the covariance of two additive functions on $U(n)$ as the error in such an approximation. The class of functions on $\mathbb{T}$ which occurs in Theorem 1 consists of convolutions of pairs of functions, which in the applications might be piecewise smooth with jump discontinuities. The convolution of a pair of such functions is then continuous and piecewise smooth but its derivative can have jump discontinuities.

For the analysis of the asymptotics of the convolution of a function f with the Fejér kernel, it will be useful to parametrize the circle by the interval $(-\pi,\pi]$. We therefore denote

$$k_n(x) = \frac{\sin^2(nx/2)}{n\sin^2(x/2)} = \frac{1-\cos(nx)}{2n\sin^2(x/2)}, \qquad K_n(e^{ix}) = k_n(x). \tag{27}$$

We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(x)\,dx = 1. \tag{28}$$

The convolution of k_n with a 2π-periodic function $f(\theta)$ is defined by

$$(f * k_n)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(x) f(\theta + x)\,dx\ . \tag{29}$$

We rewrite (29), making use of (28), in the form

$$(f * k_n)(\theta) = f(\theta) + \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(x)\,[f(\theta + x) - f(\theta)]\,dx. \tag{30}$$

First we assume that the derivative $f'(\theta)$ exists. Let

$$R(x, \theta; f) = f(x + \theta) - f(\theta) - f'(\theta)x. \tag{31}$$

Theorem 7. *Let f be a 2π-periodic integrable function such that $f'(\theta)$ exists, and such that $R(x, \theta; f) = x^2\phi_\theta(x)$ where $\phi_\theta(x)$ is integrable as a function of x. Then*

$$(f * k_n)(\theta) = f(\theta) + \frac{1}{4\pi n}\int_{-\pi}^{\pi} \frac{R(x, \theta; f)}{\sin^2(x/2)}\,dx + o(n^{-1}). \tag{32}$$

If $\phi'_\theta(x)$ exists and is integrable then the error term in (32) is $O(n^{-2})$.

Proof. Using (30),

$$(f * k_n)(\theta) = f(\theta) + \frac{f'(\theta)}{2\pi}\int_{-\pi}^{\pi} x\,k_n(x)\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(x)\,R(x, \theta; f)\,dx.$$

The first integral on the right vanishes since $x\,k_n(x)$ is odd. By using the second expression in (27), we may write

$$(f * k_n)(\theta) = f(\theta) + \frac{1}{4\pi n}\int_{-\pi}^{\pi} \frac{R(x, \theta; f)}{\sin^2(x/2)}\,dx - \frac{1}{4\pi n}\int_{-\pi}^{\pi} \frac{x^2}{\sin^2(x/2)}\,\phi_\theta(x)\,\cos(nx)\,dx.$$

By the Riemann-Lebesgue Lemma, the last term is $o(n^{-1})$. If ϕ'_θ is integrable, then integration by parts shows that this term is $O(n^{-2})$. The theorem follows. ∎

Lemma 1. *We have*

$$\frac{1}{2\pi}\int_0^\pi x\,k_n(x)\,dx = \frac{1}{n\pi}\big(1+\gamma+\log(2n)\big) + O(n^{-2}). \tag{33}$$

Proof. Using the second expression in (27) converts the integral in (33) into

$$\begin{aligned}\frac{1}{4\pi n}\int_0^\pi x\big(1-\cos(nx)\big)&\left[\frac{4}{x^2}+\left(\frac{1}{\sin^2(x/2)}-\frac{1}{(x/2)^2}\right)\right]dx = \\ &\frac{1}{n\pi}\int_0^\pi \frac{1-\cos(nx)}{x}\,dx + \frac{1}{4n\pi}\int_0^\pi x\left(\frac{1}{\sin^2(x/2)}-\frac{1}{(x/2)^2}\right)dx \\ &\qquad -\frac{1}{4n\pi}\int_0^\pi x\left(\frac{1}{\sin^2(x/2)}-\frac{1}{(x/2)^2}\right)\cos(nx)\,dx.\end{aligned}$$

The first integral on the right hand side is $\frac{1}{n\pi}(\log n\pi+\gamma-\mathrm{Ci}(n\pi))$, where by (8) we have $\mathrm{Ci}(n\pi)=O(n^{-2})$. As for the second integral,

$$\frac{1}{4n\pi}\int_0^\pi x\left(\frac{1}{\sin^2(x/2)}-\frac{1}{(x/2)^2}\right)dx = \frac{1-\log(\pi/2)}{n\pi}.$$

The last integral is $O(n^{-2})$, as can be proved by integration by parts, and putting these together, we obtain the Lemma. ∎

Next we assume that f' may be discontinuous at $\theta=\theta_0$, but that its right and left derivatives $f'_+(\theta_0)$ and $f'_-(\theta_0)$ both exist. Let

$$f(x+\theta_0) = \begin{cases} f(\theta_0)+f'_+(\theta_0)x+R_+(x,\theta_0;f), & x>0,\\ f(\theta_0)+f'_-(\theta_0)x+R_-(x,\theta_0;f), & x<0.\end{cases}$$

We obtain the asymptotics of $(f*k_n)(\theta)$ first when $\theta=\theta_0$, and later when θ is near to but different from θ_0. Let

$$R(x,\theta_0;f) = \begin{cases} R_+(x,\theta_0;f) & \text{if } x>0,\\ R_-(x,\theta_0;f) & \text{if } x<0.\end{cases} \tag{34}$$

Theorem 8. *Let $\theta=\theta_0$, where f' has a jump discontinuity. Assume that $R(x,\theta_0;f)=x^2\,\phi_{\theta_0}(x)$, where $\phi_{\theta_0}(x)$ is integrable as a function of x. Then*

$$\begin{aligned}(f*k_n)(\theta_0) = f(\theta_0) + \frac{f'_+(\theta_0)-f'_-(\theta_0)}{n\pi}&\big(1+\gamma+\log(2n)\big)\\ &+\frac{1}{4\pi n}\int_{-\pi}^{\pi}\frac{R(x,\theta_0;f)}{\sin^2(x/2)}\,dx + o(n^{-1})\end{aligned} \tag{35}$$

Proof. We have

$$
\begin{aligned}
(f * k_n)(\theta_0) = f(\theta_0) + \frac{1}{2\pi}\int_{-\pi}^{0} k_n(x)\, R_-(x,\theta_0;f)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} k_n(x)\, R_+(x,\theta_0;f)\,dx \\
+ \frac{f'_-(\theta_0)}{2\pi}\int_{-\pi}^{0} x\,k_n(x)\,dx + \frac{f'_+(\theta_0)}{2\pi}\int_{0}^{\pi} x\,k_n(x)\,dx = \\
f(\theta_0) + \left[f'_+(\theta_0) - f'_-(\theta_0)\right]\frac{1}{2\pi}\int_0^{\pi} x\,k_n(x)\,dx \\
+ \frac{1}{4\pi n}\left\{\int_{-\pi}^{0}\frac{R_-(x,\theta_0;f)}{\sin^2(x/2)}\,dx + \int_0^{\pi}\frac{R_+(x,\theta_0;f)}{\sin^2(x/2)}\,dx\right\} \\
- \frac{1}{4\pi n}\left\{\int_{-\pi}^{0}\frac{R_-(x,\theta_0;f)}{\sin^2(x/2)}\cos(nx)\,dx + \int_0^{\pi}\frac{R_+(x,\theta_0;f)}{\sin^2(x/2)}\cos(nx)\,dx\right\}
\end{aligned}
$$

The result follows from Lemma 1 and the Riemann-Lebesgue Lemma. ■

The result of Theorem 7 is valid when $f'(\theta)$ exists. Its leading term is consistent with Theorem 8, but the subsequent terms differ. Our goal is to obtain a uniform expression. Let us now assume that θ is different from θ_0, where θ_0 is a discontinuity of f'. We will again rely on the Riemann-Lebesgue Lemma, of which we note the following refinement:

Lemma 2. *If ϕ is an integrable function on $[-\pi,\pi]$, then $\int_a^b \cos(nx)\,\phi(x)\,dx \to 0$ as $n\to\infty$, uniformly in a and b for $-\pi \le a \le b \le \pi$.*

Proof. It is easy to see that if $\chi_{[a,b]}$ denotes the characteristic function of $[a,b]$ then $(a,b) \to \chi_{[a,b]}\phi$ is a continuous map into $L^1([-\pi,\pi])$, so the set of such functions is compact, and the uniformity now follows from the remark to Theorem 2.8 on p. 13 of Katznelson [Ka]. ■

Let

$$
K = \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{x^2}{\sin^2(x/2)}\,dx = 2.77259\cdots. \tag{36}
$$

Lemma 3. *Assume $\theta \neq 0$. We have, uniformly in θ:*

$$\frac{1}{2\pi}\int_\theta^\pi k_n(x)\,dx = \begin{cases} \frac12 - \frac{1}{n\pi}\left(\frac{\cos n\theta}{\theta} + n\,\mathrm{Si}(n\theta) - \frac12\cot\left(\frac{\theta}{2}\right)\right) + O(n^{-2}), & \text{if } \theta > 0,\\ \frac12 + \frac{1}{n\pi}\left(\frac{\cos n\theta}{\theta} + n\,\mathrm{Si}(n\theta) - \frac12\cot\left(\frac{\theta}{2}\right)\right) + O(n^{-2}), & \text{if } \theta < 0,\end{cases} \tag{37}$$

$$\frac{1}{2\pi}\int_\theta^\pi x\,k_n(x)\,dx = \frac{1}{\pi n}\left(\mathrm{Ci}(n|\theta|) + \frac{|\theta|}{2}\cot\left(\frac{|\theta|}{2}\right) - \log\sin\left(\frac{|\theta|}{2}\right)\right) + O(n^{-2}), \tag{38}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^\pi x^2\,k_n(x)\,dx = \frac{K}{n} + O(n^{-2}). \tag{39}$$

Proof. Assume that $\theta > 0$. By (27) we have

$$\begin{aligned}\frac{1}{2\pi}\int_\theta^\pi k_n(x)\,dx &= \frac{1}{4\pi n}\int_\theta^\pi \left[\frac{1}{\sin^2(x/2)} - \frac{\cos(nx)}{(x/2)^2} + \cos(nx)\left(\frac{1}{(x/2)^2} - \frac{1}{\sin^2(x/2)}\right)\right] dx\\ &= \frac{1}{2\pi n}\cot\left(\frac{\theta}{2}\right) - \frac{1}{\pi}\int_{n\theta}^{n\pi}\frac{\cos(t)}{t^2}\,dt + O(n^{-2})\\ &= \frac{1}{2\pi n}\cot\left(\frac{\theta}{2}\right) - \frac{1}{\pi}\left[-\frac{\cos(n\pi)}{n\pi} + \frac{\cos(n\theta)}{n\theta} - \mathrm{Si}(n\pi) + \mathrm{Si}(n\theta)\right] + O(n^{-2}).\end{aligned}$$

Similarly (38) is an even function of θ since $x\,k_n(x)$ is an odd function of x, so we may assume $\theta > 0$ in evaluating it. Then

$$\begin{aligned}&\frac{1}{2\pi}\int_\theta^\pi x\,k_n(x)\,dx = \frac{1}{4\pi n}\int_\theta^\pi x\left(1-\cos(nx)\right)\left[\frac{4}{x^2} + \frac{1}{\sin^2(x/2)} - \frac{1}{(x/2)^2}\right] dx =\\ &\frac{1}{\pi n}\left(\log(\pi) - \log(\theta) - \mathrm{Ci}(n\pi) + \mathrm{Ci}(n\theta)\right) + \frac{1}{4\pi n}\int_\theta^\pi x\left(\frac{1}{\sin^2(x/2)} - \frac{1}{(x/2)^2}\right) dx\\ &\quad - \frac{1}{4\pi n}\int_\theta^\pi x\left(\frac{1}{\sin^2(x/2)} - \frac{1}{(x/2)^2}\right)\cos(nx)\,dx =\\ &\quad \frac{1}{\pi n}\left(\mathrm{Ci}(n\theta) - \mathrm{Ci}(n\pi) + \frac{\theta}{2}\cot\left(\frac{\theta}{2}\right) - \log\sin\left(\frac{\theta}{2}\right)\right)\\ &\quad - \frac{1}{4\pi n}\int_\theta^\pi x\left(\frac{1}{\sin^2(x/2)} - \frac{1}{(x/2)^2}\right)\cos(nx)\,dx =\\ &\qquad \frac{1}{\pi n}\left(\mathrm{Ci}(n\theta) + \frac{\theta}{2}\cot\left(\frac{\theta}{2}\right) - \log\sin\left(\frac{\theta}{2}\right)\right) + O(n^{-2})\end{aligned}$$

Finally, the difference between the left and right sides of (39) is

$$\frac{1}{4\pi n}\int_{-\pi}^\pi \frac{x^2}{\sin^2(x/2)}\cos(nx)\,dx = O(n^{-2}).$$

since the integrand is regular at its endpoints. ■

We will make use of the following function:

$$H(x) = \begin{cases} \frac{1}{2}(x+\pi)^2 & -2\pi \le x \le 0; \\ \frac{1}{2}(x-\pi)^2 & 0 \le x \le 2\pi. \end{cases}$$

We note that this function satisfies $H(x+2\pi) = H(x)$ when x and $x+2\pi$ are both in its range, so H has an extension to a 2π periodic function. Its derivative has a discontinuity at 0, which is unique modulo 2π.

Lemma 4. *We have*

$$(H * k_n)(\theta) = H(\theta) + \frac{K}{2n} - \frac{1}{n}\Phi_n(\theta) + O(n^{-2})\,.$$

Proof. We have, using the fact that k_n is even, and Lemma 3:

$$\begin{aligned}(H * k_n)(\theta) = \frac{1}{4\pi}\int_{-\pi}^{-\theta} (x+\theta+\pi)^2\, k_n(x)\,dx + \frac{1}{4\pi}\int_{-\theta}^{\pi} (x+\theta-\pi)^2\, k_n(x)\,dx = \\ \frac{1}{4\pi}\int_{-\pi}^{\pi} x^2\, k_n(x)\,dx - \int_{\theta}^{\pi} x\, k_n(x)\,dx \\ + \tfrac{1}{2}(\theta^2+\pi^2)\left[\frac{1}{2\pi}\int_{\theta}^{\pi} k_n(x)\,dx + \frac{1}{2\pi}\int_{-\theta}^{\pi} k_n(x)\,dx\right] \\ - \pi\theta\left[\frac{1}{2\pi}\int_{\theta}^{\pi} k_n(x)\,dx + \frac{1}{2\pi}\int_{-\theta}^{\pi} k_n(x)\,dx\right] = \\ \frac{K}{2n} + \frac{1}{2}(\theta^2+\pi^2) - \frac{2}{n}\left(\mathrm{Ci}(n\theta) - \log\sin\left(\frac{\theta}{2}\right) + \cos(n\theta) + n\theta\,\mathrm{Si}(n\theta)\right) + O(n^{-2}).\end{aligned}$$

Adding $\pi\theta$ to the second term and subtracting it from the third gives $\frac{K}{2n} + H(\theta) - \frac{1}{n}\Phi_n(\theta) + O(n^{-2})$. ■

Theorem 9. *Suppose that f has jump discontinuities in its derivative at θ_i, and that these are the only discontinuities in f' modulo 2π. Let*

$$\alpha_i = \frac{1}{2\pi}\big(f'_+(\theta_i) - f'_-(\theta_i)\big).$$

Assume that $\phi_\theta(x) = x^{-2}R(x,\theta;f)$ is integrable on $(-\pi,\pi)$, where $R(x,\theta;f)$ is defined by (31), or by (34) if θ is a θ_i. Then with Φ_n as in (10), we have

$$(f * k_n)(\theta) = f(\theta) + \sum_i \alpha_i\, n^{-1}\Phi_n(\theta-\theta_i) + \frac{1}{4\pi n}\int_{-\pi}^{\pi} \frac{R(x,\theta;f)}{\sin^2(x/2)}\,dx + o(n^{-1}). \tag{40}$$

If $\theta \mapsto \phi_\theta$ is a continuous map of $[-\pi,\pi]$ to $L^1([-\pi,\pi])$, then (40) is uniform in θ. If ϕ'_θ exists and is integrable, the error in (40) is $O(n^{-2})$.

In applying this theorem, note that as we move θ around the interval, we want to keep $|\theta_i - \theta| \le \pi$. This means that representatives θ_i are chosen differently depending on the location of θ. We've done this implicitly by defining $\Phi_n(\theta)$ by (10) when $|\theta| \le \pi$, and extending it to a 2π periodic function.

Proof. Let $f_0 = f + \sum_i \alpha_i H_i$, where $H_i(x) = H(x - \theta_i)$. The function f_0 is continuous and has a continuous first derivative, so we may apply Theorem 7. We have

$$(f_0 * k_n)(\theta) - f_0(\theta) = \frac{1}{4\pi n} \int_{-\pi}^{\pi} \frac{R(x, \theta; f_0)}{\sin^2(x/2)}\, dx + o(n^{-1}), \tag{41}$$

If $\phi'_\theta(x)$ exists is integrable, then Theorem 7 further asserts that the error is $O(n^{-2})$. This estimate may be shown to be uniform in θ along the lines of Lemma 2. By Lemma 4, the left side of (41) equals

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+\theta)\, k_n(x)\, dx - f(\theta) + \sum_i \frac{\alpha_i\, K}{2n} - \sum_i \alpha_i n^{-1}\, \Phi_n(\theta - \theta_i) + O(n^{-2}).$$

We check easily that for the quadratic functions H_i we have $R(x, \theta; H_i) = \frac{1}{2}x^2$ independent of θ and θ_i, so using (36) we have

$$\frac{1}{4\pi n} \int_{-\pi}^{\pi} \frac{R(x, \theta; f_0)}{\sin^2(x/2)}\, dx = \frac{1}{4\pi n} \int_{-\pi}^{\pi} \frac{R(x, \theta; f)}{\sin^2(x/2)}\, dx + \sum_i \frac{\alpha_i K}{2n}. \tag{42}$$

Comparing, we obtain (40). ∎

In an important special case, the result can be made more explicit.

Theorem 10. *Let f be a continuous, piecewise linear 2π periodic function, and let θ_i and α_i be as in Theorem 9. Then*

$$(f * k_n)(\theta) = f(\theta) + \sum_i \alpha_i\, n^{-1} \Phi_n(\theta - \theta_i) + O(n^{-2}).$$

uniformly in θ.

Proof. It is easy to see that ϕ'_θ exists and is integrable. The theorem will thus follow from Theorem 9 provided we show that

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{R(x, \theta; f)}{\sin^2(x/2)}\, dx = 0. \tag{43}$$

Since $f'_+(\theta_i) = f'_-(\theta_{i+1})$, we have $\sum_i \alpha_i = 0$. Furthermore on each interval (θ_i, θ_{i+1}) the function f_0 defined in the proof of Theorem 9 is polynomial of degree ≤ 2, and the coefficient of x^2 is $\frac{1}{2}\sum_i \alpha_i = 0$, so f_0 is piecewise linear with continuous derivative and 2π periodic; therefore f_0 is constant. Thus $R(x, \theta; f_0) = 0$, and (43) now follows from (42). ∎

Let λ be a partition, and let $k_{n,\lambda}(\theta) = K_{n,\lambda}(e^{i\theta})$.

Theorem 11. *There exists a function $\Phi_{n,\lambda}(\theta)$ such that if f is a piecewise linear and continuous 2π periodic function, then with α_i as in Theorems 9 and 10, we have*

$$(f * k_{n,\lambda})(\theta) = f(\theta) + \sum_i \alpha_i \, n^{-1} \Phi_{n,\lambda}(\theta - \theta_i) + O(n^{-2}).$$

The function $\Phi_{n,\lambda} - \Phi_n$ is a trigonometric polynomial, and is independent of n.

Proof. Let $f_0 = f + \sum_i \alpha_i H_i$, where $H_i(\theta) = H(\theta - \theta_i)$ as in the proof of Theorems 9 and 10. It was shown in the proof of Theorem 10 that f_0 is constant. By Theorem 10, we have

(44)
$$\begin{aligned}(f * k_{n,\lambda} - f)(\theta) = \sum \alpha_i \, n^{-1} \, \Phi_n(\theta - \theta_i) + (f_0 - \sum \alpha_i H_i) * (k_{n,\lambda} - k_n)(\theta) + O(n^{-2}) = \\ \sum_i \alpha_i \, n^{-1} \, \Phi_n(\theta - \theta_i) - \sum_i \alpha_i H_i * (k_{n,\lambda} - k_n)(\theta) + O(n^{-2}),\end{aligned}$$

since f_0 is constant and $k_{n,\lambda} - k_n$ has mean value 0. By Theorem 3(ii), there exist polynomials f_λ and g_λ in t and t^{-1} such that

$$(k_{n,\lambda} - k_n)(\theta) = \tfrac{1}{n}\big(f_\lambda(e^{i\theta}) + e^{in\theta} g_\lambda(e^{i\theta}) + e^{-in\theta} g_\lambda(e^{-i\theta})\big).$$

Moreover $f(t) = f(t^{-1})$, so the latter expression may be written as an even trigonometric polynomial—a finite linear combination of functions $\cos_k(\theta) = \cos(k\theta)$. Substituting this into the right hand side of (44), the convolution may be worked out using

$$(H_i * \cos_k)(\theta) = \frac{1}{k^2} \cos_k(\theta - \theta_i).$$

Thus f_λ contributes $n^{-1} \sum \alpha_i G(\theta - \theta_i)$, where G is a fixed trigonometric polynomial, while g_λ contributes terms of order $O(n^{-2})$ which may be discarded. The theorem follows. ∎

References

[BD] D. Bump and P. Diaconis, Toeplitz Minors, *J. Combin. Theory Ser. A*, **97**, 252–271 (2002).

[CD] M. Coram and P. Diaconis, New tests of the correspondence between unitary eigenvalues and the zeros of Riemann's zeta function, to appear in *Ann. Statist.*

[DE] P. Diaconis and S. Evans, Linear functionals of eigenvalues of random matrices, *Trans. Amer. Math. Soc.* **353** (2001), 2615–2633.

[DS] P. Diaconis and M. Shahshahani, On the eigenvalues of random matrices, in *Studies in applied probability*, a special volume of *J. Appl. Probab.* **31A**, (1994) 49–62.

[Dy1] F. Dyson, Statistical Theory of the Energy Levels of Complex Systems, I, II and III, *J. Math. Phys.* **3** (1962), 140–156, 157–165 and 166–175.

[Dy2] F. Dyson, Correlations between Eigenvalues of a Random Matrix, *Commun. math. Phys.* **19** (1970), 235–250.

[F] L. Fejér, Untersuchungen über Fouriershe Reihen, *Math. Anal.* **58** (1904), 501–569.

[GW] R. Goodman and N. Wallach, Representations and Invariants of the Classical Groups, Cambridge (1998).

[HFGO] D. Hejhal, J. Friedman, M. Gutzwiller and A. Odlyzko, ed., *Emerging applications of number theory*, Springer-Verlag (1999).

[Hu] C. Hughes, On the characteristic polynomial of a random unitary matrix and the Riemann zeta function, Dissertation, University of Bristol (2001).

[HKO] C. Hughes, J. Keating and N. O'Connell, On the characteristic polynomial of a random unitary matrix, *Comm. Math. Phys.* **220** (2001), 429–451.

[Ka] Y. Katznelson, *An Introduction to Harmonic Analysis*, 2nd edition, Dover (1976).

[KS1] J. Keating and N. Snaith, Random matrix theory and $\zeta(\frac{1}{2}+it)$, *Comm. Math. Physics* **214** (2000), 57–89.

[KS2] J. Keating and N. Snaith, Random matrix theory and L-functions at $s=\frac{1}{2}$, *Comm. Math. Physics* **214** (2000), 91–110.

[Ko] T. Körner, *Fourier Analysis*, Cambridge (1988).

[M] I. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford (1995).

[Me] M. Mehta, *Random matrices*, Second edition, Academic Press (1991).

[P] M. Pinsky, Fejér asymptotics and the Hilbert transform, to appear in the Proceedings of the Mt. Holyoke Conference on Harmonic Analysis.

[R] E. Rains, High powers of random elements of compact Lie groups, *Probab. Theory and Related Fields* **107** (1997), no. 2, 219–241.

[So1] A. Soshnikov, Level spacings distribution for large random matrices: Gaussian fluctuations. *Ann. of Math.* (2) **148** (1998), 573–617.

[So2] A. Soshnikov, Central limit theorem for local linear statistics in classical compact groups and related combinatorial identities, *Ann. Probability* **28** 1353–1370 (2000).

[St] R. Stanley, *Enumerative combinatorics*, Cambridge (1986, 1997 and 1999).

[T] M. Taylor, Multi-dimensional Fejér kernel asymptotics, Preprint (2001), Deparment of Mathematics, University of North Carolina at Chapel Hill.

[TW1] C. Tracy and H. Widom, Introduction to random matrices, *Geometric and quantum aspects of integrable systems (Scheveningen, 1992)*, 103–130, Lecture Notes in Phys., **424** (1993), Springer Verlag.

[TW2] C. Tracy and H. Widom, Correlation functions, cluster functions, and spacing distributions for random matrices, *J. Statist. Phys.* **92** (1998), 809–835.

[W] K. Wieand, Eigenvalue distributions of random matrices in the permutation group and compact Lie groups, Harvard PhD. Dissertation (1998).

[Z] A. Zygmund, Trigonometric series, Second Edition, Cambridge (1959).