Residues of Weyl Group Multiple Dirichlet Series Associated to $\widetilde{\mathrm{GL}}_{n+1}$

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1 Introduction

Friedberg, Hoffstein and Lieman [11] considered a pair of double Dirichlet series formed with n-th order L-functions, where $n \geqslant 3$. They showed that these Dirichlet series have meromorphic continuation to all of $\mathbb{C}^2$, and the pair satisfies an interesting group of 32 functional equations. The purpose of this paper is to explain how to fit their result into the framework of Weyl group multiple Dirichlet series as developed in Brubaker, Bump, Chinta, Friedberg and Hoffstein [3] and Brubaker, Bump and Friedberg [4]. We recall that if Φ is a root system and n a sufficiently large integer, then working over a field K containing the group μ_n of n-th roots of unity [3] and [4] constructed a multiple Dirichlet series whose group of functional equations is the Weyl group of Φ, and whose coefficients involve n-th order Gauss sums. We will say that the *type* of this multiple Dirichlet series is $\Phi^{(n)}$.

In this paper we consider the case where $n = 3$ and $K = \mathbb{Q}(e^{2\pi i/3})$ with ring of integers $\mathfrak{o}_K = \mathbb{Z}[e^{2\pi i/3}]$. We will show that the Weyl group multiple Dirichlet series of type $A_3^{(3)}$ has the pair of double Dirichlet series considered by Friedberg, Hoffstein and Lieman as residues, and that we can understand their functional equations as a consequence of this fact. We will give the definition of this series $Z(s_1, s_2, s_3; A_3^{(3)})$ and its normalization $Z^*(s_1, s_2, s_3; A_3^{(3)})$ later in Section 4.

The functional equations found by Friedberg, Hoffstein and Lieman involve the following two double Dirichlet series. The first (see equations (1.2), (1.3), and (1.4) in [11]) is

$$Z_1(s,w) = \sum_{\substack{d \in \mathfrak{o}_K \\ d \equiv 1 \bmod 3}} L(s, \chi^{(3)}_{d_0})\, P(s,d)\, \mathbb{N}d^{-w}, \tag{1}$$

where we have factored $d = d_0 R^3$ with d_0 cube-free, and where in terms of the cubic symbol

$$\chi^{(3)}_d(c) = \left(\frac{c}{d}\right).$$

In (1) the factor $P(s,d)$ is an Eulerian Dirichlet polynomial in divisors of d, which we will now define. Let μ be the Moebius function on $\mathfrak{o}$. Following Friedberg, Hoffstein and Lieman [11] let

$$P(s,d) = \sum_{R_1 R_2 R_3 = R} \frac{\chi_{d_0}(R_3)\mu(R_3)}{\mathbb{N}R_3^s\, \mathbb{N}R_1^{3s-2}}. \tag{2}$$

We note that $P(s,d) = 1$ if d is cube-free. Second, we have (see equation (2.2) and the Dirichlet series of page 324 in [11])

$$Z_2(s,w) = \sum_{\substack{m \in \mathfrak{o}_K \\ m \equiv 1 \bmod 3}} D(w,m)\, \mathbb{N}m^{-s}, \tag{3}$$

where

$$D(w,m) = \zeta_3(3w - 1/2) \sum_{\substack{d \in \mathfrak{o}_K \\ d \equiv 1 \bmod 3}} G(m,d)\, \mathbb{N}d^{-w}.$$

Throughout this paper ζ is the Dedekind zeta function of K, and here ζ_3 is the partial zeta function, omitting the Euler factor at the ternary prime. Also

$$G(m,d) = \mathbb{N}d^{-1/2} \sum_{c \bmod d} \left(\frac{c}{d}\right) e\left(\frac{mc}{d}\right)$$

is a normalized Gauss sum, with $e(x) = e^{2\pi i \,\mathrm{tr}(x)}$. The definitions of Z_1 and Z_2 can be made with any n-th power residue symbol, but for simplicity, we focus on the case $n = 3$.

Our main result is the following.

Theorem 1 *With notation as above, we have*

$$\mathrm{Res}_{s_1=2/3}\ Z^*(s_1,s_2,s_3;A_3^{(3)}) = Z_2\left(2s_2-\frac{1}{3},2s_3-\frac{1}{2}\right), \tag{4}$$

$$\mathrm{Res}_{s_2=2/3}\ Z^*(s_1,s_2,s_3;A_3^{(3)}) = Z_1\left(2s_1-\frac{1}{3},2s_3-\frac{1}{3}\right). \tag{5}$$

We will prove this in Sections 6 and 7 (see Theorems 4 and 3).

We will give now an argument suggesting that the two-variable double Dirichlet series constructed in [11] can be expressed as a residue of either the type $A_n^{(n)}$ or $A_{2n-2}^{(n)}$ Weyl group multiple Dirichlet series. Although, in this paper, we will take $n=3$, and realize this double Dirichlet series as a residue of the $A_3^{(3)}$ Weyl group multiple Dirichlet series, the heuristic we will now explain shows that we should also be able to construct it (or perhaps a similar but slightly different object) as a residue of the $A_4^{(3)}$ Weyl group multiple Dirichlet series also, but this conclusion should be regarded as provisional, since the construction remains uninvestigated.

Our starting point is the assumption, discussed in [5], that the Weyl group multiple Dirichlet series $A_r^{(n)}$ corresponds to the Whittaker function of an Eisenstein series of minimal parabolic type on the n-fold metaplectic cover of GL_{r+1}. We will refer to this statement as the *Eisenstein series conjecture*. We say "the" metaplectic cover but there is an abuse of language in the use of the definite article here, since a few different covers were described by Kazhdan and Patterson [12]. If $r \geqslant t$, then the standard embedding $\mathrm{GL}_t \longrightarrow \mathrm{GL}_r$ induces a map $H^2(\mathrm{GL}_t(\mathbb{A}),\mu_n) \longrightarrow H^2(\mathrm{GL}_r(\mathbb{A}),\mu_n)$. We will describe metaplectic covers of t and r as *compatible* if the corresponding cohomology classes match in this restriction map. If the restriction of the cohomology class on GL_t is the inverse of the class on GL_r, we say that they are *inverse* covers. We will choose a compatible family of covers and denote them by $\widetilde{\mathrm{GL}}_r$; in these, we may take the parameter c of Kazhdan and Patterson [12] to be 0. If f is an automorphic form on $\widetilde{\mathrm{GL}}_r$ then its complex conjugate $\bar{f}$ is an automorphic form on the inverse cover.

We recall that Jacquet, Shalika and Piatetski-Shapiro gave Rankin-Selberg constructions which, applied to a pair of automorphic forms ϕ_r and ϕ_t on GL_r and GL_t respectively, unfold to Whittaker integrals representing the L-function $L(s,\phi_r\times\phi_t)$. The fact that these constructions were also useful when ϕ_r and ϕ_t are automorphic forms on the metaplectic group, even in cases where the uniqueness of Whittaker models fails, was emphasized by Bump and Hoffstein, and T. Suzuki. Usually these Rankin-Selberg constructions do not produce Euler products when ϕ_r and ϕ_t are metaplectic, but they do produce Dirichlet series having functional equations, and by abuse of notation, we will write these Dirichlet series $L(s, \phi_r \times \phi_t)$ even if this is not an Euler product.

Except in the special case $t = r = 2$, the constructions of Jacquet, Piatetski-Shapiro and Shalika only work if the metaplectic covers on which ϕ_r and ϕ_t live are inverse covers. Thus if instead ϕ_t and ϕ_r live on compatible covers, we can replace ϕ_t with its complex conjugate $\bar{\phi}_t$, which may naturally be regarded as an automorphic form on the inverse cover, and we will denote the corresponding Jacquet-Piatetski-Shapiro construction as $L(s, \phi_r \times \overline{\phi_s})$. It is a Dirichlet series with a functional equation but generally not an Euler product.

Let θ_r denote the generalized theta series on $\widetilde{\mathrm{GL}}_r$. (Since n is fixed in this discussion, we suppress it from the notation.) We are mainly interested in the case $r \leqslant n$, since if $r > n$ the function θ_r does not have a Whittaker model. If $r \leqslant n$, it does have a Whittaker model; this is only unique if $r = n$ or $n - 1$. It is conjectured in [7] and [9] that if f is an automorphic form on the n-fold cover of GL_k then $L(s, f \times \bar{\theta}_r)$ agrees with a Whittaker coefficient of an Eisenstein series on the n-fold cover of GL_{k+n-r}, induced from the automorphic forms f and θ_{n-r}. We will refer to this as the *Bump-Hoffstein conjecture*.

It was shown by Bump and Lieman [10] that an Eisenstein series on the n-fold cover of GL_n has unique Whittaker models (in a slightly modified sense). Let us call this Eisenstein series $E_{n-1,1}(s)$, since it is the Eisenstein series of maximal parabolic type formed with the theta function on the n-fold cover of GL_{n-1}; it involves a single parameter, which we will denote s. The spherical Whittaker vectors were computed by Banks, Bump and Lieman [1], resulting in the fact that these Whittaker coefficients involve n-th order L-series $L(s, \chi_D^{(n)})$. In the case $n = 3$, such a phenomenon was noted much earlier by Bump and Hoffstein [8].

Now by the conjecture of Bump and Hoffstein, we may consider the Mellin transform of $E_{n-1,1}(s)$. This is a double Dirichlet series whose coefficients are $L(s, \chi_D^{(n)})$, one hopes the same double Dirichlet series as in [11]. This Mellin transform should be regarded as the Rankin-Selberg convolution $L(w, E_{n-1,1}(s) \times \bar{\theta}_1)$, which by the conjecture agrees with a Whittaker coefficient of an Eisenstein series on $\widetilde{\mathrm{GL}}_{2n-1}$ induced from $E_{n-1,1}$ and θ_{n-1}. The Eisenstein series $E_{n-1,1}$ of [1] and [10] is a residue of the Eisenstein series of minimal parabolic type on $\widetilde{\mathrm{GL}}_n$, and θ_{n-1} is a residue of the Eisenstein series of minimal parabolic type on $\widetilde{\mathrm{GL}}_{n-1}$, and so the Eisenstein series on $\widetilde{\mathrm{GL}}_{2n-1}$ is a residue of the minimal parabolic Eisenstein series. This means that the double Dirichlet series of [11] should be a residue of the Whittaker coefficient of the Eisenstein series of minimal parabolic type on $\widetilde{\mathrm{GL}}_{2n-1}$, that is, assuming the Eisenstein series conjecture, of the $A_{2n-2}^{(n)}$ Weyl group multiple Dirichlet series.

The argument that the double Dirichlet series of [11] should also be a residue of the $A_n^{(n)}$ Weyl group multiple Dirichlet series is similar. One considers instead $L(w, E_{n-1,1} \times \bar{\theta}_{n-1})$, which according to the Bump-Hoffstein conjecture should be a residue of an Eisenstein series on the n-fold cover of GL_{n+1}, By the Eisenstein series conjecture, it should therefore be a residue of the $A_n^{(n)}$ Weyl group multiple Dirichlet series. Kazhdan and Patterson [12] proved that the Whittaker coefficients of θ_{n-1} are n-th order Gauss sums, so one obtains a similar double Dirichlet series, except that the L-series $L(s, \chi_D^{(n)})$ appears paired with an n-th order Gauss sum $\overline{g(1,D)}$. This Gauss sum disappears when one takes the functional equation in s, and so one ends up with the same multiple Dirichlet series.

We will mostly use the notations of [3] for Gauss sums, etc. If c is squarefree and prime to m, we will also use the normalized Gauss sum $G(m,c) = \mathbb{N}m^{-1/2}\, g(m,c)$, whose absolute value is 1. We will also denote $g(1,p)$ by $g(p)$ or $G(1,p) = G(p)$.

Regarding foundations, we will take some liberties with regard to Hilbert symbols. Our approach is to work over the field $\mathbb{Q}(e^{2\pi i/3})$, whose ring $\mathfrak{o} = \mathbb{Z}[e^{2\pi i/3}]$ of integers is a principal ideal domain. Any ideal $\mathfrak{a}$ of $\mathfrak{o}$ that is prime to 3 has a unique generator α such that $\alpha \equiv 1$ modulo 3. If $\alpha \equiv \beta \equiv 1$ modulo 3, and if $\gcd(\alpha, \beta) = 1$, then the cubic reciprocity law has the simple form

$$\left(\frac{\alpha}{\beta}\right) = \left(\frac{\beta}{\alpha}\right) \tag{6}$$

We can thus make the convention that $\sum_d$ means that we sum over ideals $d\mathfrak{o}$, and for each such ideal, we take the generator $d \equiv 1$ modulo 3, as a consequence of which Hilbert symbols will not appear in any of our computations. A better point of view would be to carefully work out the Hilbert symbols in the manner of [3]. We will not do this, though if one wanted to extend our discussion to general n one would want to do this.

There is another liberty that we will take. We will ruthlessly suppress mention of what happens at the ternary prime. This means that all functional equations involve scattering matrices, which we have not worked out. Thus when we write an identity such as

$$Z(s_1, s_2, s_3; A_3^{(3)}) = Z\left(1 - s_1, s_1 + s_2 - \frac{1}{2}, s_3; A_3^{(3)}\right)$$

we really mean is (in the notation of [3])

$$Z_\Psi(s_1, s_2, s_3; A_3^{(3)}) = Z_{\sigma_1\Psi}\left(1 - s_1, s_1 + s_2 - \frac{1}{2}, s_3; A_3^{(3)}\right).$$

We hope that our suppression of the scattering matrix (which appears in this formula through the action of σ_1 on the parametrizing data Ψ) will serve to make the paper more readable, at the expense of being slightly imprecise.

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2 Heuristic Approach

In this portion, we demonstrate a connection between cubic twists of GL(1) L-functions and Weyl group multiple Dirichlet series on the heuristic level. We obtain the objects in [11] by taking residues of the $A_3^{(3)}$ multiple Dirichlet series. As noted in the introduction, we will work in the field $K = \mathbb{Q}(e^{2\pi i/3})$ in order to streamline and simplify the necessary foundations.

We may try to study these objects as residues of a Weyl group multiple Dirichlet series associated to the A_3 root system. Rather than writing down the precise formulation of this series, we begin with an approximation to the series. In what follows, we assume that our sums are over square-free integers which are pairwise relatively prime. Of course, these collections no longer possess perfect functional equations, but understanding how the functional equations act on these restricted sums gives an indication of what may be true for the sum over all integers.

In this restricted sum over square-free, relatively prime integers, we have the following simplified form for the $A_3^{(3)}$ multiple Dirichlet series, a so-called heuristic multiple Dirichlet series as in Section 1 of [3]:

$$Z(s_1, s_2, s_3; A_3^{(3)}) =$$
$$\sum_{d_1,d_2,d_3} g(d_1)g(d_2)g(d_3)\left(\frac{d_1}{d_2}\right)^{-1}\left(\frac{d_2}{d_3}\right)^{-1}\mathbb{N}d_1^{-2s_1}\mathbb{N}d_2^{-2s_2}\mathbb{N}d_3^{-2s_3}. \tag{7}$$

The Gauss sums in this expression are unnormalized so that $|g(1,p)| = |g(p)| = \mathbb{N}p^{1/2}$. Again, we emphasize that this is merely an approximate definition of Z in this case, and we refrain from writing conditions on integers in the sum. The precise definition will be given in Section 4. As we will now show, this approximation is good enough to study residues and resulting functional equations.

We will give heuristic justification in this section for the two identities (4) and (5). Later we will give proper justifications for these in Theorem 3 and Theorem 4.

First we want to take the residue of $Z(s_1, s_2, s_3; A_3^{(3)})$ in s_1 at the right-most pole. Rewriting (7) so that the inner sum is over d_1, we have

$$\sum_{d_2,d_3} g(d_2)g(d_3)\left(\frac{d_2}{d_3}\right)^{-1}\mathbb{N}d_2^{-2s_2}\mathbb{N}d_3^{-2s_3}\sum_{d_1} g(d_1)\left(\frac{d_1}{d_2}\right)^{-1}\mathbb{N}d_1^{-2s_1}.$$

Assuming that d_1 and d_2 are relatively prime, the inner sum can be rewritten as

$$\sum_{d_1} g(d_1)\left(\frac{d_1}{d_2}\right)^{-1}\mathbb{N}d_1^{-2s_1} = \sum_{d_1}\frac{g(d_2, d_1)}{\mathbb{N}d_1^{2s_1}}. \tag{8}$$

We recognize this inner sum as part of the Fourier coefficient of a metaplectic Eisenstein series on the 3-fold cover of GL_2. Patterson [13], [14] proved that

$$\mathrm{Res}_{s_1=2/3}\sum_{d_1}\frac{g(d_2, d_1)}{\mathbb{N}d_1^{2s_1}}\mathbb{N}d_2^{1/6} = \tau(d_2), \tag{9}$$

the (d_2)-th Fourier coefficient of the cubic theta function. This is true because the Dirichlet series on the left is the (d_2)-th Fourier coefficient of a metaplectic Eisenstein series whose residue is the theta function. Moreover, Patterson evaluated τ finding that

$$\tau(n) = \begin{cases} \mathbb{N}d^{1/2}\,\overline{G(1,c)} & \text{if } n = cd^3 \text{ with } c \text{ is squarefree,} \\ 0 & \text{if } n \text{ is not of this form.} \end{cases} \tag{10}$$

Inserting the right-hand side of (9) into the expression for $\mathrm{Res}_{s_1=2/3}Z(s_1,s_2,s_3)$, we get (assuming d_2 is square-free)

$$\begin{aligned}\sum_{d_2,d_3} g(d_2)g(d_3)\left(\frac{d_2}{d_3}\right)^{-1}\tau(d_2)\mathbb{N}d_2^{-2s_2-1/6}\mathbb{N}d_3^{-2s_3} &= \\ \sum_{d_2,d_3} G(1,d_3)\left(\frac{d_2}{d_3}\right)^{-1}\mathbb{N}d_2^{-2s_2+1/3}\mathbb{N}d_3^{-2s_3+1/2} &= Z_2(2s_2-1/3,2s_3-1/2).\end{aligned}$$

This gives a heuristic version of (4).

By a similar argument, taking the residue in $s_2=2/3$ we get

$$\begin{aligned}\mathrm{Res}_{s_2=2/3}\left(\sum_{d_1,d_3} g(d_1)g(d_3)\mathbb{N}d_1^{-2s_1}\mathbb{N}d_3^{-2s_3}\sum_{d_2} g(d_1d_3,d_2)\mathbb{N}d_2^{-2s_2}\right) &= \\ \sum_{d_1,d_3} g(d_1)g(d_3)\tau(d_1d_3)\mathbb{N}d_1^{-2s_1-1/6}\,\mathbb{N}d_3^{-2s_3-1/6}. & \qquad (11)\end{aligned}$$

Since we are taking d_1 and d_3 to be squarefree and coprime, we have

$$\tau(d_1d_3)=\overline{G(d_1d_3)}=\overline{G(1,d_1)}\ \ \overline{G(1,d_3)}\left(\frac{d_1}{d_3}\right)^{-1}\left(\frac{d_3}{d_1}\right)^{-1}.$$

Recalling that our representatives d_i are restricted so that cubic reciprocity holds in the form (6), the right-hand side of (11) equals

$$\sum\left(\frac{d_1}{d_3}\right)d_1^{-2s_1+1/3}d_3^{-2s_3+1/3}=Z_1(2s_1-1/3,2s_3-1/3).$$

From this we get a heuristic version of (5).

This heuristic approach suggests that residues of the $A_3^{(3)}$ Weyl group multiple Dirichlet series realize the Dirichlet series of Friedberg, Hoffstein, and Lieman. It also shows the appropriate changes of variables for the residues of the A_3 series in order to exactly realize the series from [11]. We show that these are consistent with functional equations coming from the Weyl group multiple Dirichlet series in the next section.

3 Functional Equations

Friedberg, Hoffstein and Lieman [11] proved the following functional equations for Z_1 and Z_2.

$$Z_1(s,w)=Z_1(w,s), \qquad (12)$$

$$Z_2(1-s, w+s-1/2) = Z_1(s,w), \tag{13}$$

$$Z_2(s+w-1/2, 1-w) = Z_2(s,w). \tag{14}$$

Let us show how these follow from (4) and (5).

The A_3 functional equations are generated by the following transformations:

$$\begin{aligned}
\sigma_1 : (s_1, s_2, s_3) &\longmapsto \left(1-s_1, s_1+s_2+\frac{1}{2}, s_3\right), \\
\sigma_2 : (s_1, s_2, s_3) &\longmapsto \left(s_1+s_2+\frac{1}{2}, 1-s_2, s_2+s_3+\frac{1}{2}\right), \\
\sigma_3 : (s_1, s_2, s_3) &\longmapsto \left(s_1, s_2+s_3+\frac{1}{2}, 1-s_3\right).
\end{aligned}$$

Thus (ignoring, as we promised in the introduction, a finite scattering matrix) $Z(s_1, s_2, s_3; A_3^{(3)})$ is invariant under these transformations. They generate the A_3 Weyl group, which is the symmetric group of order 24. Since the A_3 Dynkin diagram has a symmetry, there is a larger group of order 48 which may be regarded as the full group of automorphisms of $Z(s_1, s_2, s_3; A_3^{(3)})$.

Invariance under σ_3 gives the relationship

$$Z(s_1, s_2, s_3; A_3^{(3)}) = Z\left(s_1, s_2+s_3+\frac{1}{2}, 1-s_3; A_3^{(3)}\right)$$

Taking the residue at $s_1 = \frac{2}{3}$ and using (5) gives

$$Z_2\left(2s_2-\frac{1}{3}, 2s_3-\frac{1}{2}\right) = Z_2\left(2s_2+2s_3-\frac{4}{3}, \frac{3}{2}-2s_3\right)$$

which, taking $s = 2s_2 - \frac{1}{3}$ and $w = 2s_3 - \frac{1}{2}$ gives (14). Similarly invariance under $\sigma_1\sigma_2$ gives

$$Z(s_1, s_2, s_3; A_3^{(3)}) = Z\left(\frac{3}{2}-s_1-s_2, s_1, s_2+s_3-\frac{1}{2}; A_3^{(3)}\right).$$

Taking residues at $s_1 = \frac{2}{3}$ and using both (4) and (5) gives

$$Z_2\left(2s_2-\frac{1}{3}, 2s_3-\frac{1}{2}\right) = Z_1\left(\frac{4}{3}-2s_2, 2s_2+2s_3-\frac{4}{3}\right),$$

which implies (13). Finally, (12) follows by taking the residue at $s_2 = \frac{2}{3}$ in the relation

$$Z(s_1, s_2, s_3; A_3^{(3)}) = Z(s_3, s_2, s_1; A_3^{(3)}), \tag{15}$$

which is the non-Weyl group symmetry corresponding to the symmetry of the Dynkin diagram.

We now consider the following paradox. The A_3 Weyl group W, together with the "outer" symmetry (15) generate a group $\tilde{W}$ of order 48. Yet Friedberg, Hoffstein and Lieman [11] describe the group of functional equations generated by (12), (13) and (14) as a nonabelian group of order 32. Since $32 \nmid 48$, this group is not a subgroup of the group of order 48. How can this be?

When we start with several objects $Z_i(s_1, s_2, \cdots, s_r)$, $(i = 1, \cdots, k)$ which are interchanged among themselves by functional equations, we must carefully describe what we mean by a functional equation. We can take several copies of the variables, s_{ij} $(i = 1, \cdots, k, j = 1, \cdots, r)$ and consider the map $Z : \mathbb{C}^{rk} \longrightarrow \mathbb{C}^k$ obtained by putting these together; that is,

$$Z(s_{11}, \cdots, s_{kr}) = (Z_1(s_{11}, \cdots, s_{1r}), \cdots, Z_k(s_{k1}, \cdots, s_{kr})).$$

A "functional equation" from this point of view is any linear (or affine) transformation L of $\mathbb{C}^{rk}$ such that there exists a transformation L' of $\mathbb{C}^k$ such that the diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^{rk} & \xrightarrow{L} & \mathbb{C}^{rk} \\ \downarrow Z & & \downarrow Z \\ \mathbb{C}^{k} & \xrightarrow{L'} & \mathbb{C}^{k} \end{array}$$

In the examples we have seen such groups have been wreath products, typically of the form $H^k \rtimes S_k$, where H is a group of linear (or affine) transformations just of $\mathbb{C}^r$, and S_k is the symmetric group. This means that if $|H| = h$, then the order of the group of functional equations is $k!\, h^k$.

In the case at hand, the group H can be taken to be the group of functional equations satisfied by Z_1 alone. This group is generated by (12) and the symmetry

$$Z_1(s, w) = Z_1(1 - w, 1 - s),$$

which may be obtained by taking the residue in the Weyl group symmetry (corresponding to $\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$)

$$Z(s_1, s_2, s_3; A_3^{(3)}) = Z\left(\frac{3}{2} - s_2 - s_3, s_2, \frac{3}{2} - s_2 - s_1; A_3^{(3)}\right),$$

then taking the residue when $s_2 = \frac{2}{3}$ and taking $s = 2s_1 - \frac{1}{3}$, $w = 2s_3 - \frac{1}{3}$. The group $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the cyclic group of order two. There is a similar group of functional equations satisfied by Z_2 alone; this group, also isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Combining these two groups as described above produces the wreath product $H^2 \rtimes S_2$, a nonabelian group of order 32. The S_2, interchanging the two copies of $\mathbb{C}^2$, is generated by a transformation corresponding to (13), relating Z_1 and Z_2. The paradox is thus resolved, since while the group H itself is a subgroup of the group $\tilde{W}$, this wreath product is not.

Two more examples of wreath products occurring as groups of functional equations are the group of order $384 = 4^3 \cdot 3!$ which occurs in Brubaker, Friedberg and Hoffstein [6], and the group of order $7{,}962{,}624 = 24^4 \cdot 4!$ in the example of Brubaker [2]. The latter group is $H^4 \rtimes S_4$, where the group $H \cong S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has order 24 but is not isomorphic to S_4. The second wreath product can be understood by realizing the multiple Dirichlet series as triple residues of the Weyl group multiple Dirichlet series $Z(s_1, \cdots, s_6; E_6^{(3)})$ and the former is a variant involving a GL_2 metaplectic cusp form.

4 The Precise Definition of $Z(s_1, s_2, s_3; A_3^{(3)})$

Recall, as in [3], that to each reduced root system Φ of rank r and each integer n sufficiently large, we have an associated Dirichlet series of the form

$$Z(s_1, \cdots, s_r) = \sum_{C_1} \cdots \sum_{C_r} \frac{H(C_1, \cdots, C_r)\Psi(C_1, \cdots, C_r)}{\mathbb{N}C_1^{2s_1} \cdots \mathbb{N}C_r^{2s_r}}.$$

We will recall this definition here, but for simplicity, we will assume that Φ is *simply-laced*, meaning that all roots have the same length; we normalize them to have length 1.

The function $\Psi(C_1, \cdots, C_r)$ satisfies a transformation property when C_i is changed by a unit or an n-th power. This property interacts with the twisted multiplicativity of $H(C_1, \cdots, C_r)$ in order to guarantee that the sum is well-defined up to units. In view of the convention, stated in the introduction, that $\sum_C$ means summing over ideals $C\mathfrak{o}$ and choosing the generator $C \equiv 1$ modulo 3, we could omit Ψ from the following discussion, taking it to be 1. However we will include it in order to facilitate comparison of this paper with [4].

We remark that the foundations in [3] are different from those in Friedberg, Hoffstein and Lieman [11]. The function Ψ corresponds roughly to the ray class characters built into the definitions of Z_1 and Z_2 in [11]. As explained in the introduction, we will not try to match these up, since they have only to do with the ternary prime.

This series exhibits a twisted multiplicativity described in the introduction to [4], involving n-th order residue symbols. In the case of the A_3 root system, this multiplicativity is given, for $\gcd(C_1C_2C_3, C_1'C_2'C_3') = 1$, by

$$\frac{H(C_1C_1', C_2C_2', C_3C_3')}{H(C_1, C_2, C_3)H(C_1', C_2', C_3')} =$$

$$\left(\frac{C_1}{C_2'}\right)^{-1}\left(\frac{C_1'}{C_2}\right)^{-1}\left(\frac{C_2}{C_3'}\right)^{-1}\left(\frac{C_2'}{C_3}\right)^{-1}\prod_{i=1}^{3}\left(\frac{C_i}{C_i'}\right)\left(\frac{C_i'}{C_i}\right). \tag{16}$$

This multiplicativity reduces the specification of $H(C_1, \cdots, C_r)$ to values at prime powers $H(p^{k_1}, \cdots, p^{k_r})$ for $k_1, \cdots, k_r$ non-negative integers and p any prime. The non-zero values at these prime powers are in one-to-one correspondence with the Weyl group associated to Φ. More precisely, $H(p^{k_1}, \cdots, p^{k_r})$ for $k_1, \cdots, k_r$ will be non-zero if and only if there exists a $w \in W$ such that

$$\rho - w\rho = \sum_{\alpha_i \in \Delta} k_i \alpha_i$$

where ρ is the Weyl vector, equal to half the sum of the positive roots, and Δ is the collection of simple positive roots. In this case,

$$H(p^{k_1}, \cdots, p^{k_r}) = \prod_{\alpha \in \Phi_w} g(p^{d(\alpha)-1}, p^{d(\alpha)})$$

where Φ_w is the set of positive roots α such that $w(\alpha)$ is a negative root and $d(\alpha)$ is the height of the root α, given by the expression

$$d(\alpha) = \frac{2\langle \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

Given any fixed prime p, Table 1 gives the p-power values $H(p^{k_1}, p^{k_2}, p^{k_3})$ for the $Z(s_1, s_2, s_3; A_3^{(3)})$, where we list the element of the Weyl group S_4, the symmetric group, in the left column and the corresponding non-zero monomial in the right column. The resulting 24 terms depend implicitly on

$w \in S_4$	$\prod_{w\in\Phi_w} g(p^{d(\alpha)-1}, p^{d(\alpha)})\mathbb{N}p^{-2k_1s_1-2k_2s_2-2k_3s_3}$
1	1
(12)	$g(1,p)\ \mathbb{N}p^{-2s_1}$
(23)	$g(1,p)\mathbb{N}p^{-2s_2}$
(123)	$g(1,p)g(p,p^2)\mathbb{N}p^{-4s_1-2s_2}$
(34)	$g(1,p)\mathbb{N}p^{-2s_3}$
(12)(34)	$g(1,p)^2\mathbb{N}p^{-2s_1-2s_3}$
(132)	$g(1,p)g(p,p^2)\mathbb{N}p^{-2s_1-4s_2}$
(13)	$g(1,p)^2g(p,p^2)\mathbb{N}p^{-4s_1-4s_2}$
(24)	$g(1,p)^2g(p,p^2)\mathbb{N}p^{-4s_2-4s_3}$
(124)	$g(1,p)^2g(p,p^2)g(p^2,p^3)\mathbb{N}p^{-6s_1-4s_2-4s_3}$
(243)	$g(1,p)g(p,p^2)\mathbb{N}p^{-2s_2-4s_3}$
(1243)	$g(1,p)g(p,p^2)^2\mathbb{N}p^{-4s_1-2s_2-4s_3}$
(234)	$g(1,p)g(p,p^2)\mathbb{N}p^{-4s_2-2s_3}$
(1234)	$g(1,p)g(p,p^2)g(p^2,p^3)\mathbb{N}p^{-6s_1-4s_2-2s_3}$
(14)	$g(1,p)^2g(p,p^2)^2g(p^2,p^3)\mathbb{N}p^{-6s_1-6s_2-6s_3}$
(142)	$g(1,p)^2g(p,p^2)g(p^2,p^3)\mathbb{N}p^{-2s_1-6s_2-6s_3}$
(143)	$g(1,p)^2g(p,p^2)g(p^2,p^3)\mathbb{N}p^{-4s_1-4s_2-6s_3}$
(1432)	$g(1,p)g(p,p^2)g(p^2,p^3)\mathbb{N}p^{-2s_1-4s_2-6s_3}$
(134)	$g(1,p)^2g(p,p^2)g(p^2,p^3)\mathbb{N}p^{-6s_1-6s_2-2s_3}$
(1342)	$g(1,p)^2g(p^2,p^3)\mathbb{N}p^{-2s_1-6s_2-2s_3}$
(1423)	$g(1,p)^2g(p,p^2)^2g(p^2,p^3)\mathbb{N}p^{-4s_1-8s_2-6s_3}$
(14)(23)	$g(1,p)^3g(p,p^2)^2g(p^2,p^3)\mathbb{N}p^{-6s_1-8s_2-6s_3}$
(1324)	$g(1,p)^2g(p,p^2)^2g(p^2,p^3)\mathbb{N}p^{-6s_1-8s_2-4s_3}$
(13)(24)	$g(1,p)g(p,p^2)^2g(p^2,p^3)\mathbb{N}p^{-4s_1-8s_2-4s_3}$

Table 1: All term in the p-part of $Z(s_1, s_2, s_3; A_3^{(3)})$.

the degree of the cover $n \geqslant 3$ as the Gauss sums are made of n-th order residue symbols.

We now recall the main theorem of [4] which gives the functional equations and analytic continuation of $Z(s_1, \ldots, s_r; \Phi^{(n)})$. First we need to attach additional zeta and gamma factors to our definition of Z. If α is a positive root, write $\alpha = \sum k_i \alpha_i$ as before. Let

$$\zeta_\alpha(\mathbf{s}) = \zeta\left(1 + 2n \sum_{i=1}^{r} k_i (s_i - \frac{1}{2})\right) \tag{17}$$

and

$$\boldsymbol{G}_\alpha(\mathbf{s}) = \boldsymbol{G}_n\left(\frac{1}{2} + \sum_{i=1}^{r} k_i \left(s_i - \frac{1}{2}\right)\right)$$

where $\boldsymbol{G}_n(s)$ is defined by

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}. \tag{18}$$

We define the *normalized* multiple Dirichlet series by

$$Z^*_\Psi(\mathbf{s}) = \left[\prod_{\alpha \in \Phi^+} G_\alpha(\mathbf{s})\, \zeta_\alpha(\mathbf{s})\right] Z_\Psi(\mathbf{s}). \tag{19}$$

In the case of $\Phi = A_3$ and $n = 3$, we have six positive roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ corresponding to the zeta factors

$$\zeta(6s_1 - 2)\zeta(6s_2 - 2)\zeta(6s_3 - 2)$$
$$\zeta(6s_1 + 6s_2 - 5)\zeta(6s_2 + 6s_3 - 5)\zeta(6s_1 + 6s_2 + 6s_3 - 8).$$

Then we have the following theorem from [4], stated here in the case where the associated root system Φ is simply-laced, as in the case of GL_r, for any r. Let Δ denote a base of positive simple roots for Φ.

Theorem 2 *The function $Z^*_\Psi(\mathbf{s})$ has meromorphic continuation to the complex space $\mathbb{C}^r$ and satisfies the functional equation*

$$Z^*_{w\Psi}(w\mathbf{s}) = Z^*_\Psi(\mathbf{s})$$

for each $w \in W$, where the action of w on functions Ψ is given as in [4]. It is analytic except along the hyperplanes $\mu(\alpha; \mathbf{s}) = \frac{1}{2} + \frac{1}{2n}$, where α runs through Φ, n is the degree of the cover, and μ is defined by

$$\mu(\alpha; \mathbf{s}) = \sum_{\alpha_i \in \Delta} k_i s_i \quad \textit{where} \quad \alpha = \sum_{\alpha_i \in \Delta} k_i \alpha_i.$$

Along these hyperplanes it can have simple poles.

In particular, for $\Phi = A_3$ and $n = 3$, the leading pole in each s_i occurs at $s_i = 2/3$.

5 Multiplicativity in Residual Dirichlet Series

In this section we will confirm that the multiplicativity of the two sides in (4) and (5) are consistent. In subsequent sections, this allows us to verify (4) and (5) by comparing the p-parts of the two sides.

Lemma 1 *If $C_2, C_3 \equiv 1$ mod 3 in $\mathfrak{o}$, define coefficients $\phi(C_2, C_3)$ by*

$$Res_{s_1=2/3} Z(s_1, s_2, s_3; A_3^{(3)}) = \sum \phi(C_2, C_3)\, \mathbb{N}C_2^{-2s_1} \mathbb{N}C_3^{-2s_2}.$$

Then if $\gcd(C_2C_3, C_2'C_3') = 1$ *we have*

$$\frac{\phi(C_2C_2', C_3C_3')}{\phi(C_2, C_3)\, \phi(C_2', C_3')} = \left(\frac{C_3}{C_3'}\right)\left(\frac{C_3'}{C_3}\right)\left(\frac{C_2}{C_3'}\right)^{-1}\left(\frac{C_2'}{C_3}\right)^{-1}. \tag{20}$$

This is what we want, since this is the same multiplicativity satisfied by the coefficients of $Z_2(s, w)$, where $s = 2s_2 - \frac{1}{3}$, $w = 2s_3 - \frac{1}{2}$. Note that by reciprocity we can write $\left(\frac{C_3}{C_3'}\right)\left(\frac{C_3'}{C_3}\right) = \left(\frac{C_3}{C_3'}\right)^{-1}$, but we are writing it as above to facilitate comparison with [4].

Proof As in [4], we may rewrite

$$Z(s_1, s_2, s_3; A_3^{(3)}) \sum_{C_1, C_2, C_3} \frac{H(C_1, C_2, C_3)\Psi(C_1, C_2, C_3)}{\mathbb{N}C_1^{2s_1} \mathbb{N}C_2^{2s_2} \mathbb{N}C_3^{2s_3}}$$

as a collection of Kubota Dirichlet series in s_1 having poles at $s_1 = 2/3$ multiplied by monomials in the variable s_i, $i = 1, 2, 3$. This is somewhat difficult, but carried out in [4]. Performing this, we are left with

$$\mathrm{Res}_{s_1=2/3} \sum_{\substack{C_2, C_3 \\ C_1 \text{ reduced}}} \frac{H(C_1, C_2, C_3)}{\mathbb{N}C_1^{2s_1}\mathbb{N}C_2^{2s_2}\mathbb{N}C_3^{2s_3}} \mathcal{D}(s_1, \Psi^{C_1,C_2,C_3}, C_1^{-2}C_2) \tag{21}$$

where the definition of "reduced" is given in Definition 5.1 of [4], coming from the manner in which we assemble pairs of terms like those in the table into Kubota Dirichlet series. In particular, the definition guarantees that $C_1^{-2}C_2$ is an integer. (The definition of "reduced" will be discussed in more detail in the following section, as needed.) We may write (9) in the form

$$\mathrm{Res}_{s=2/3} \ \mathbb{N}C_0^{s-1/2}\mathcal{D}(s, \Psi, C_0) = \tau(C_0).$$

Again, this residue should depend on the choice of function Ψ in the Dirichlet series. We may choose Ψ to correspond to Patterson's Eisenstein series defined in [13] to obtain the above equality. Then taking the residue gives

$$\sum_{\substack{C_2, C_3 \\ C_1 \text{ reduced}}} \frac{H(C_1, C_2, C_3)}{\mathbb{N}C_2^{2s_2}\mathbb{N}C_3^{2s_3}} \mathbb{N}C_1^{-1}\mathbb{N}C_2^{-1/6}\tau(C_1^{-2}C_2).$$

Now we check the multiplicativity of resulting object. We have

$$\phi(C_2, C_3) = \sum_{C_1 \text{ reduced}} H(C_1, C_2, C_3)\mathbb{N}C_1^{-1}\mathbb{N}C_2^{-1/6}\tau(C_1^{-2}C_2).$$

If C_1 is reduced and C_1' is coprime to C_1, then C_1C_1' is also reduced, so we may now compute (20). As shown in [4], the integer $C_0 = C_1^{-2}C_2$ is cube-free (and in general, n-th power free for A_n). Using (10),

$$\tau(C_0) = \tau(C_1^{-2}C_2) = \overline{G_1(1, C_1^{-2}C_2)}.$$

By elementary properties of the Gauss sum, and cubic reciprocity, we have

$$\frac{G(1, C_1^{-2}(C_1')^{-2}C_2C_2')}{G(1, C_1^{-2}C_2)G(1, (C_1')^{-2}C_2')} = \left(\frac{C_1^{-2}C_2}{(C_1')^{-2}C_2'}\right)\left(\frac{(C_1')^{-2}C_2'}{C_1^{-2}C_2}\right) = \left(\frac{C_1'C_2'}{C_1C_2}\right)^{-1}, \tag{22}$$

whenever $\gcd(C_1C_2, C_1'C_2') = 1$. Upon taking conjugates of the Gauss sums, we obtain conjugate characters. If the reduced C_1 has value 1 or a single prime p, then the multiplicativity of the two numerators agrees. We multiply (16) by the conjugate of (22), then use reciprocity and the multiplicativity of the power residue symbol to obtain the right-hand side of (20). $\square$

Lemma 2 *If $C_2, C_3 \equiv 1$ mod 3 in $\mathfrak{o}$, define coefficients $\rho(C_1, C_3)$ by*

$$\mathit{Res}_{s_2=2/3} Z(s_1, s_2, s_3; A_3^{(3)}) = \sum \rho(C_1, C_3)\, \mathbb{N}C_1^{-2s_1} \mathbb{N}C_3^{-2s_2}.$$

Then if $\gcd(C_1C_3, C_1'C_3') = 1$ *we have*

$$\frac{\rho(C_1C_1', C_3C_3')}{\rho(C_1, C_3)\, \rho(C_1', C_3')} = \left(\frac{C_1}{C_3'}\right)\left(\frac{C_1'}{C_3}\right). \tag{23}$$

Proof Again, we want to rewrite this series as a collection of Kubota Dirichlet series, now in s_2 having leading poles at $s_2 = 2/3$ multiplied by monomials in the variable s_i, $i = 1, 2, 3$. We obtain

$$\mathrm{Res}_{s_2=2/3} \sum_{\substack{C_1, C_3 \\ C_2 \text{ reduced}}} \frac{H(C_1, C_2, C_3)}{\mathbb{N}C_1^{2s_1} \mathbb{N}C_2^{2s_2} \mathbb{N}C_3^{2s_3}} \mathcal{D}(s_2, \Psi^{C_1,C_2,C_3}, C_1C_2^{-2}C_3). \tag{24}$$

Taking the residue, we find that

$$\rho(C_1, C_3) = \sum_{C_2 \text{ reduced}} H(C_1, C_2, C_3)\, \mathbb{N}C_2^{-1} (\mathbb{N}C_1 \mathbb{N}C_3)^{-1/6} \tau(C_1C_2^{-2}C_3).$$

To analyze the multiplicativity of ρ, we use (16), (10) and the fact that

$$\frac{G(C_1C_1'C_2^{-2}(C_2')^{-2}C_3C_3')}{G(C_1C_2^{-2}C_3)G(C_1'(C_2')^{-2}C_3')} = \left(\frac{C_1C_2^{-2}C_3}{C_1'(C_2')^{-2}C_3'}\right)\left(\frac{C_1'(C_2')^{-2}C_3'}{C_1C_2^{-2}C_3}\right) = \left(\frac{C_1'C_2'C_3'}{C_1C_2C_3}\right)^{-1},$$

which then needs to be conjugated to obtain the result coming from the definition of the theta function. We obtain

$$\left(\frac{C_1'C_2'C_3'}{C_1C_2C_3}\right) \cdot \left(\frac{C_1}{C_2'}\right)^{-1} \left(\frac{C_1'}{C_2}\right)^{-1} \left(\frac{C_2}{C_3'}\right)^{-1} \left(\frac{C_2'}{C_3}\right)^{-1} \prod_{i=1}^{3} \left(\frac{C_i}{C_i'}\right)^2,$$

This simplifies to give (23). $\square$

We will show now that the multiplicativity (23) agrees with that satisfied by the coefficients of $Z_1(s,w)$, where $s = 2s_1 - \frac{1}{3}$, $w = 2s_3 - \frac{1}{3}$. Recall that the character χ_{C_3} is the primitive cubic residue symbol associated to the integer $C_3 = bc^2d^3$ with b, c, and d square-free according to

$$\chi_{C_3}(m) = \chi_b(m)\overline{\chi}_c(m).$$

The character χ_{C_3} is the only constituent of Z_1, defined by (1) and (2), that is not completely multiplicative, and this contributes

$$\frac{\left(\frac{C_1C_1'}{C_3C_3'}\right)}{\left(\frac{C_1}{C_3}\right)\left(\frac{C_1'}{C_3'}\right)} = \left(\frac{C_1}{C_3'}\right)\left(\frac{C_1'}{C_3}\right)$$

whenever the characters are non-zero for choices of C_1, C_1', C_3, C_3', matching the multiplicativity (23).

6 First main theorem: the s_2 residue

In order to take the residue in the variable s_2 of $Z(s_1, s_2, s_3; A_3^{(3)})$, we organize the Dirichlet series as in (24), in terms of Kubota Dirichlet series coming from a GL_2 metaplectic Eisenstein series. As shown in the last section, because the multiplicativity of the residue matches that of the series in [11], it is enough to consider only the prime power coefficients. Using (24), we see that prime powered terms appear in the sum when C_1, C_2, C_3 are all powers of a single prime p with C_2 reduced.

Let us recall the precise definition of "reduced" as in [4], Section 5, in the case $\Phi = A_3$.

Recall that ρ is defined to be half the sum of the positive roots, so for A_3, the positive roots are

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$$

and $\rho = \frac{3}{2}\alpha_1 + 2\alpha_2 + \frac{3}{2}\alpha_3$. We say that (C_1, C_2, C_3) is *admissible* if for each prime p, there is a w_p depending on p, such that the following equality holds:

$$\rho - w_p(\rho) = \mathrm{ord}_p(C_1)\alpha_1 + \mathrm{ord}_p(C_2)\alpha_2 + \mathrm{ord}_p(C_3)\alpha_3.$$

For such admissible triples, C_i is reduced if, for every prime p,

$$l(\sigma_i w_p) = l(w_p) + 1$$

Series with p-power terms	Residue at $s_2 = \frac{2}{3}$
$\psi_1(s_2)$	1
$g(1,p)\mathbb{N}p^{1/2-2s_1-s_2}\psi_p(s_2)$	$g(1,p)\mathbb{N}p^{-1/6-2s_1}\tau(p)$
$g(1,p)g(p,p^2)\mathbb{N}p^{-4s_1-2s_2}\psi_1(s_2)$	$g(1,p)g(p,p^2)\mathbb{N}p^{-4/3-4s_1}$
$g(1,p)\mathbb{N}p^{1/2-s_2-2s_3}\psi_p(s_2)$	$g(1,p)\mathbb{N}p^{-1/6-2s_3}\tau(p)$
$g(1,p)^2\mathbb{N}p^{1-2s_1-2s_2-2s_3}\psi_{p^2}(s_2)$	$g(1,p)^2\mathbb{N}p^{1/3-2s_1-2s_3}\tau(p^2)$
$g(1,p)g(p,p^2)\mathbb{N}p^{-2s_2-4s_3}\psi_1(s_2)$	$g(1,p)g(p,p^2)\mathbb{N}p^{-4/3-4s_3}$
$g(1,p)^2g(p,p^2)g(p^2,p^3)$ $\times\mathbb{N}p^{1/2-6s_1-5s_2-4s_3}\psi_p(s_2)$	$g(1,p)^2g(p,p^2)g(p^2,p^3)$ $\times\mathbb{N}p^{-17/6-6s_1-4s_3}\tau(p)$
$g(1,p)g(p,p^2)^2$ $\times\mathbb{N}p^{1-4s_1-4s_2-4s_3}\psi_{p^2}(s_2)$	$g(1,p)g(p,p^2)^2$ $\times\mathbb{N}p^{-5/3-4s_1-4s_3}\tau(p^2)$
$g(1,p)g(p,p^2)g(p^2,p^3)$ $\times\mathbb{N}p^{-6s_1-4s_2-2s_3}\psi_1(s_2)$	$g(1,p)g(p,p^2)g(p^2,p^3)$ $\times\mathbb{N}p^{-8/3-6s_1-2s_3}$
$g(1,p)^2g(p,p^2)^2g(p^2,p^3)$ $\times\mathbb{N}p^{-6s_1-6s_2-6s_3}\psi_1(s_2)$	$g(1,p)^2g(p,p^2)^2g(p^2,p^3)$ $\times\mathbb{N}p^{-4-6s_1-6s_3}$
$g(1,p)g(p,p^2)g(p^2,p^3)$ $\times\mathbb{N}p^{-2s_1-4s_2-6s_3}\psi_1(s_2)$	$g(1,p)g(p,p^2)g(p^2,p^3)$ $\times\mathbb{N}p^{-8/3-2s_1-6s_3}$
$g(1,p)^2g(p,p^2)g(p^2,p^3)$ $\times\mathbb{N}p^{1/2-4s_1-5s_2-6s_3}\psi_p(s_2)$	$g(1,p)^2g(p,p^2)g(p^2,p^3)$ $\times\mathbb{N}p^{-17/6-4s_1-6s_3}\tau(p)$

Table 2: All terms in the p-part of the residue at $s_2 = \frac{2}{3}$

where $l(\alpha)$ denotes the length of a root α as a product of simple positive roots.

In short, admissible triples (C_1, C_2, C_3) are those for which $H(C_1, C_2, C_3)$ is non-zero. At each prime p, these admissible triples come in pairs corresponding to the elements $(w_p, \sigma_i(w_p))$. These pairs form the p-part of a Kubota Dirichlet series in s_i (up to a twisted multiplicativity). We enumerate these pairs by choosing the reduced member in each case (the triple in the pair with $\mathrm{ord}_p(C_i)$ minimal).

Note that if $(C_1, C_2, C_3) = (p^{k_1}, p^{k_2}, p^{k_3})$ for some fixed prime p, $k_i \geqslant 0$, then for each prime $q \neq p$, $w_q = 1$ and the triple automatically has C_2 reduced at every prime $q \neq p$. At the distinguished prime p, we have our table of 24 non- zero triples, of which 12 necessarily have C_2 reduced according to the definition. Then by (24), the prime power coefficients in the residue come from the following terms in the left-hand column in Table 2.

In Table 2 we use Kubota's notation

$$\psi_m(s_2) = \mathbb{N}m^{(1/2-s_2)}\mathcal{D}(s_2, \Psi, m), \tag{25}$$

where as in [3] and [4] we define

$$\mathcal{D}(s, \Psi, m) = \sum_{0 \neq c \in \mathfrak{o}_S/\mathfrak{o}_S^\times} g(m, c)\, \Psi(c)\, \mathbb{N}(c)^{-2s} .$$

Thus

$$\mathrm{Res}_{s_2=2/3}\psi_{p^k}(s_2) = \tau(p^k),$$

the p^k-th coefficient of the cubic theta function. The results of this are shown in the right-hand column of the above table.

Since $n = 3$ we may use the identities $\tau(p) = \overline{G(1,p)}$, $\tau(p^2) = 0$, and $g(p^2, p^3) = -\mathbb{N}p^2$ to simplify the resulting 10 non-zero terms:

1	$\mathbb{N}p^{\frac{1}{3}-2s_1}$
$\mathbb{N}p^{2/3-4s_1}$	$\mathbb{N}p^{1/3-2s_3}$
$\mathbb{N}p^{2/3-4s_3}$	$-\mathbb{N}p^{5/3-6s_1-4s_3}$
$-\mathbb{N}p^{4/3-6s_1-2s_3}$	$-\mathbb{N}p^{4/3-2s_1-6s_3}$
$-\mathbb{N}p^{5/3-4s_1-6s_3}$	$-\mathbb{N}p^{2-6s_1-6s_3}$

The above collection of 10 monomials is the sum total of terms which involve p-powers in the residue at $s_2 = 2/3$. We are now armed with the data to prove the first of our main theorems.

Theorem 3 *With notation as above, and P as in (2), we have*

$$\mathit{Res}_{s_2=2/3} \quad \zeta(6s_1-1)\zeta(6s_3-1)\zeta(6s_1+6s_3-4)Z(s_1,s_2,s_3;A_3^{(3)}) \quad = \sum_d \frac{L(s,\chi_{d_0})P(s,d)}{\mathbb{N}d^w}$$

under the change of variables $s = 2s_1 - \frac{1}{3}$, $w = 2s_3 - \frac{1}{3}$.

This is (5). Note that the original definition of $Z(s_1, s_2, s_3; \Phi^{(n)})$ has accompanying zeta factors in one-to-one correspondence with positive roots of Φ. In the case of A_3 this gives six zeta factors. In the residue at $s_2 = 2/3$

one of these zeta factors reduces to a constant $\zeta(2)$ which we ignore, and five zeta factors in s_1 and s_3 of the form

$$\zeta(6s_1-2)\zeta(6s_3-2)\zeta(6s_1-1)\zeta(6s_3-1)\zeta(6s_1+6s_3-4).$$

In the statement of the theorem above, we have omitted the zeta factors

$$\zeta(3s-1)=\zeta(6s_1-2), \qquad \zeta(3w-1)=\zeta(6s_3-2). \tag{26}$$

These are interchanged by the functional equation $(s,w)\mapsto(w,s)$. The gamma factors, which we have also omitted, are worth discussing. Using the triplication formula, we have

$$\boldsymbol{G}_3(s)=(2\pi)^{-(4s-1)}\,3^{6s-\frac{7}{2}}\Gamma\left(2s-\frac{2}{3}\right)\Gamma\left(2s-\frac{1}{3}\right). \tag{27}$$

Using this, the gamma factor in (19) produces 12 gamma functions, two of which are constant in the residue; the remaining 10 include

$$\Gamma(s), \qquad \Gamma(w), \qquad \Gamma\left(s+w-\frac{1}{3}\right), \qquad \Gamma\left(s+w-\frac{2}{3}\right),$$

which are exactly the gamma factors associated to Z_1 in [11], and six others, which, using the triplication formula again, produce

$$\Gamma(3s-1)\Gamma(3w-1).$$

These are the gamma factors that correspond to our "spare" zeta functions (26).

Proof Having checked that the multiplicativity agrees in the previous section, it remains to check the equality stated in the theorem at prime- powered terms with respect to a single prime p. Writing out the left-hand side of the equation using the ten non-zero terms from the residue in s_2 and the accompanying zeta factors, all under the change of variables $(2s_1-1/3, 2s_3-1/3)\mapsto(s,w)$ gives

$$\begin{aligned}\zeta^{(p)}(3s)\zeta^{(p)}(3w)\zeta^{(p)}(3s+3w-2)&\\ \big(1+\mathbb{N}p^{-s}+\mathbb{N}p^{-2s}+\mathbb{N}p^{-w}+\mathbb{N}p^{-2w}&\\ -\mathbb{N}p^{-3s-2w}-\mathbb{N}p^{-3s-w}-\mathbb{N}p^{-s-3w}-\mathbb{N}p^{-2s-3w}-\mathbb{N}p^{-3s-3w}\big)&\end{aligned} \tag{28}$$

where the superscript (p) in the zeta factors indicates we are only considering the Euler factor at the prime p. We would like to compare this to the Dirichlet series on the right-hand side of the equality in the theorem, previously labeled as $Z_1(s,w)$. Note that $P(s,d)$ is Eulerian and can be written explicitly as

$$P(s,d) = \prod_{p^k || R} \left(1 - \chi_{d_0}(p)\mathbb{N}p^{-s} + \mathbb{N}p^{2-3s} - \cdots + \mathbb{N}p^{2k-3ks}\right)$$

At prime powers $d = p^m$, we can break the sum over d into three sums depending on the residue class of $m \pmod 3$. We call the resulting sums $Z_{1,0}^{(p)}(s,w)$, $Z_{1,1}^{(p)}(s,w)$, and $Z_{1,2}^{(p)}(s,w)$, where the second subindex denotes the residue class of m.

Since $d_0 = 1$ if $m \equiv 0 \quad (3)$, then we may write

$$Z_{1,0}^{(p)}(s,w) = \zeta^{(p)}(s) \sum_{l=0}^{\infty} \mathbb{N}p^{-3lw} P^{(p)}(s, \mathbb{N}p^{3l}) \tag{29}$$

where

$$P^{(p)}(s, \mathbb{N}p^{3l}) = \left(1 - \mathbb{N}p^{-s} + \mathbb{N}p^{2-3s} - \cdots + \mathbb{N}p^{2l-3ls}\right).$$

We will show that (29) is equal to the following piece from (28),

$$\begin{aligned} &\zeta^{(p)}(3s)\zeta^{(p)}(3w)\zeta^{(p)}(3s+3w-2) \\ &\times \left(1 + \mathbb{N}p^{-s} + \mathbb{N}p^{-2s} - \mathbb{N}p^{-s-3w} - \mathbb{N}p^{-2s-3w} - \mathbb{N}p^{-3s-3w}\right) \end{aligned}$$

or, upon factoring out $(1 + \mathbb{N}p^{-s} + \mathbb{N}p^{-2s})$,

$$\zeta^{(p)}(s)\zeta^{(p)}(3w)\zeta^{(p)}(3s+3w-2)\left(1 + \mathbb{N}p^{-s-3w}\right).$$

Comparing with (29), we can immediately cancel the zeta Euler factors $\zeta^{(p)}(s)$, so it remains to show

$$\begin{aligned} \sum_{l=0}^{\infty} \mathbb{N}p^{-3lw} P^{(p)}(s, \mathbb{N}p^{3l}) &= \zeta^{(p)}(3w)\zeta^{(p)}(3s+3w-2)\left(1 + \mathbb{N}p^{-s-3w}\right) \\ &= \left(1 + \mathbb{N}p^{-s-3w}\right) \sum_{k=0}^{\infty} \mathbb{N}p^{-3kw} \sum_{j=0}^{\infty} \mathbb{N}p^{(2-3s-3w)j} \end{aligned}$$

Setting $k + j = l$ and then rewriting as a sum over l and inner sum over $0 \leqslant j \leqslant l$ gives the equality with $Z_{1,0}(s,w)$.

Now if $d = p^m$ and $m \not\equiv 0$ (3), then $L^{(p)}(s, \chi_{d_0}) = 1$ and we have

$$Z_{1,1}^{(p)}(s,w) = \sum_{l=0}^{\infty} \mathbb{N}p^{-(3l+1)w} P^{(p)}(s, \mathbb{N}p^{3l+1}),$$
$$Z_{1,2}^{(p)}(s,w) = \sum_{l=0}^{\infty} \mathbb{N}p^{-(3l+2)w} P^{(p)}(s, \mathbb{N}p^{3l+2}),$$

with, for example,

$$P^{(p)}(s, \mathbb{N}p^{3l+1}) = \left(1 + \mathbb{N}p^{2-3s} + \mathbb{N}p^{4-6s} + \cdots + \mathbb{N}p^{2l-3ls}\right).$$

We now compare $Z_{1,1}^{(p)}(s,w)$ with the following terms from (28):

$$\begin{aligned} & \zeta^{(p)}(3s)\zeta^{(p)}(3w)\zeta^{(p)}(3s+3w-2)\left(\mathbb{N}p^{-w} - \mathbb{N}p^{-3s-w}\right) \\ = \; & \mathbb{N}p^{-w}\zeta^{(p)}(3w)\zeta^{(p)}(3s+3w-2) \\ = \; & \mathbb{N}p^{-w}\sum_{k=0}^{\infty}\mathbb{N}p^{-3kw}\sum_{j=0}^{\infty}\mathbb{N}p^{2-3s-3w} \end{aligned}$$

Again letting $k + j = l$ and summing over l and j gives the result. The comparison between $Z_{1,2}^{(p)}(s,w)$ and the terms from (28) consisting of

$$\zeta^{(p)}(3s)\zeta^{(p)}(3w)\zeta^{(p)}(3s+3w-2)\left(\mathbb{N}p^{-2w} - \mathbb{N}p^{-3s-2w}\right)$$

follows similarly, and completes the proof of the theorem. □

7 Second main theorem: the s_1 residue

In this section, we repeat the calculation from the previous section. As before, we can obtain all the prime-power terms (for a fixed prime p) in the residue by reorganizing the Dirichlet series into Kubota Dirichlet series, summed over triples (C_1, C_2, C_3) now with C_1 reduced as in (21). Using the table of 24 non-zero terms, we pick out the twelve for which C_1 is reduced, and can compute the resulting residue at $s_1 = \frac{2}{3}$:

$$\begin{array}{l}
1 \\
g(1,p)\mathbb{N}p^{-\frac{1}{6}-2s_2}\tau(p) \\
g(1,p)\mathbb{N}p^{-2s_3} \\
g(1,p)g(p,p^2)\mathbb{N}p^{-\frac{4}{3}-4s_2} \\
g(1,p)^2g(p,p^2)\mathbb{N}p^{-\frac{1}{3}-4s_2-4s_3}\tau(p^2) \\
g(1,p)g(p,p^2)\mathbb{N}p^{-\frac{1}{6}-2s_2-4s_3}\tau(p) \\
g(1,p)g(p,p^2)\mathbb{N}p^{-\frac{1}{3}-4s_2-2s_3}\tau(p^2) \\
g(1,p)^2g(p,p^2)g(p^2,p^3)\mathbb{N}p^{-\frac{3}{2}-6s_2-6s_3}\tau(p) \\
g(1,p)g(p,p^2)g(p^2,p^3)\mathbb{N}p^{-\frac{4}{3}-4s_2-6s_3} \\
g(1,p)^2g(p^2,p^3)\mathbb{N}p^{-\frac{3}{2}-6s_2-2s_3}\tau(p) \\
g(1,p)^2g(p,p^2)^2g(p^2,p^3)\mathbb{N}p^{-\frac{8}{3}-8s_2-6s_3} \\
g(1,p)g(p,p^2)^2g(p^2,p^3)\mathbb{N}p^{-\frac{8}{3}-8s_2-4s_3}
\end{array}$$

Since $n = 3$ using the same identities as before, we have the following 10 non-zero terms:

$$\begin{array}{ll}
1 & \mathbb{N}p^{\frac{1}{3}-2s_2} \\
g(1,p)\mathbb{N}p^{-2s_3} & \mathbb{N}p^{2/3-4s_2} \\
g(p,p^2)\mathbb{N}p^{1/3-2s_2-4s_3} & -\mathbb{N}p^{3-6s_2-6s_3} \\
\mathbb{N}p^{8/3-4s_2-6s_3} & -g(1,p)\mathbb{N}p^{1-6s_2-2s_3} \\
-\mathbb{N}p^{10/3-8s_2-6s_3} & -g(p,p^2)\mathbb{N}p^{4/3-8s_2-4s_3}
\end{array}$$

Just as before, we are now prepared to assert the other main equality of Dirichlet series.

Theorem 4 *With notation as above,*

$$Res_{s_1=2/3} \quad \zeta(6s_2-1)\zeta(6s_3-2)\zeta(6s_2+6s_3-4)Z(s_1,s_2,s_3;A_3^{(3)}) = \zeta(3w-1/2)\sum_{m,d}\frac{G(m,d)}{\mathbb{N}m^s\mathbb{N}d^w},$$

under the change of variables $s = 2s_2 - \frac{1}{3}$ and $w = 2s_3 - \frac{1}{2}$.

This is (4). We again have 5 zeta factors upon taking residues at $s_1 = 2/3$. The two not on the list above are

$$\zeta(3s-1) = \zeta(6s_2-2), \qquad \zeta(3s+3w-\frac{5}{2}) = \zeta(6s_2+6s_3-5). \tag{30}$$

Under the substitution $(2s_2 - 1/3, 2s_3 - 1/2) \mapsto (s, w)$ these are invariant under the functional equation $(s, w) \mapsto (s + w - 1/2, 1 - w)$ and map to the previously excluded two zeta factors in the residue of s_2 (discussed in the remark after Theorem 1) under $(s, w) \mapsto (1 - s, w + s - 1/2)$. Using (27), the gamma factor in (19) produces two constants and 10 gamma functions in the residue. These include

$$\Gamma(s), \qquad \Gamma\left(s + w - \frac{1}{2}\right), \qquad \Gamma\left(w - \frac{1}{6}\right), \qquad \Gamma\left(w + \frac{1}{6}\right),$$

which are the gamma functions associated to Z_2 in [11], and six other, which, using the triplication formula, produce $\Gamma(3s-1)\,\Gamma\left(3s + 3w - \frac{5}{2}\right)$, which are the gamma factors associated with the two spare zeta functions (30).

Proof As argued before, we only need to check prime power contributions for a single fixed prime p, having shown that the two Dirichlet series possess the same multiplicativity. On performing the residue and change of variables on the left-hand side of the equality, we have the prime power contribution:

$$\begin{aligned}\zeta^{(p)}(3w-1/2)\zeta^{(p)}(3s)\zeta^{(p)}(3s+3w-1/2)\big(1 + \mathbb{N}p^{-s} + G(1,p)\mathbb{N}p^{-w} \\ +\mathbb{N}p^{-2s} + \overline{G(1,p)}\mathbb{N}p^{1/2-s-2w} - \mathbb{N}p^{1/2-3s-3w} - \mathbb{N}p^{1/2-2s-3w} \\ -G(1,p)\mathbb{N}p^{-3s-w} - \mathbb{N}p^{1/2-4s-3w} - \overline{G(1,p)}\mathbb{N}p^{1/2-4s-2w}\big)\end{aligned} \quad (31)$$

where we have repeatedly used the identity that $g(m,d) = \mathbb{N}d^{1/2}G(m,d)$. We want to compare this with

$$Z_2^{(p)}(s,w) = \zeta^{(p)}(3w-1/2)\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{G(p^r,p^k)}{\mathbb{N}p^{rs}\mathbb{N}p^{kw}}$$

and we can begin by canceling the zeta factors $\zeta^{(p)}(3w-1/2)$ on both sides. We proceed as in the proof of the previous theorem, separating $Z_2^{(p)}(s,w)$ into three separate sums according to the residue class of k mod 3. We call the resulting Dirichlet series $Z_{2,0}^{(p)}(s,w)$, $Z_{2,1}^{(p)}(s,w)$, and $Z_{2,2}^{(p)}(s,w)$, respectively.

We first consider

$$Z_{2,0}^{(p)}(s,w) = \sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{G(p^r,p^{3l})}{\mathbb{N}p^{rs}\mathbb{N}p^{3lw}} \quad (32)$$

We want to show that this is equal to the set of terms on the left hand side of our desired equality with $\mathbb{N}p^{-kw}$, $k \equiv 0 \pmod 3$. These are

$$\begin{aligned}
& \zeta^{(p)}(3s)\zeta^{(p)}(3s+3w-1/2) \\
& \times \left(1+\mathbb{N}p^{-s}+\mathbb{N}p^{-2s}-\mathbb{N}p^{1/2-2s-3w}-\mathbb{N}p^{1/2-3s-3w}-\mathbb{N}p^{1/2-4s-3w}\right) \\
= \; & \zeta^{(p)}(s)\zeta^{(p)}(3s+3w-3/2)\ \left(1-\mathbb{N}p^{1/2-2s-3w}\right)
\end{aligned}$$

Since $n=3$, the Gauss sum $G(p^r,p^{3l})$ reduces to a Ramanujan sum, with the following evaluation:

$$g(p^r,p^{3l}) = \begin{cases} 1 & \text{if } l=0, \\ -\mathbb{N}p^{3l-1} & \text{if } r=3l-1, l>0, \\ \phi(\mathbb{N}p^{3l}) & \text{if } r \geqslant 3l, l>0, \\ 0 & \text{otherwise.} \end{cases}$$

Upon making the translation for the normalized Gauss sum $G(p^r,p^{3l})$, this allows us to rewrite (32) as

$$\begin{aligned}
& \zeta^{(p)}(s)+\sum_{l=1}^{\infty}\mathbb{N}p^{-3lw}\left(-\mathbb{N}p^{3l/2-1-(3l-1)s}+\sum_{r=3l}^{\infty}\phi(p^{3l})\mathbb{N}p^{-3l/2-rs}\right) \\
= \; & \zeta^{(p)}(s)+\sum_{l=1}^{\infty}\mathbb{N}p^{-3lw}\Big(-\mathbb{N}p^{3l/2-1-(3l-1)s} \\
& +(\mathbb{N}p^{3l/2}-\mathbb{N}p^{3l/2-1})\mathbb{N}p^{-3ls}\sum_{r=0}^{\infty}\mathbb{N}p^{-rs}\Big) \\
= \; & \zeta^{(p)}(s)+p^{3/2-3s-3w}\zeta^{(p)}(s)\zeta^{(p)}(3s+3w-3/2) \\
& -p^{1/2-2s-3w}\zeta^{(p)}(3s+3w-3/2)-\mathbb{N}p^{1/2-3s-3w}\zeta^{(p)}(3s+3w-3/2)\zeta^{(p)}(s) \\
= \; & \zeta^{(p)}(s)\zeta^{(p)}(3s+3w-1/2)\left(1-\mathbb{N}p^{1/2-2s-3w}\right).
\end{aligned}$$

This completes the case when $d=p^k$ with $k \equiv 0 \pmod 3$. We now consider the series

$$Z_{2,1}^{(p)}(s,w)=\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{G(p^r,p^{3l+1})}{\mathbb{N}p^{rs}\mathbb{N}p^{(3l+1)w}},$$

matching it to terms in (31) containing $\mathbb{N}p^{kw}$ with $k \equiv 1 \pmod 3$. This is much simpler than the previous case since

$$G(p^r,p^{3l+1}) = \begin{cases} \mathbb{N}p^{3l/2}G(1,p) & \text{if } r=3l, \\ 0 & \text{otherwise,} \end{cases}$$

owing to the presence of the non-trivial residue character in the Gauss sum. This gives

$$Z_{2,1}^{(p)}(s,w) = \sum_{l=0}^{\infty} \mathbb{N}p^{3l/2-3ls-(3l+1)w} G(1,p) = G(1,p)\mathbb{N}p^{-w}\zeta^{(p)}(3s+3w-3/2).$$

Comparing with the terms from (31) (after canceling $\zeta^{(p)}(3w-1/2)$) of form

$$\begin{aligned}\zeta^{(p)}(3s)\zeta^{(p)}(3s+3w-3/2)(G(1,p)\mathbb{N}p^{-w} - G(1,p)\mathbb{N}p^{-3s-w})\\ = \zeta^{(p)}(3s+3w-3/2)(G(1,p)\mathbb{N}p^{-w})\end{aligned}$$

which completes this case. The equality

$$Z_{2,2}^{(p)}(s,w) = \sum_{l=0}^{\infty}\sum_{r=0}^{\infty} \frac{G(p^r, p^{3l+2})}{\mathbb{N}p^{rs}\mathbb{N}p^{(3l+2)w}} = \overline{G(1,p)}\mathbb{N}p^{1/2-s-2w}\zeta^{(p)}(3s+3w-3/2)$$

follows similarly and comparing with (31) gives the theorem. □

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