On Kubota's Dirichlet Series

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Kubota [19] showed how the theory of Eisenstein series on the higher metaplectic covers of SL_2 (which he discovered) can be used to study the analytic properties of Dirichlet series formed with n-th order Gauss sums. In this paper we will prove a functional equation for such Dirichlet series in the precise form required by the companion paper [2]. Closely related results are in Eckhardt and Patterson [10].

The Kubota Dirichlet series are the entry point to a fascinating universe. Their residues, for example, are mysterious if $n > 3$, though there is tantalizing evidence that these residues exhibit a rich structure that can only be partially glimpsed at this time. When $n = 4$ the residues are the Fourier coefficients of the biquadratic theta function that were studied by Suzuki [23]. Suzuki found that he could only determine *some* of the coefficients. This failure to determine all the coefficients was explained in terms of the failure of uniqueness of Whittaker models for the generalized theta series by Deligne [9] and by Kazhdan and Patterson [15]. On the other hand, Patterson [22] conjectured that the mysterious coefficients are essentially square roots of Gauss sums. Evidence for Patterson's conjecture is discussed in Bump and Hoffstein [6] and in Eckhardt and Patterson [10], where the conjecture is refined in light of numerical data. Partial proofs were given by Suzuki in [24] and [25].

Another set of conjectures relevant to the mysterious coefficients of n-th order theta functions were given by Bump and Hoffstein, who considered theta functions on the n-fold covers of GL_r for arbitrary r. They are expressed as identities between Rankin-Selberg convolutions of generalized theta series and Whittaker coefficients of Eisenstein series on the metaplectic group, but they boil down to properties of the residues of Kubota Dirichlet series, and their higher rank generalizations. See Bump and Hoffstein [6], Bump [4] and Hoffstein [12]. These conjectures are different from the Patterson conjecture, and there are other considerations which suggest that there may be further unproved relations beyond those described in the conjectures of Patterson and Bump and Hoffstein.

Recently Brubaker, Bump, Chinta, Friedberg and Hoffstein [2] and Brubaker, Bump and Friedberg [3] have considered multiple Dirichlet series whose coefficients involve Gauss sums. These *Weyl group multiple Dirichlet series* and their residues are potentially a tool for studying the hidden structures suggested by the conjectures of Patterson and Bump and Hoffstein, and may also have applications to analytic number theory. For example the multiple Dirichlet series used by Chinta [8] to study the zeta functions of biquadratic fields are of this type.

The basic building blocks in the Weyl group multiple Dirichlet series are the Kubota Dirichlet series. The method of analytic continuation of the Weyl group multiple Dirichlet series, starting with the analytic properties of the one variable Kubota Dirichlet series is described in [5]. To develop this theory, we need versions of Kubota's Dirichlet series slightly different from those available in the literature. The purpose of this paper is to prove such results in the form required for [2] and other papers in preparation. The following points regarding our Theorem 1 are noteworthy.

- We exhibit concrete finite-dimensional families of Dirichlet series that are closed under the functional equations. The description of these Dirichlet series is as simple as we can imagine. They have the form

$$\mathcal{D}(s,\Psi,\alpha)=\sum_{0\neq c\in\mathfrak{o}_S/\mathfrak{o}_S^\times} g(\alpha,c)\,\Psi(c)\,\mathbb{N}(c)^{-2s}\ , \tag{1}$$

  where $\mathfrak{o}_S$ is the ring of S-integers in a number field, $g(\alpha,c)$ a Gauss sum and the function Ψ is restricted to a finite-dimensional vector space of functions making the Dirichlet series well defined if c is replaced by εc, where $\varepsilon\in\mathfrak{o}_S^\times$. (See Sections 1 and 2 for notations.)

- We construct certain Dirichlet polynomials P_η that appear in the functional equation, and prove the remarkable economy noted in Remark 2, that the same finite set of polynomials of P_η appear in the functional equations of $\mathcal{D}(s,\Psi,\alpha)$ for all α.

See Theorem 1 for details, and [2] for the way these results are used.

Three ideas make these results possible. First, there is the S-integer formulation, long advocated by Patterson as a way of obtaining the benefits of the adelic treatment in a classical setting. Second, we take as our Eisenstein series a particular vector in an induced representation, depending on Ψ in (1), chosen to have small support in the big Bruhat cell, to produce (1) as a Fourier coefficient. Finally, in the functional equation one obtains the composition of a Whittaker functional with an intertwining integral applied to this vector, and this may be studied by a method adapted from Banks [1]. See Proposition 7 below.

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1 Statement of the Theorem

Let $n > 1$ be an integer and let F be an algebraic number field. We will assume that F contains the group μ_n of n-th roots of unity, and moreover that -1 is an n-th power in F. If n is even this implies that F actually contains μ_{2n}, and that F is totally complex.

Remark 1 One expects that in some form our results (perhaps with modification) will be true with just the assumption that $\mu_n \subset F$. However the assumption that $-1 \in (F^\times)^n$ is made for more than just convenience. In particular, Theorem 5 defining the Kubota symbol without congruence conditions would not be true without this assumption.

If S is a finite set of places of F containing the archimedean ones, and let $F_S = \prod_{v \in S} F_v$, and let $\mathfrak{o}_S$ be the set of S-integers. We recall that $\mathfrak{o}_S$ is the set of elements $\alpha \in F$ such that $|\alpha_v|_v \leqslant 1$ for all places v of F not in S.

We will fix a particular S. We assume that S contains all nonarchimedean primes which are ramified over $\mathbb{Q}$ (in particular those dividing n) and enough others that the ring $\mathfrak{o}_S$ of S-integers is a principal ideal domain. It is easy to see that such a set S of places exists; to give $\mathfrak{o}_S$ class number one, we have only to include all places corresponding to primes dividing a given set of generators of the ideal class group of F.

We will denote

$$F_\infty = \prod_{v \in S_\infty} F_v, \qquad F_{\mathrm{fin}} = \prod_{v \in S_{\mathrm{fin}}} F_v,$$

where S_∞ is the set of archimedean places in S, and S_{fin} is the set of nonarchimedean ones. We embed $\mathfrak{o}_S$ in F_S along the diagonal. It is discrete and cocompact. Let $\psi : F_S \longrightarrow \mathbb{C}^\times$ be an additive character such that the restriction ψ_v of ψ to F_v is nontrivial for all $v \in S$. Then $\psi(x) = \prod_{v \in S} \psi_v(x_v)$.

Lemma 1 *We may choose ψ so that for all $v \in S_\infty$ we have $\psi_v(x) = e^{2\pi i\, \mathrm{tr}(x)}$ for all $x \in F_v$, where $\mathrm{tr} : \mathbb{C} \longrightarrow \mathbb{R}$ is the trace map, and so that if $x \in F_S$ we have*

$$\psi(x\mathfrak{o}_S) = 1 \text{ if and only if } x \in \mathfrak{o}_S.$$

Proof Following Tate [27], we will define a local character ψ_v of F_v for *all* places v of S, whether $v \in S$ or not. Let v_0 be a place of $\mathbb{Q}$. We define a character $\tau_{v_0} : \mathbb{Q}_{v_0} \longrightarrow \mathbb{C}$ as follows. If v_0 is the archimedean place, so $\mathbb{Q}_{v_0} = \mathbb{R}$, we take $\tau_{v_0}(x) = e^{2\pi i x}$. On the other hand if v is nonarchimedean, then $\mathbb{Z}_{v_0} + \mathbb{Q} = \mathbb{Q}_{v_0}$, so any element x of $\mathbb{Q}_{v_0}$ can be written as $x_1 + x_2$ where $x_1 \in \mathbb{Z}_{v_0}$ and $x_2 \in \mathbb{Q}$. We define $\tau_{v_0}(x) = e^{-2\pi i x_2}$. Note that x_2 is determined modulo $\mathbb{Z}_{v_0} \cap \mathbb{Q} = \mathbb{Z}$, so this is well-defined. Now if v is a place of F above v_0 we take $\psi_v = \tau_{v_0} \circ \mathrm{tr}$, where $\mathrm{tr} : F_v \longrightarrow \mathbb{Q}_{v_0}$ is the trace map. Note that this ψ_v has the prescribed description for all archimedean places v.

It is shown by Tate [27] that $\prod_v \psi_v(x_v) = 1$ for $x \in F$. Now if $x \in \mathfrak{o}_S$, this means that

$$\psi(x) = \prod_{v \in S} \psi_v(x_v) = \prod_{v \notin S} \psi_v(x_v)^{-1} = 1$$

since for each $v \notin S$, $x_v \in \mathfrak{o}_v$, and so $\psi_v(x_v) = 1$.

Now suppose that $\mathrm{tr}(x\mathfrak{o}_S) = 1$; we must show that $x \in \mathfrak{o}_S$. The first step is to show that $x \in F$. We call a subset Λ of F_S a *lattice* if it is of the form $\Lambda_\infty \times \Lambda_{\mathrm{fin}}$, where $\Lambda_\infty \subseteq F_\infty$ is discrete and cocompact, and $\Lambda_{\mathrm{fin}} \in F_{\mathrm{fin}}$ is compact and open. We note that

$$\Lambda = \{x \in F_S \mid \psi(x\mathfrak{o}_S) = 1\}$$

is a lattice containing $\mathfrak{o}_S$ and so it contains $\mathfrak{o}_S$ as a subgroup of finite codimension. This means that $\Lambda \subseteq \frac{1}{N}\mathfrak{o}_S$ for some N, and so $\Lambda \subseteq F$.

Now that we know $\Lambda \subseteq F$, we may embed it in F_v for all places v. If $x \notin \mathfrak{o}_S$, then $x_w \in \mathfrak{o}_w$ for some $w \notin S$. Let w_0 be the place of $\mathbb{Q}$ below w. Since F is unramified over $\mathbb{Q}$ outside S, the local different of F_w over $\mathbb{Q}_{w_0}$ is $\mathfrak{o}_w$ and so $\psi_w(x_w\xi_w) \neq 1$ for some $\xi_w \in \mathfrak{o}_w$. Let S_1 be a set of places containing S such that $x_v \in \mathfrak{o}_v$ for all $v \in S_1$. By strong approximation (Cassels [7]) we may find $y \in F$ such that y_w is sufficiently near ξ_w that $\psi_w(x_w y_w) \neq 1$, while y_v is sufficiently near 1 for all $v \in S_1 - S$ that $\psi_v(x_v y_v) = 1$ for all such v, and $y_v \in \mathfrak{o}_v$ for all $v \in S_1$. Then

$$\psi(xy) = \prod_{v \notin S} \psi_v(x_v y_w)^{-1},$$

but there is exactly one factor on the right that is not equal to 1, namely the contribution of $\psi_w(x_w y_w)$. Hence $\psi(xv) \neq 1$, which is a contradiction since $y \in \mathfrak{o}_S$. $\square$

Let $(\ ,\ ) : F_S \times F_S \longrightarrow \mu_n$ be the n-th order Hilbert symbol (Section 2). Thus in terms of the local Hilbert symbols

$$(\alpha, \beta) = \prod_{v \in S} (\alpha_v, \beta_v)_v = \prod_{v \notin S} (\alpha_v, \beta_v)^{-1},$$

where the first equality is the definition of $(\ ,\ )$ and the second equality is the Hilbert reciprocity law. We also denote by $\left(\frac{\alpha}{\beta}\right)$ the power residue symbol, defined when $\alpha, \beta \in \mathfrak{o}_S$. Properties of the Hilbert symbol and power residue symbol are collected together in Section 2.

If $0 \neq c \in \mathfrak{o}_S$, define the Gauss sum

$$g(\alpha, c) = \sum_{d \bmod c} \left(\frac{d}{c}\right) \psi\left(\frac{\alpha d}{c}\right).$$

We will say that a subgroup Ω of $F_S^\times$ is *isotropic* if the Hilbert symbol $(\varepsilon, \delta) = 1$ for all $\varepsilon, \delta \in \Omega$. For example, let

$$\Omega_0 = \mathfrak{o}_S^\times (F_S^\times)^n.$$

Lemma 2 *The group Ω_0 is maximal isotropic.*

Proof The Hilbert symbol is clearly trivial on n-th powers, so it is sufficient to prove this when ε and δ are units. In this case we can use the second expression in (1), and note that $(\varepsilon_v, \delta_v) = 1$ for $v \in S$ by Proposition 1 (vi) below. The maximality assertion may be deduced from Proposition 8 in Section XIII.5 of Weil [28] (page 262) together with our assumption on the class number of the ring of S-integers. $\square$

We will assume that $\Omega = \Omega_0$. The purely local result of Suzuki [26] discusses cases where a subring R of F_v can be found such that $\Omega = R^\times (F_v^\times)^n$ is a maximal isotropic subgroup.

Let $\mathcal{M}(\Omega)$ be the vector space of functions Ψ on $F_{\text{fin}}^\times$ that satisfy

$$\Psi(\varepsilon c) = (c, \varepsilon)^{-1} \Psi(c), \tag{2}$$

when $\varepsilon \in \Omega$.

Lemma 3 *The dimension of $\mathcal{M}(\Omega)$ is the cardinality of $F_S^\times/\Omega$. It is finite.*

Proof It is easy to check (using the isotropy of Ω) that an element of $\mathcal{M}(\Omega)$ can be assigned arbitrary values on a set of coset representatives of $F_S^\times/\Omega$, and that it is uniquely determined by these values. Thus the dimension of $\mathcal{M}(\Omega)$ is thus the cardinality of $F_S^\times/\Omega$. This is $\leqslant |F_S^\times/(F_S^\times)^n| < \infty$. $\square$

Let

$$\boldsymbol{G}(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

We note that the multiplication formula for the Gamma function implies that

$$\boldsymbol{G}(s) = (2\pi)^{-(n-1)(2s-\frac{1}{2})}\, n^{-1/2+n(2s-1)} \prod_{j=1}^{n-1} \Gamma\left(2s-1+\frac{j}{n}\right).$$

Define

$$\mathcal{D}^*(s,\Psi,\alpha) = \boldsymbol{G}(s)^r\, \zeta_F(2ns-n+1)\, \mathcal{D}(s,\Psi,\alpha) \tag{3}$$

where $r = \frac{1}{2}[F:\mathbb{Q}]$ is the number of complex places of F and ζ_F is the Dedekind zeta function of F.

If $v \in S_{\text{fin}}$ let q_v denote the cardinality of the residue class field $\mathfrak{o}_v/\mathfrak{p}_v$, where $\mathfrak{o}_v$ is the local ring in F_v and $\mathfrak{p}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}.

If $\Psi \in \mathcal{M}(\Omega)$ and $\eta \in F_S^\times$ denote

$$\tilde{\Psi}_\eta(c) = (\eta, c)\, \Psi(c^{-1}\eta^{-1}). \tag{4}$$

It is easy to check that $\tilde{\Psi}_\eta \in \mathcal{M}(\Omega)$ and that it depends only on the class of η in $F_S^\times/(F_S^\times)^n$.

Theorem 1 *Let $\Psi \in \mathcal{M}(\Omega)$, and let $\alpha \in \mathfrak{o}_S$. Then $\mathcal{D}^*(s,\Psi,\alpha)$ has meromorphic continuation to all s, analytic except possibly at $s = \frac{1}{2} \pm \frac{1}{2n}$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta(s)$ that depend only on the image of η in $F_S^\times/(F_S^\times)^n$ such that*

$$\mathcal{D}^*(s,\Psi,\alpha) = \mathbb{N}(\alpha)^{1-2s} \sum_{\eta \in F_S^\times/(F_S^\times)^n} P_{\alpha\eta}(s) \mathcal{D}^*(1-s, \tilde{\Psi}_\eta, \alpha). \tag{5}$$

The proof will occupy the rest of the paper.

Remark 2 Note that the P_η do not depend on Ψ. It is also very remarkable that α intervenes only as a permutation of the finite set of polynomials P_η.

2 Hilbert symbols, power residues and Gauss sums

For reference we collect the most important properties of the Hilbert symbol, power residue symbol and Gauss sums.

Let n be a positive integer. We will assume that all fields that occur in this section have characteristic zero, or positive characteristic prime to n. If F is any such field, we will denote by μ_n the group of n-th roots of unity in the algebraic closure of F, and our assumption implies that μ_n has cardinality n. We will almost always assume that $\mu_n \subset F$. Moreover, we will also impose the condition that -1 is an n-th power. If n is even, this means that $\mu_{2n} \subset F$.

Remark 3 We will find it convenient to identify μ_n with the group of roots of unity in $\mathbb{C}$. This involves fixing an isomorphism $\boldsymbol{j} : \mu_n \longrightarrow \{x \in \mathbb{C}^\times \,|\, x^n = 1\}$. We will suppress this isomorphism from the notation.

Let F be a local field containing μ_n. The Hilbert symbol is map $F^\times \times F^\times \longrightarrow \mu_n$. We will take the symbol to be the *inverse* of the symbol defined by Neukirch [21].

Proposition 1 *(i) The Hilbert symbol is a skew-symmetric bilinear pairing. That is,*

$$\begin{aligned} (aa', b) &= (a, b)(a', b), \\ (a, bb') &= (a, b)(a, b'), \\ (a, b) &= (b, a)^{-1}. \end{aligned}$$

(ii) We have $(a, b) = 1$ if and only if a is a norm from $F(b^{1/n})$.
(iii) We have $(a, -a) = 1$, and if $a \neq 1$ we have $(a, 1-a) = 1$.
(iv) We have $(a, b) = 1$ if and only if a is a norm from $F(b^{1/n})$.
(v) If F is complex, then $(a, b) = 1$ for all $a, b \in F^\times$, while if F is real and $n = 2$ we have

$$(a, b) = \begin{cases} 1 & \text{if } a > 0 \text{ or } b > 0; \\ -1 & \text{if } a, b < 0. \end{cases}$$

(vi) If F is nonarchimedean and n does not divide the residue characteristic, then $(a, b) = 1$ when a, b are both elements of the group $\mathfrak{o}_F^\times$ of units.

Proof See Neukirch [21], Chapter V Section 3, particularly Proposition 3.2 on p. 334. $\square$

Now let F be a global field containing μ_n. For each place v of F, let $(a, b)_v$ denote the local Hilbert symbol on $F_v^\times$.

Theorem 2 (Hilbert) *If $a, b \in F^\times$, then*

$$\prod_v (a, b)_v = 1. \tag{6}$$

This is the *Hilbert reciprocity law.*

Proof See Neukirch [21] Chapter VI, Theorem 8.1 on p. 414. □

Let S be a finite set of places of F. We assume that S contains the archimedean places and all finite places dividing n. If $a \in \mathfrak{o}_S$ and $\mathfrak{b}$ is an ideal of $\mathfrak{o}_S$ then the *power residue symbol* $\left(\frac{a}{\mathfrak{b}}\right)$ is defined as follows. If a is not coprime to $\mathfrak{b}$ then it is defined to be zero. If $\mathfrak{b} = \mathfrak{p}$ is prime and a is coprime to $\mathfrak{p}$, then $\left(\frac{a}{\mathfrak{p}}\right)$ is defined to be the unique element of μ_n such that

$$\left(\frac{a}{\mathfrak{p}}\right) \equiv a^{(\mathbb{N}\mathfrak{p}-1)/n} \mod \mathfrak{p}.$$

Note that $(\mathbb{N}\mathfrak{p} - 1)/n$ is an integer since we are assuming that $\mu_n \subset F$. Finally, the definition is extended to all $\mathfrak{a}$ by multiplicativity, that is, by the condition

$$\left(\frac{a}{\mathfrak{b}\mathfrak{c}}\right) = \left(\frac{a}{\mathfrak{b}}\right)\left(\frac{a}{\mathfrak{c}}\right).$$

If $a, b \in \mathfrak{o}_S$ we will also denote by $\left(\frac{a}{b}\right) = \left(\frac{a}{b\mathfrak{o}_S}\right)$ where $b\mathfrak{o}_S$ is the principal ideal generated by b.

Proposition 2 *We have*

$$\left(\frac{a}{\varepsilon b}\right) = \left(\frac{a}{b}\right) \qquad \varepsilon \in \mathfrak{o}_S^\times$$

and

$$\left(\frac{a}{b}\right) = \left(\frac{a'}{b}\right) \text{ if } a \equiv a' \text{ modulo } b.$$

Proof This is clear. □

If $a, b \in \mathfrak{o}_S$ let $F_S = \prod_{v \in S} F_v$. Define a pairing $(\ ,\ ) : F_S^\times \times F_S^\times \longrightarrow \mu_n$ by

$$(a, b) = \prod_{v \in S} (a, b)_v = \prod_{v \in S} (a, b)_v^{-1},$$

where the last equality is the Hilbert reciprocity law (6).

Theorem 3 (Gauss, Eisenstein, Kummer, Hilbert) *If a and b are coprime then*

$$\left(\frac{\alpha}{\beta}\right) = (\beta, \alpha)\left(\frac{\beta}{\alpha}\right). \tag{7}$$

This is the *n-th power reciprocity law.*

Proof See Neukirch [21], Theorem 8.3 of Chapter 6. (Recall that our Hilbert symbol is the inverse of his.) $\square$

We turn next to properties of Gauss sums. Let F, S and ψ be as in Section 1.

Proposition 3 *Let*

$$g_t(m,c) = \sum_{d \text{ mod } c} \left(\frac{d}{c}\right)^t \psi\left(\frac{md}{c}\right).$$

This Gauss sum has the following properties.

(i) We have

$$g_t(m,cc') = \left(\frac{c}{c'}\right)^t \left(\frac{c'}{c}\right)^t g_t(m,c)\, g_t(m,c'), \qquad \textit{if } c, c' \textit{ are coprime;}$$

(ii) We have

$$g_t(am,c) = \left(\frac{a}{c}\right)^{-t} g_t(m,c) \qquad \textit{if } a, c \textit{ are coprime};$$

(iii) If $c \in \mathfrak{o}_S$ let $\phi(c)$ denote the cardinality of $(\mathfrak{o}_S/c\mathfrak{o}_S)^\times$. If p is prime, then

$$g_t(p^k,p^l) = \begin{cases} \mathbb{N}(p)^k\, g_{tl}(1,p) & \textit{if } l = k+1; \\ \phi(p^l) & \textit{if } n|tl \textit{ and } k \geqslant l; \\ 0 & \textit{otherwise.} \end{cases}$$

(iv) If t and n are coprime, and a and p are coprime, then

$$|g_t(a,p)| = \sqrt{\mathbb{N}p}.$$

Proof These properties of the Gauss sums are well known. See, for example, Ireland and Rosen [13]. $\square$

We will denote $g_1(m,c)$ by $g(m,c)$.

Lemma 4 *If $\varepsilon \in \mathfrak{o}_S^\times$ then $g(\alpha, \varepsilon c) = (c, \varepsilon)\, g(\alpha, c)$.*

Proof We have

$$\sum_{d \text{ mod } c} \left(\frac{d}{\varepsilon c}\right) \psi\left(\frac{\alpha d}{\varepsilon c}\right) = \left(\frac{\varepsilon}{c}\right) \sum_{d \text{ mod } c} \left(\frac{d}{c}\right) \psi\left(\frac{\alpha d}{c}\right).$$

The statement follows from (7) since ε is a unit and so $\left(\frac{c}{\varepsilon}\right) = 1$. $\square$

3 The Kubota symbol

In this section we will recall results of Kubota defining the higher metaplectic covers of SL_2, and the "Kubota symbol." For the latter, our setup is different from Kubota's, and we prove from scratch that the symbol we define is a character of the preimage of $\mathrm{SL}_2(\mathfrak{o}_S)$ in the metaplectic group.

Let $G = \mathrm{SL}_2$. A map $X : G(F_S) \longrightarrow F_S^\times$ is defined by

$$X\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0; \\ d & \text{otherwise.} \end{cases}$$

Let $\sigma : G(F_S) \times G(F_S) \longrightarrow \mu_n$ be the map

$$\sigma(g_1, g_2) = \left(\frac{X(g_1 g_2)}{X(g_1)}, \frac{X(g_1 g_2)}{X(g_2)} \right).$$

Theorem 4 (Kubota) *We have*

$$\sigma(g_1, g_2)\, \sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\, \sigma(g_2, g_3).$$

Thus σ is a cocycle defining a cohomology class in $H^2(G(F_S), \mu_n)$.

Proof See Kubota [17], where the cocycle σ is given in a slightly different form. Kazhdan and Patterson [15] pointed out the simpler equivalent form, which is easily obtained from Kubota's expression by Proposition 1. $\square$

The metaplectic double cover $\tilde{G}(F_S) = \widetilde{\mathrm{SL}}_2(F_S)$ consists of ordered pairs $[a, \zeta]$ with $a \in G(F_S)$ and $\zeta \in \mu_n$, with group law

$$[a, \zeta][b, \zeta'] = [ab, \zeta\zeta'\sigma(a, b)].$$

We will denote the standard section by $\boldsymbol{i} : G(F_S) \longrightarrow \tilde{G}(F_S)$, so $\boldsymbol{i}(g) = [g, 1]$.

Theorem 5 *Define $\kappa : \mathrm{SL}_2(\mathfrak{o}_S) \longrightarrow \mu_n$ by*

$$\kappa\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{c}\right) & \text{if } c \neq 0; \\ 1 & \text{if } c = 0. \end{cases}$$

Then

$$\kappa(\gamma\gamma') = \sigma(\gamma, \gamma')\, \kappa(\gamma)\, \kappa(\gamma'). \tag{8}$$

This result is closely related to Kubota [18], Proposition 2 on p. 25, and the related result of Kubota [16]. Since our setup is slightly different, we give a proof.
Proof Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \qquad \gamma'' = \gamma\gamma' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.$$

First assume that c, c' and c'' are all nonzero. We have

$$c'' \equiv dc' \mod c \tag{9}$$

$$d'' \equiv dd' \mod c \tag{10}$$

and since

$$\gamma = \gamma''(\gamma')^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} d'' & -b'' \\ -c'' & a'' \end{pmatrix},$$

we also have

$$c = c'd'' - d'c''. \tag{11}$$

Let g be the greatest common divisor of d' and d''; write $d' = gd'_0$ and $d'' = gd''_0$ so that d'_0 and d''_0 are coprime. By (11) we have

$$c_0 = c'd''_0 - d'_0c'' \tag{12}$$

where $c = gc_0$, and we may rewrite (10) as

$$d''_0 \equiv dd'_0 \mod c_0. \tag{13}$$

Applying Proposition 2 and the reciprocity law Theorem 3 repeatedly. Since $d'' = gd''_0$

and since $-1 \in \mathfrak{o}_S^\times$, we have

$$\begin{aligned}
\left(\frac{d''}{c''}\right) &= \left(\frac{g}{c''}\right)\left(\frac{d_0''}{-d_0'c''}\right)\left(\frac{d_0''}{d_0'}\right)^{-1} \\
\text{[reciprocity]} &= (-d_0'c'', d_0'')\left(\frac{g}{c''}\right)\left(\frac{-d_0'c''}{d_0''}\right)\left(\frac{d_0''}{d_0'}\right)^{-1} \\
\text{[using (12)]} &= (-d_0'c'', d_0'')\left(\frac{g}{c''}\right)\left(\frac{c_0}{d_0''}\right)\left(\frac{d_0''}{d_0'}\right)^{-1} \\
\text{[reciprocity]} &= (-d_0'c'', d_0'')(d_0'', c_0)\left(\frac{g}{c''}\right)\left(\frac{d_0''}{c_0}\right)\left(\frac{d_0''}{d_0'}\right)^{-1} \\
\text{[using (10)]} &= (-d_0'c'', d_0'')(d_0'', c_0)\left(\frac{g}{c''}\right)\left(\frac{d}{c_0}\right)\left(\frac{d_0'}{c_0}\right)\left(\frac{d_0''}{d_0'}\right)^{-1} \\
\text{[reciprocity]} &= (-d_0'c'', d_0'')(d_0'', c_0)(c_0, d_0')\left(\frac{g}{c''}\right)\left(\frac{d}{c_0}\right)\left(\frac{c_0}{d_0'}\right)\left(\frac{d_0''}{d_0'}\right)^{-1} \\
\text{[using (12)]} &= (-d_0'c'', d_0'')(d_0'', c_0)(c_0, d_0') \\
&\qquad \times \left(\frac{g}{c''}\right)\left(\frac{d}{c_0}\right)\left(\frac{c'}{d_0'}\right)\left(\frac{d_0''}{d_0'}\right)\left(\frac{d_0''}{d_0'}\right)^{-1} \\
\text{[reciprocity]} &= (-d_0'c'', d_0'')(d_0'', c_0)(c_0, d_0')(d_0', c')\left(\frac{g}{c''}\right)\left(\frac{d}{c_0}\right)\left(\frac{d_0'}{c'}\right).
\end{aligned}$$

Now (9) implies that

$$\left(\frac{g}{c''}\right) = (c'', g)\left(\frac{c''}{g}\right) = (c'', g)\left(\frac{d}{g}\right)\left(\frac{c'}{g}\right) = (c'', g)(c', g)^{-1}\left(\frac{d}{g}\right)\left(\frac{g}{c'}\right).$$

Thus (8) will follow if we show that

$$(-d_0'c'', d_0'')(d_0'', c_0)(c_0, d_0')(c'', g)(c', g)^{-1} = \sigma(\gamma, \gamma').$$

We begin by noting that by Proposition 1 (ii) and (12) we have

$$1 = \left(\frac{c_0}{c'd_0''}, 1 - \frac{c_0}{c'd_0''}\right) = \left(\frac{c_0}{c'd_0''}, \frac{d_0'c''}{c'd_0''}\right).$$

Thus

$$\sigma(\gamma, \gamma') = \left(\frac{c''}{c}, \frac{c''}{c'}\right) = \left(\frac{c''}{gc_0}, \frac{c''}{c'}\right)\left(\frac{c_0}{c'd_0''}, \frac{d_0'c''}{c'd_0''}\right)$$

which, after simplification, equals

$$(-d_0'c'', d_0'')(d_0'', c_0)(c_0, d_0')(d_0', c')(c'', g)(c', g)^{-1} \times (c', c')(c'', c'').$$

The statement follows since we are assuming that -1 is an n-th power, so $(x, x) = (x, -x) = 1$ identically.

We have assumed that c, c' and c'' are nonzero. We leave the remaining cases to the reader. $\square$

Let $K = \prod_{v \in S} K_v$ where

$$K_v = \begin{cases} \mathrm{SU}_2 & \text{if } v \text{ is complex;} \\ \mathrm{SL}_2(\mathfrak{o}_v) & \text{if } v \text{ is nonarchimedean.} \end{cases}$$

The group K has an open subgroup K^s over which the cover splits. Indeed, let $K_s = \prod_{v \in S} K_v^s$ where

$$K_v^s = \begin{cases} K_v(\mathfrak{p}_v^{2\,\mathrm{ord}_v(n)}) & \text{if } v \in S_{\mathrm{fin}}; \\ K_v & \text{if } v \in S_\infty, \end{cases}$$

where if $\mathfrak{a}$ is an ideal of $\mathfrak{o}_v$ then $K_v(\mathfrak{a})$ is the principal congruence subgroup of K_v consisting of elements of K_{fin} which are congruent to the identity modulo $\mathfrak{a}$.

Lemma 5 (Kubota) *Let $v \in S$. If $v \in S_{\mathrm{fin}}$ let $\beta_v : K_v^s \longrightarrow \mu_n$ be the map*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \beta_v \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c, d)^{-1} & \text{if } cd \neq 0 \text{ and } n \nmid \mathrm{ord}(c); \\ 1 & \text{otherwise,} \end{cases}$$

while if $v \in S_\infty$, let $\beta_v : K_v^s \longrightarrow \mu_n$ be the constant map $g \longmapsto 1$. Then if g_1 and $g_2 \in K_v^s$ we have

$$\sigma(g_1, g_2) = \beta_v(g_1 g_2)^{-1} \beta_v(g_1)\, \beta(g_2). \tag{14}$$

Proof See Kubota [18], Theorem 2 on p. 19. $\square$

This means that, denoting $\beta(g) = \prod_{v \in S} \beta_v(g_v)$, the map

$$g \longmapsto [g, \beta(g)]$$

is a homomorphism $K^s \longrightarrow \widetilde{\mathrm{SL}}_2(F_S)$. We will denote by $\widetilde{K}^s$ the image of this homomorphism.

4 Local functional equations

In this section we use as data $\Psi \in \mathcal{M}(\Omega)$ to construct a vector with small support in the big Bruhat cell of an induced representation of the metaplectic group. By studying the effect of the intertwining integral on such a vector we will obtain a local functional equation leading to the phenomenon described in Remark 2.

Define $I(s)$ to be the space of K-finite functions $f : \widetilde{\mathrm{SL}}_2(F_S) \longrightarrow \mathbb{C}$ such that

$$f\left(\left[\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix}, \zeta\right] \tilde{g}\right) = |a|^{2s} \zeta\, f(\tilde{g}), \qquad a \in \Omega, x \in F, \zeta \in \mu_n.$$

Here ζ on the right-hand side actually means $\boldsymbol{j}(\zeta)$, as explained in Remark 3.

We have an action of $\widetilde{\mathrm{SL}}_2(F_v)$ for all nonarchimedean $v \in S$, and an $(\mathfrak{sl}_2, K_v)$-module structure when v is archimedean. It may be checked using Kazhdan and Patterson [15], particularly Section 0.3 or Flicker [11] Section 2.1 that since Ω is maximal isotropic, $I(s)$ is irreducible when s is in general position.

Since we are assuming that F is totally complex, the metaplectic cover splits over the archimedean places. This means that we may decompose $S = S_\infty \cup S_{\mathrm{fin}}$ into the union of its complex and nonarchimedean places and

$$F_S = F_\infty F_{\mathrm{fin}}, \qquad \widetilde{\mathrm{SL}}_2(F_S) \cong \mathrm{SL}_2(F_\infty) \times \widetilde{\mathrm{SL}}_2(F_{\mathrm{fin}}),$$

where $F_\infty = \prod_{v \in S_\infty} F_v$ and $F_{\mathrm{fin}} = \prod_{v \in S_{\mathrm{fin}}} F_v$. We can factor

$$\Omega = \Omega_{\mathrm{fin}} \cdot F_\infty^\times,$$

where Ω_{fin} is the projection of Ω on $F_{\mathrm{fin}}^\times$. Indeed, Ω contains $(F_\infty^\times)^n = F_\infty^\times$ since F is totally complex.

The space $I(s)$ can thus be identified with $I_\infty(s) \otimes I_{\mathrm{fin}}(s)$, where $I_\infty(s)$ and $I_{\mathrm{fin}}(s)$ are defined analogously to $I(s)$. In particular the space $I_\infty(s)$ consists of all K_∞-finite functions f_∞ on $\mathrm{SL}_2(F_\infty)$ such that

$$f_\infty\left(\left[\begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix}, \zeta\right] \tilde{g}\right) = \zeta\, |a|^{2s}\, f(\tilde{g}), \qquad a \in \mathbb{C}^\times$$

where $K_\infty = \prod_{v \in S_\infty} K_v$. The local Hilbert symbol is trivial on $\mathbb{C}$, and the metaplectic cover of $\mathrm{SL}(2, F_\infty)$ is trivial. We will denote by $f_\infty^\circ = f_{s,\infty}^\circ$ the normalized spherical vector in I_∞, whose restriction to K_∞ is identically 1.

If $f \in I(s)$ and $re(s) > \frac{1}{2}$ let

$$M(s) f(\tilde{g}) = \int_{F_S} f\left(\boldsymbol{i}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \tilde{g}\right) dx.$$

This integral is then absolutely convergent. It is well-known that the integral has meromorphic continuation (in a suitable sense) to all s with a simple pole at $s = \frac{1}{2}$. We also define

$$M_{\mathrm{fin}}(s) : I_{\mathrm{fin}}(s) \longrightarrow I_{\mathrm{fin}}(1-s)$$

by

$$M_{\mathrm{fin}}(s) f_{\mathrm{fin}}(\tilde{g}) = \int_{F_{\mathrm{fin}}} f\left(\boldsymbol{i}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\tilde{g}, s\right)\, dx, \qquad \tilde{g} \in \widetilde{\mathrm{SL}}_2(F_{\mathrm{fin}}),$$

and $M_\infty(s) : I_\infty(s) \longrightarrow I_\infty(1-s)$ by the same formula.We fix some particular fractional ideals in $\mathfrak{o}_{\mathrm{fin}}$. First, let $\mathfrak{D}$ be the conductor of ψ_{fin}, that is, the largest fractional ideal on which it is trivial. Next, let

$$\mathfrak{N} = \prod_v \mathfrak{p}_v^{2\,\mathrm{ord}_v(n)+1}, \tag{15}$$

where v runs through S_{fin}. Finally, let $\mathfrak{M} = \mathfrak{N}^{-1}\mathfrak{D}$. It is chosen so that if $y \notin \mathfrak{M}$ then $\int_{\mathfrak{N}} \psi(uy)\, du = 0$.

Let $\mathfrak{o}_{\mathrm{fin}} = \prod_{v \in S_{\mathrm{fin}}} \mathfrak{o}_v$. By a *fractional ideal* of $\mathfrak{o}_{\mathrm{fin}}$ we mean a product of fractional ideals of the $\mathfrak{o}_v$. It is an open and compact $\mathfrak{o}_{\mathrm{fin}}$-submodule of F_{fin}. If $\alpha, c \in F_S^\times$ and $\Psi \in \mathcal{M}(\Omega)$ we will denote, for every fractional ideal $\mathfrak{X}$ of $\mathfrak{o}_{\mathrm{fin}}$

$$J(\mathfrak{X}, \alpha, c) = J_\Psi(\mathfrak{X}, \alpha, c) = \int_{y \in \mathfrak{X}} (y, c)\, |y|_{\mathrm{fin}}^{2s-2} \Psi(c^{-1}y^{-1}) \psi_{\mathrm{fin}}(\alpha y)\, dy.$$

The function Ψ will be fixed in this discussion and we usually denote J_Ψ as simply J. The value of $J(\mathfrak{X}, \alpha, c)$ obviously depends only on the projections of α and c on F_{fin} and we will use the same notation if α and c are given as elements of $F_{\mathrm{fin}}^\times$.

Lemma 6 *Let $\Psi \in \mathcal{M}(\Omega)$ and $\alpha, c \in F_S^\times$. The value of $J_\Psi(\mathfrak{X}, c)$ is constant for all sufficiently large fractional ideals $\mathfrak{X}$ in $\mathfrak{o}_{\mathrm{fin}}$. More precisely if $\mathfrak{X} \supseteq \alpha^{-1}\mathfrak{M}$ then*

$$J_\Psi(\mathfrak{X}, \alpha, c) = J_\Psi(\alpha^{-1}\mathfrak{M}, \alpha, c).$$

Proof We have

$$J(\mathfrak{X}, \alpha, c) - J(\alpha^{-1}\mathfrak{M}, \alpha, c) = \int_{y \in \mathfrak{X} - \alpha^{-1}\mathfrak{M}} (y, c)\, |y|_{\mathrm{fin}}^{2s-2} \Psi(c^{-1}y^{-1}) \psi_{\mathrm{fin}}(\alpha y)\, dy.$$

It follows from Hensel's Lemma that $1 + \mathfrak{N} \subset (\mathfrak{o}_S^\times)^n$. Hence if $u \in 1 + \mathfrak{N}$ we have

$$(y, c)\, |y|_{\mathrm{fin}}^{2s-2} \Psi(c^{-1}y^{-1}) = (uy, c)\, |uy|_{\mathrm{fin}}^{2s-2} \Psi(c^{-1}u^{-1}y^{-1}).$$

Thus for any $n \in \mathfrak{N}$ the variable change $y \longmapsto (1+n)y$ shows that

$$J(\mathfrak{X}, c) - J(\alpha^{-1}\mathfrak{M}) = \psi(\alpha n y) \int_{y \in \mathfrak{X} - \alpha^{-1}\mathfrak{M}} (y, c)\, |y|_{\mathrm{fin}}^{2s-2} \Psi(c^{-1}y^{-1}) \psi_{\mathrm{fin}}(\alpha y)\, dy.$$

Now we may integrate over $n \in \mathfrak{N}$ and obtain zero since $\alpha y \notin \mathfrak{M}$. $\square$

Denote the stable value $J_\Psi(\alpha^{-1}\mathfrak{M}, \alpha, c)$ of $J_\Psi(\mathfrak{X}, \alpha, c)$ by $J_\Psi(\alpha, c)$ or (more often) $J(\alpha, c)$.

Proposition 4 *Let $\eta \in F_S^\times$ be given. There exists an S-Dirichlet polynomials $p_\eta(s, \alpha)$ such that, with $\tilde{\Psi}_\eta$ as in (4), we have*

$$\prod_{v \in S_{\mathrm{fin}}} (1 - q_v^{-n(2s-1)}) J_\Psi(\alpha, c) = \sum_\eta p_\eta(s, \alpha)\, \tilde{\Psi}_\eta(c). \tag{16}$$

If $\lambda \in F_S^\times$ we have

$$p_\eta(s, \lambda\alpha) = |\lambda|_{\mathrm{fin}}^{1-2s} p_{\lambda\eta}(s, \alpha). \tag{17}$$

Also p_η depends only on the coset of η in $F_S/(F_S^\times)^n$.

Proof Splitting the integral defining J into cosets of $(F_S^\times)^n$ we may write

$$J(\alpha, c) = \sum_{\eta \in F_S^\times/(F_S^\times)^n} J_\eta(\alpha, c).$$

where

$$J_\eta(\alpha, c) = \int_{y \in \alpha^{-1}\mathfrak{M} \cap \eta(F_S^\times)^n} (y, c)\, |y|_{\mathrm{fin}}^{2s-2}\, \Psi(c^{-1}y^{-1})\, \psi_{\mathrm{fin}}(\alpha y)\, dy.$$

Observe that J_η, like $\tilde{\Psi}_\eta$ depends only on η in $F_S^\times/(F_S^\times)^n$. If $y \in \eta(F_S^\times)^n$ we have

$$(y, c)\, \Psi(c^{-1}y^{-1}) = \tilde{\Psi}_\eta(c).$$

Thus

$$\prod_{v \in S_{\mathrm{fin}}} (1 - q_v^{-n(2s-1)})\, J_\eta(\alpha, c) = p_\eta(s, \alpha)\, \tilde{\Psi}_\eta(c).$$

where

$$p_\eta(s, \alpha) = \prod_{v \in S_{\mathrm{fin}}} (1 - q_v^{-n(2s-1)}) \int_{y \in \alpha^{-1}\mathfrak{M} \cap \eta(F_S^\times)^n} |y|_{\mathrm{fin}}^{2s-2} \psi_{\mathrm{fin}}(\alpha y)\, dy. \tag{18}$$

We may write

$$p_\eta(s,\alpha) = \prod_{v\in S_{\text{fin}}} p_{\eta,v}(s,\alpha),$$

where the $p_{\eta,v}$ are defined as follows. Let $\mu(v) = \operatorname{ord}_v(\alpha^{-1}\mathfrak{M})$. Then

$$p_{\eta,v}(s,\alpha) = (1-q_v^{-n(2s-1)})\left[\int_{y\in\eta(F_v^\times)^n\,\cap\,\mathfrak{p}_v^{\mu(v)}} |y|_v^{2s-2}\,\psi_v(\alpha y)\,dy\right]. \tag{19}$$

For $v\in S_{\text{fin}}$ let $\mathfrak{U}_v$ be a sufficiently small ideal in $\mathfrak{o}_v$ such that $\mathfrak{U}_v\subset\eta^{-1}\mathfrak{p}^{\mu(v)}$, and $\psi_v(\alpha\eta y)=1$ for $y\in\mathfrak{U}_v$. (We hold η fixed and suppress it from the notation, for example not indicating the dependence of $\mathfrak{U}_v$ on it.) There is no harm in assuming $\mathfrak{U}_v=\mathfrak{p}_v^{nN_v}$ is an n-th power. With these conditions, we choose $\mathfrak{U}_v$ as large as possible, so that $|\mu(v)-nN_v|_v$ remains bounded uniformly as α varies, a point we will need at the end of the proof.

Note that

$$(1-q_v^{-n(2s-1)})\,|\eta|_v^{2s-1}\int_{y\in(F_v^\times)^n\,\cap\,\mathfrak{U}_v}|y|_v^{2s-2}\,\psi_v(\alpha y)\,dy =$$
$$(1-q_v^{-n(2s-1)})\,|\eta|_v^{2s-1}\int_{y\in(F_v^\times)^n\,\cap\,\mathfrak{p}^{nN_v}}|y|_v^{2s-1}\,\frac{dy}{|y|}.$$

Summing a geometric series, this equals

$$V_v\,|\eta|_v^{2s-1}\,q_v^{-nN_v(2s-1)},\qquad \text{where } V_v=\text{vol}\,(\mathfrak{o}_v^\times)^n.$$

Using this we split the integral to obtain

$$p_{\eta,v}(s,\alpha) = (1-q_v^{-n(2s-1)})\left[\int_{\substack{y\,\in\,\eta(F_v^\times)^n\,\cap\,\mathfrak{p}_v^{\mu(v)}\\ y\,\notin\,\mathfrak{U}_v}} |y|_v^{2s-1}\,\psi_v(\alpha y)\,\frac{dy}{|y|_v}\right]$$
$$+V_v\,|\eta|_v^{2s-1}\,q_v^{-nN_v(2s-1)}. \tag{20}$$

The first integration is over a compact set, so $p_{\eta,v}(s,\alpha)$ is a polynomial in q_v^{-s}, and so $p_\eta(s,\alpha)$ is an S-Dirichlet polynomial. This expression for $p_{\eta,v}(s,\alpha)$ is valid even if $\text{re}(s)$ is no longer assumed to be $>\frac{1}{2}$.

Finally, let us prove (17). We have

$$p_{\eta,v}(s,\alpha) = (1-q_v^{-n(2s-1)})\left[\int_{y\in\eta(F_v^\times)^n\,\cap\,\mathfrak{a}} |y|_v^{2s-2}\,\psi_v(\alpha y)\,dy\right]$$

for all sufficiently large fractional ideals $\mathfrak{a}$. So taking $\mathfrak{a}$ sufficiently large, we also have

$$p_{\eta,v}(s,\lambda\alpha) = (1-q_v^{-n(2s-1)})\left[\int_{y\in\eta(F_v^\times)^n\cap\lambda^{-1}\mathfrak{a}} |y|_v^{2s-2}\,\psi_v(\lambda\alpha y)\,dy\right].$$

The variable change $y\longmapsto\lambda^{-1}y$ now proves (17). $\square$

If $f\in I(s)$ and $c\in F_{\text{fin}}$, then denote

$$\Psi_{f,\alpha,s}(c) = |c|_{\text{fin}}^{2s}\int_{F_{\text{fin}}} f_{\text{fin}}\left(\boldsymbol{i}\begin{pmatrix} & -c^{-1}\\ c & \end{pmatrix}\boldsymbol{i}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)\psi_{\text{fin}}(\alpha x)\,dx. \tag{21}$$

This integral is convergent if $\mathrm{re}(s)>\frac{1}{2}$. We wish to extend this definition to all s, which we accomplish by the following lemma.

Lemma 7 *Let $f\in I_{\text{fin}}(s)$. Then there exists a fractional ideal $\mathfrak{X}_0$ of $\mathfrak{o}_{\text{fin}}$ (depending on f) such that if $\mathfrak{X}$ is a fractional ideal containing $\mathfrak{X}_0$ then*

$$\int_{\mathfrak{X}} f\left(\boldsymbol{i}\begin{pmatrix} & -c^{-1}\\ c & \end{pmatrix}\boldsymbol{i}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)\psi_{\text{fin}}(\alpha x)\,dx = \int_{\mathfrak{X}_0} f\left(\boldsymbol{i}\begin{pmatrix} & -c^{-1}\\ c & \end{pmatrix}\boldsymbol{i}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)\psi_{\text{fin}}(\alpha x)\,dx. \tag{22}$$

Proof We have, in $\widetilde{\mathrm{SL}}_2(F_S)$

$$\boldsymbol{i}\begin{pmatrix} & -c^{-1}\\ c & cx\end{pmatrix} = (c,x)\,\boldsymbol{i}\begin{pmatrix}(cx)^{-1} & -c^{-1}\\ & cx\end{pmatrix}\boldsymbol{i}\begin{pmatrix}1 & \\ -x^{-1} & 1\end{pmatrix}.$$

Then the difference between the right- and left-hand sides in (22) is thus

$$\sum_{\eta\in F_S^\times/(F_S^\times)^n}\int_{(\mathfrak{X}-\mathfrak{X}_0)\cap(\eta F_S^\times)^n}(c,x)\,f\left(\boldsymbol{i}\begin{pmatrix}(cx)^{-1} & -c^{-1}\\ & cx\end{pmatrix}\boldsymbol{i}\begin{pmatrix}1 & \\ x^{-1} & 1\end{pmatrix}\right)\psi(\alpha x)\,dx.$$

Since f is a smooth vector, that is, invariant by elements sufficiently close to the identity, and since $\boldsymbol{i}\begin{pmatrix}1 & \\ x^{-1} & 1\end{pmatrix}$ is near the identity in the maximal compact subgroup if $|x|$ is sufficiently large we may omit this $\boldsymbol{i}\begin{pmatrix}1 & \\ x^{-1} & 1\end{pmatrix}$ from the last expression when x is in the complement of $\mathfrak{X}_0$ provided $\mathfrak{X}_0$ is sufficiently large. It is easy to see that if $\mathfrak{X}_0$ is sufficiently large then

$$\int_{(\mathfrak{X}-\mathfrak{X}_0)\cap(\eta F_S^\times)^n}(c,x)|x|^{-2s}\,\psi(\alpha x)\,dx = 0,$$

and the statement follows. $\square$

Consequently we may extend the definition of Ψ_f by the formula

$$\Psi_{f,\alpha,s}(c) = |c|_{\text{fin}}^{2s} \lim_{\mathfrak{X}} \int_{\mathfrak{X}} f_{\text{fin}} \left(\boldsymbol{i} \begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix} \boldsymbol{i} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right) \psi_{\text{fin}}(\alpha x)\, dx, \tag{23}$$

where it is understood that the notation means that the expression is exact for sufficiently large fractional ideals $\mathfrak{X}$.

Proposition 5 *The function $\Psi_{f,\alpha,s}(c)$ is analytic as a function of s in the sense that if $f = f_s$ varies in a family parametrized by $s \in \mathbb{C}$ such that $f_s|K$ is independent of s, then $\Lambda_{\text{fin}}(f_s)$ is an analytic function of s.*

Proof This is clear. $\square$

The content of the next Lemma is to construct a function f with small support contained in the big Bruhat cell from Ψ. We will see later in Proposition 6 that $\Psi_{f,\alpha,s}$ matches Ψ for sufficiently many α, depending on the choice of an auxiliary ideal $\mathfrak{a}$.

Lemma 8 *Let $\Psi \in \mathcal{M}(\Omega)$. If $\mathfrak{a}$ is an ideal of $\mathfrak{o}_{\text{fin}}$ such that $\mathfrak{N}$ (defined in (15)) divides $\mathfrak{a}$, then*

$$f_{s,\text{fin}}^{\mathfrak{a}} \left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta \right] \right) = \begin{cases} \text{vol}(\mathfrak{a})^{-1}\, \zeta |c|_{\text{fin}}^{-2s}\, \Psi(c) & \textit{if } d/c \in \mathfrak{a} \\ 0 & \textit{otherwise.} \end{cases},$$

defines an element of $I_{\text{fin}}(s)$.

Proof In this proof denote $f = f_{s,\text{fin}}^{\mathfrak{a}}$. First note the following matrix identity:

$$f_{s,\text{fin}} \left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta \right] \right) = f_{s,\text{fin}} \left(\left[\begin{pmatrix} c^{-1} & a \\ & c \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & d/c \\ & 1 \end{pmatrix}, \zeta \right] \right) =$$
$$f_{s,\text{fin}} \left(\left[\begin{pmatrix} c^{-1} & a \\ & c \end{pmatrix} \begin{pmatrix} 1 & \\ -d/c & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \zeta \right] \right) =$$
$$\begin{cases} \text{vol}(\mathfrak{a})^{-1}\, \zeta |c|_v^{-2s}\, \Psi(c) & \text{if } d/c \in \mathfrak{a} \\ 0 & \text{otherwise.} \end{cases}$$

To show that $f \in I_{\text{fin}}(s)$, we must show that it satisfies

$$f \left(\left[\begin{pmatrix} t & * \\ & t^{-1} \end{pmatrix}, \zeta \right] \tilde{g} \right) = \zeta |t|_{\text{fin}}^{2s} f(\tilde{g}), \qquad t \in \Omega, \tag{24}$$

and that it is smooth. Using the above matrix identity and (2) it is easy to check (24).

For smoothness, we first show that the cocycle is trivial over $K_{\mathrm{fin}}(\mathfrak{a})$. By (14) it is sufficient to show that if $g \in K_v(\mathfrak{p}_v^N)$, then $\beta(g) = 1$. If $N \geqslant 2\,\mathrm{ord}_v(n) + 1$ then since $d \equiv 1 \bmod \mathfrak{p}_v^N$, it follows from Hensel's Lemma that d is an n-th power.

With $\mathfrak{a}$ chosen this way, we claim that f_{fin} is right invariant under $K_{\mathrm{fin}}(\mathfrak{a})$. If $\kappa \in K_{\mathrm{fin}}(\mathfrak{a})$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \kappa = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

where $d'/c' \in \mathfrak{a}$ if and only if $d/c \in \mathfrak{a}$. Assuming this, we have $\Psi(c) = \Psi(c')$ because

$$\begin{pmatrix} -d' & c' \end{pmatrix} = \begin{pmatrix} -d & c \end{pmatrix} \kappa', \qquad \kappa' = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \kappa \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

so $c'/c = \delta - \beta d/c$ is near 1, and Ψ is continuous since it is constant on cosets of the n-th powers, which comprise an open subgroup of $F_{\mathrm{fin}}^\times$. □

Proposition 6 *Let $\Psi \in \mathcal{M}(\Omega)$, let $\mathfrak{D}$ be the conductor of ψ_{fin} and let $\mathfrak{N}$ be as in (15). Let $\mathfrak{a}$ be a fractional ideal of $\mathfrak{o}_{\mathrm{fin}}$ such that $\mathfrak{N}|\mathfrak{a}$, and let $f_{\mathrm{fin}} = f_{\mathrm{fin}}^{\mathfrak{a}}$ Then for all $c \in F_{\mathrm{fin}}^\times$ we have*

$$\Psi_{f,\alpha,s}(c) = \begin{cases} \Psi(c) & \text{if } \alpha \in \mathfrak{a}^{-1}\mathfrak{D}, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

In particular $\Psi_{f,\alpha,s}(c)$ is independent of s for all α.

Proof Let $0 \neq c \in \mathfrak{o}_S$. Factor $c = c_\infty c_{\mathrm{fin}}$ corresponding to the decomposition $S = S_\infty \cup S_{\mathrm{fin}}$, where S_∞ and S_{fin} are the archimedean and nonarchimedean places in S. We note that $\Psi(c)$ depends only on c_{fin}, because Ω contains $F_\infty^\times$, and in (2) if $\varepsilon \in F_\infty^\times$ then, since F is totally complex, $(\varepsilon, c) = 1$ identically, and the Hilbert symbol is trivial on $\mathbb{C}$. In the integral

$$|c|_{\mathrm{fin}}^{2s} \int_{F_{\mathrm{fin}}} f_{\mathrm{fin}} \left(\boldsymbol{i} \begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix} \boldsymbol{i} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s \right) \psi_{\mathrm{fin}}(\alpha x)\, dx$$

we have

$$f_{\mathrm{fin}} \left(\boldsymbol{i} \begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix} \boldsymbol{i} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s \right) = \begin{cases} \mathrm{vol}(\mathfrak{a})^{-1}\, |c|_{\mathrm{fin}}^{-2s} & \text{if } x \in \mathfrak{a} \\ 0 & \text{otherwise.} \end{cases}$$

The statement follows. □

Proposition 7 *Let $\Psi \in \mathcal{M}(\Omega)$. Let $\mathfrak{a}$ be a fractional ideal of $\mathfrak{o}_{\mathrm{fin}}$ which is contained in $\alpha^{-1}\mathfrak{D} \cap \mathfrak{N}$, so that by Proposition 6 we have*

$$\Psi_{f,\alpha,s}(c) = \Psi(c), \qquad f = f^{\mathfrak{a}}_{s,\mathrm{fin}} \tag{26}$$

Then, with $\tilde{\Psi}_\eta$ and p_η as in Proposition 4, we have

$$\prod_{v \in S_{\mathrm{fin}}} (1 - q_v^{-n(2s-1)})\, \Psi_{M_{\mathrm{fin}}(s)f,\alpha,1-s}(c) = \sum_\eta p_\eta(s,\alpha)\, \tilde{\Psi}_\eta(c). \tag{27}$$

Proof We will confirm this when $\mathrm{re}(s) > \frac{1}{2}$; both sides are analytic, so this suffices. Then the integral defining $M_{\mathrm{fin}}(s) : I(s) \longrightarrow I(1-s)$ is convergent, and although the integral (21) is divergent when applied to $M_{\mathrm{fin}}(s)f$, we may use instead (23). We have

$$\Psi_{M_{\mathrm{fin}}(s)f,\alpha,1-s}(c) =$$

$$|c|_{\mathrm{fin}}^{2-2s} \int_{\mathfrak{X}} M_{\mathrm{fin}}(s) f^{\mathfrak{a}}_{\mathrm{fin}}\left(\boldsymbol{i}\begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s\right)\psi(\alpha x)\,dx \;=$$

$$|c|_{\mathrm{fin}}^{2-2s} \int_{\mathfrak{X}}\int_{F_{\mathrm{fin}}} f^{\mathfrak{a}}_{\mathrm{fin}}\left(\boldsymbol{i}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}\boldsymbol{i}\begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s\right)$$
$$\times \psi_{\mathrm{fin}}(\alpha x)\; dy\, dx,$$

where $\mathfrak{X}$ is any sufficiently large fractional ideal in F_{fin}. We now interchange the order of integration and use a matrix identity to write this as

$$|c|_{\mathrm{fin}}^{2-2s}\int_{F_{\mathrm{fin}}}\int_{\mathfrak{X}} (yc, y)\, f^{\mathfrak{a}}_{\mathrm{fin}}\left(\boldsymbol{i}\begin{pmatrix} c^{-1}y^{-1} & -c \\ & cy \end{pmatrix}\boldsymbol{i}\begin{pmatrix} & -1 \\ 1 & x - c^{-2}y^{-1} \end{pmatrix}, s\right)\psi_{\mathrm{fin}}(\alpha x)\, dx\, dy.$$

Recalling that we are taking f_{fin} to be the same test function that we took in Lemma 8, by Proposition 6 this equals

$$|c|_{\mathrm{fin}}^{2-2s}\ \mathrm{vol}(\mathfrak{a})^{-1}\int_{F_{\mathrm{fin}}} (c,y)\, |cy|_{\mathrm{fin}}^{-2s}\, \Psi(cy)\, \psi(\alpha c^{-2}y^{-1})$$
$$\left[\int_{(c^{-2}y^{-1}+\mathfrak{a})\cap\mathfrak{X}} \psi_{\mathrm{fin}}(\alpha(x - c^{-2}y^{-1}))\ dx\right] dy.$$

We may choose $\mathfrak{X}$ so large that $\mathfrak{N} \subseteq \mathfrak{X}$. This implies that $\mathfrak{a} \subseteq \mathfrak{X}$, so the inner integral is over a coset of $\mathfrak{a}$ if $c^{-2}y^{-1} \in \mathfrak{X}$, and over the empty set if not. In the first case, the integrand $\psi_{\mathrm{fin}}(\alpha(x - c^{-2}y^{-1})) = 1$ since ψ is trivial on $\alpha\mathfrak{a}$. The inner integral thus produces vol($\mathfrak{a}$) times the characteristic function of the set of y such that $c^{-2}y^{-1} \in \mathfrak{X}$. Thus we obtain

$$|c|_{\mathrm{fin}}^{2-4s} \int_{c^{-2}y^{-1}\in\mathfrak{X}} (c,y)\ |y|_{\mathrm{fin}}^{-2s}\, \Psi(cy)\, \psi(\alpha c^{-2}y^{-1})\, dy.$$

Make the variable change $y \longrightarrow c^{-2}y^{-1}$ we obtain

$$\int_{y\in\mathfrak{X}} (y,c)\ |y|_{\mathrm{fin}}^{2s-2}\, \Psi(c^{-1}y^{-1})\, \psi(\alpha y)\, dy.$$

Thus

$$\Psi_{M_{\mathrm{fin}}(s)f,\alpha,1-s}(c) = J_\Psi(\alpha, c),$$

and the result follows from (16). □

We end this section with a complementary computation at the archimedean places. Let $v \in S_\infty$ so that $F_v = \mathbb{C}$, so $K_v = \mathrm{SU}(2)$, and recall that

$$I_v(s) = \left\{ K_v\text{-finite } f : \mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathbb{C} \mid f\left(\begin{pmatrix} y & * \\ & y^{-1} \end{pmatrix} g\right) = |y|_v^{2s} f(g) \right\}.$$

Let $f_v^\circ \in I_v(s)$ denote the normalized spherical vector, so that

$$f_v^\circ\left(\begin{pmatrix} y & * \\ & y^{-1} \end{pmatrix} k\right) = |y|_v^{2s}, \qquad k \in \mathrm{SU}(2),\ y \in \mathbb{C}. \tag{28}$$

Then $\Lambda_v(f_v^\circ)$ is a Bessel function; more precisely,

Proposition 8 *Let v be a complex place. If $0 \neq \alpha_v, c_v \in \mathbb{C}$ we have*

$$\begin{aligned} &\int_{\mathbb{C}} f_v^\circ\left(\begin{pmatrix} & -c_v^{-1} \\ c_v & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \psi_v(\alpha_v x)\, dx \quad = \\ &(2\pi)^{2s}\, |\alpha_v|_v^{s-1/2}\, |c|_v^{-2s}\, \Gamma(2s)^{-1}\, K_{2s-1}(4\pi\sqrt{|\alpha_v|_v}). \end{aligned} \tag{29}$$

We remind the reader when v is a complex place the absolute value $|x+iy|_v = x^2+y^2$ for $x, y \in \mathbb{R}$.

Proof Using the definition of $I_v(s)$ we reduce immediately to the case where $c_v = 1$. Then

$$\begin{aligned} &\int_{\mathbb{C}} f_v^\infty\left(\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} x\Delta_x^{-1} & \Delta_x^{-1} \\ -\Delta_x^{-1} & \bar{x}\Delta_x^{-1} \end{pmatrix}, s\right)\right) \psi_v(\alpha_v x)\, dx = \\ &\int_{\mathbb{C}} (1+x\bar{x})^{-2s}\, e^{2\pi i(\mathrm{tr}\ \alpha_v x)}\, dx, \end{aligned}$$

where $\Delta_x = \sqrt{1+x\bar{x}}$, and (we remind the reader) $\psi_v(x) = e^{2\pi i\,\mathrm{tr}(x)}$. The integral is easily evaluated. □

5 Eisenstein series

We will now consider the Eisenstein series associated with the special data in Proposition 6. We will consider the constant term and nonconstant terms in the Fourier expansions. The Kubota Dirichlet series will appear in the nonconstant terms, while (as usual) the functional equations will be controlled by the constant term.

Let $\Gamma = \mathrm{SL}(2, \mathfrak{o}_S)$, and let $\tilde{\Gamma}$ be the preimage of Γ in $\widetilde{\mathrm{SL}}(2, F_S)$. By Theorem 5

$$\widetilde{\kappa}([\gamma, \zeta]) = \zeta^{-1}\kappa(\gamma), \qquad \kappa \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{c}\right) & \text{if } c \neq 0; \\ 1 & \text{if } c = 0 \end{cases}$$

defines a character of $\tilde{\Gamma}$. Let $f_s \in I(s)$. We assume that the restriction of f_s to K is independent of s. Consider the Eisenstein series

$$E(\tilde{g}, s, f_s) = \sum_{\gamma \in B(\mathfrak{o}_S)\backslash\Gamma} \kappa(\gamma) f_s(\boldsymbol{i}(\gamma)\tilde{g}), \qquad \tilde{g} \in \widetilde{\mathrm{SL}}(2, F_S),$$

where

$$B(\mathfrak{o}_S) = \left\{ \begin{pmatrix} a & u \\ & a^{-1} \end{pmatrix} \mid a \in \mathfrak{o}_S^\times, u \in \mathfrak{o}_S \right\}.$$

We choose the function f_s to be of the form $f_{s,\infty} \otimes f_{s,\mathrm{fin}}$ where $f_{s,\infty} \in I_\infty(s)$ and $f_{s,\mathrm{fin}} \in I_{\mathrm{fin}}(s)$. Moreover we choose $f_{s,\infty} = \otimes_{v \in S_\infty} f_{s,v}^\circ$, where $f_{s,v}^\circ$ is the normalized spherical vector (28). The series is absolutely convergent if $\mathrm{re}(s) > 1$.

Proposition 9 *Let $f_s = f_{s,\infty} \otimes f_{s,\mathrm{fin}} \in I(s) = I_{s,\infty}(s) \otimes I_{s,\mathrm{fin}}(s)$, where $f_{s,\infty} = f_{s,\infty}^\circ$ is the normalized spherical vector. Then*

$$\int_{F_S/\mathfrak{o}_S} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \tilde{g}, s, f_s \right) dx =$$

$$f_s(\tilde{g}) + \left(n\pi \frac{\Gamma(2ns - n)}{\Gamma(2ns - n + 1)} \right)^r \frac{\zeta_S(2ns - n)}{\zeta_S(2ns - n + 1)} \left(M_{\mathrm{fin}}(s) f_{s,\mathrm{fin}}\right)(\tilde{g})\, f_{1-s,\infty}^\circ(\tilde{g}),$$

where r is the number of complex pairs of embeddings and ζ_S denotes the partial Dedekind zeta function of F (with Euler factors removed for those places in S).

Proof Let

$$N(\mathfrak{o}_S) = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathfrak{o}_S \right\}.$$

The constant term of the Fourier expansion is computed in the usual way. There are two contributions from γ in the two Bruhat cells; if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the two cells are contributions of γ with $c = 0$ and $c \neq 0$ respectively. We find that

$$\int_{F_S/\mathfrak{o}_S} E\left(\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\tilde{g}, s, f_s\right)\, dx =$$

$$f_s(\tilde{g}) + \sum_{\substack{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(\mathfrak{o}_S)\backslash\Gamma/N(\mathfrak{o}_S) \\ c \neq 0}} \kappa(\gamma) \int_{F_S} f\left(\boldsymbol{i}(\gamma)\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\tilde{g}, s\right)\, dx =$$

$$f_s(\tilde{g}) + \sum_{0 \neq c \in \mathfrak{o}_S/\mathfrak{o}_S^\times} \sum_{d \bmod c} \left(\frac{d}{c}\right) \int_{F_S} f\left(\boldsymbol{i}\begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\tilde{g}, s\right)\, dx.$$

The contribution of c will be zero unless

$$\sum_{d \bmod c} \left(\frac{d}{c}\right) \neq 0.$$

This means that c is a unit times an n-th power. Since we are choosing c from a set of representatives of $\mathfrak{o}_S/\mathfrak{o}_S^\times$ we may disregard the unit and take c to be an n-th power. Replace c by c^n and c will now run through $\mathfrak{o}_S/\mathfrak{o}_S^\times$. The second term becomes

$$\frac{1}{n}\sum_{c \in \mathfrak{o}_S/\mathfrak{o}_S^\times} \phi(c^n)|c|^{-2ns}\left[\int_{F_S} f\left(\boldsymbol{i}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\tilde{g}, s\right)\, dx\right], \tag{30}$$

where $\phi(a)$ is the cardinality of $(\mathfrak{o}_S/a\mathfrak{o}_S)^\times$, that is, the Euler totient function for the ring $\mathfrak{o}_S$. Making use of the fact that $\mathfrak{o}_S$ is a principal ideal domain,

$$\frac{1}{n}\sum_{c \in \mathfrak{o}_S/\mathfrak{o}_S^\times} \phi(c^n)|c|^{-2ns} = \frac{\zeta_S(2ns-n)}{\zeta_S(2ns-n+1)}.$$

The integral in brackets in (30) factors into S_∞ and S_{fin} components. We have assumed that f_∞ is spherical at every archimedean place, so applying the intertwining integral gives a constant times the spherical vector in $I(1-s)$, that is,

$$\int_{F_\infty} f_{s,\infty}\left(\boldsymbol{i}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\tilde{g}, s\right)\, dx = c(s)\, f_{1-s,\infty}(\tilde{g}),$$

and to compute $c(s)$ we may take $\tilde{g}$ to be the identity. At each archimedean place, this reduces to

$$\int_{\mathbb{C}} (1 + x\bar{x})^{-2s} dx = \frac{\pi}{2s-1} = n\pi \frac{\Gamma(2ns-n)}{\Gamma(2ns-n+1)},$$

so we have a factor of

$$\left[\int_{\mathbb{C}} (1 + x\bar{x})^{-2s} dx\right]^r = \left[n\pi \frac{\Gamma(2ns-n)}{\Gamma(2ns-n+1)}\right]^r.$$

At the nonarchimedean places, we have a factor of

$$\int_{F_{\mathrm{fin}}} f\left(\boldsymbol{i}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\tilde{g}, s\right) dx = M_{\mathrm{fin}}(s) f(\tilde{g}).$$

□

We define

$$E^*(\tilde{g}, s, f_s) = (2\pi)^{-r(2ns-n+1)}\Gamma(2ns-n+1)^r \zeta_S(2ns-n+1) E(\tilde{g}, s, f_s)$$

where, as before, r is the number of pairs of complex embeddings. Let $D \in \mathbb{Z}$ be the discriminant of F. We will denote by $|D|$ the usual real absolute value of $\mathbb{R}$ applied to D.

Theorem 6 *The Eisenstein series has meromorphic continuation in s to all of $\mathbb{C}$ and satisfies the functional equation*

$$E^*(\tilde{g}, s, f_s) = \\ |D|^{\frac{1}{2}-2ns+n} \prod_{v \in S_{\mathrm{fin}}} \left(\frac{1-q_v^{-2ns+n}}{1-q_v^{2ns-n-1}}\right) E^*(\tilde{g}, 1-s, M_{\mathrm{fin}}(s) f_{s,\mathrm{fin}} \otimes f_{1-s,\infty}). \quad (31)$$

It can have simple poles at $\mathrm{re}(s) = \frac{1}{2} \pm \frac{1}{2n}$ *but is elsewhere holomorphic.*

Proof We have computed

$$\int_{F_S/\mathfrak{o}_S} E^*\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\tilde{g}, s, f_s\right) dx = \\ (2\pi)^{-r(2ns-n+1)}\, \Gamma(2ns-n+1)^r\, \zeta_S(2ns-n+1)\, f_s(\tilde{g}) + \\ (n/2)^r\, (2\pi)^{-r(2ns-n)}\, \Gamma(2ns-n)^r\, \zeta_S(2ns-n)\, (M_{\mathrm{fin}}(s) f_{s,\mathrm{fin}} \otimes f^{\circ}_{1-s,\infty})(\tilde{g})$$

It follows from Moeglin and Waldspurger [20] Proposition IV.1.10 (or by adapting the discussion of Jacquet [14]) that

$$
\begin{aligned}
&E^*\left(\tilde{g}, s, f_s\right) = (n/2)^r\,(2\pi)^{-r(2ns-n)}\,\Gamma(2ns-n)^r\,\zeta_S(2ns-n) \\
&\times E(\tilde{g}, 1-s, M_{\mathrm{fin}}(s) f_{s,\mathrm{fin}} \otimes f_{1-s,\infty}).
\end{aligned}
$$

We note that the fact that $I(s)$ is not in general irreducible is not a problem. (It is a finite direct sum of several copies of the same irreducible representation.) We recall the functional equation of the Dedekind zeta function. Let

$$
\Lambda(s) = |D|^{s/2}(2\pi)^{-rs}\Gamma(s)^r\zeta_F(s),
$$

where $D \in \mathbb{Z}$ is the discriminant of F and $|D|$ is the usual absolute value on $\mathbb{R}$. Then

$$
\Lambda(s) = \Lambda(1-s).
$$

Hence in terms of partial zeta functions,

$$
\begin{aligned}
&(2\pi)^{-r(2ns-n)}\,\Gamma(2ns-n)^r\,\zeta_S(2ns-n) = \\
&|D|^{\frac{1}{2}-2ns+n}\prod_{v\in S_{\mathrm{fin}}}\left(\frac{1-q_v^{-2ns+n}}{1-q_v^{2ns-n-1}}\right)(2\pi)^{-r(1+n-2ns)}\,\Gamma(1+n-2ns)^r\,\zeta_S(1+n-2ns).
\end{aligned}
$$

Using the last expression, the functional equation can be rewritten as stated in the Theorem. The two constant terms have poles at $2ns-n+1=0,1$ and $2ns-n=0,1$, that is, $s=\frac{1}{2}\pm\frac{1}{2n}$ and $\frac{1}{2}$. Both constant terms have poles at $s=\frac{1}{2}$, but these poles must cancel, since Eisenstein series never have a pole at the center of the critical strip, where the intertwining integral $M(s)$ is unitary. $\square$

Theorem 7 *Let $0 \neq \alpha \in \mathfrak{o}_S$. There is a constant $C = C(n,r)$ depending only on the degree n of the cover and the number r of complex places of F such that*

$$
\begin{aligned}
&\int_{F_S/\mathfrak{o}_S} E^*\left(\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s, f_s\right)\,\psi(\alpha x)\,dx = \\
&C\,\boldsymbol{G}(s)^r\,\zeta_S(2ns-n+1)\,\mathcal{D}(s,\Psi_{f,\alpha,s},\alpha)\,|D|^{\frac{1}{2}-s}\,|\alpha|_\infty^{s-1/2}\,K_{2s-1}(4\pi\sqrt{|\alpha_v|_v}). \qquad (32)
\end{aligned}
$$

Proof Assume that $\mathrm{re}(s) > 1$.

$$\int_{F_S/\mathfrak{o}_S} E\left(\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s, f_s\right) \psi(\alpha x)\, dx =$$

$$\sum_{\substack{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(\Omega, \mathfrak{o}_S)\backslash\Gamma/N(\mathfrak{o}_S) \\ c \neq 0}} \kappa(\gamma) \int_{F_S} f\left(\boldsymbol{i}(\gamma)\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s\right) \psi(\alpha x)\, dx =$$

$$\sum_{0 \neq c \in \mathfrak{o}_S/\mathfrak{o}_S^\times(\Omega)} g(\alpha, c) \int_{F_S} f\left(\boldsymbol{i}\begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix}\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s\right) \psi(\alpha x)\, dx.$$

At an archimedean place v, we have

$$\int_{\mathbb{C}} f_v^\circ\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \psi_v(\alpha_v x)\, dx = (2\pi)^{2s}|\alpha_v|_v^{s-1/2}\Gamma(2s)^{-1}K_{2s-1}(4\pi\sqrt{|\alpha_v|_v}). \quad (33)$$

By (21) and (29) the integral over F_S now factors into a product of $|c|_{\mathrm{fin}}^{-2s}\, \Psi_f(c, \alpha, s)$ times

$$(2\pi)^{2rs}\, \Gamma(2s)^{-r} \prod_{v \in S_\infty} |\alpha_v|_v^{s-1/2}\, |c_v|_v^{-2s}\, K_{2s-1}(4\pi\sqrt{|\alpha_v|})\ .$$

To determine an identity for E^*, we incorporate the following normalizing factor into our integral at archimedean places:

$$(2\pi)^{-r(2ns-n+1)}\, \Gamma(2ns-n+1)^r\, \zeta_S(2ns-n+1).$$

Thus rewriting the above by combining the normalizing factor, the integral over finite places of S, and the integrals at archimedean places, we have

$$\begin{aligned} &\int_{F_S/\mathfrak{o}_S} E^*\left(\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, s\right) \psi(\alpha x)\, dx = \\ &(2\pi)^{-r(2ns-n+1)}(2\pi)^{2rs}\Gamma(2s)^{-r}\, \Gamma(2ns-n+1)^r\, |D|^{\frac{1}{2}-s}\zeta_S(2ns-n+1) \times \\ &\sum_{0 \neq c \in \mathfrak{o}_S/\mathfrak{o}_S^\times(\Omega)} g(\alpha, c)\, \Psi_f(s, \alpha, c)|c|^{-2s} \prod_{v \in S_\infty} |\alpha_v|_v^{s-1/2}K_{2s-1}(4\pi\sqrt{\alpha_v}). \end{aligned} \quad (34)$$

This can be further simplified using

$$\frac{\Gamma(2ns-n+1)}{\Gamma(2s)} = n\frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}\ .$$

The result follows. □

6 Proof of Theorem 1

We can now prove Theorem 1 with

$$P_\eta(s) = |D|^{(2-2n)s+n-\frac{1}{2}} \prod_{v\in S_{\mathrm{fin}}} (1-q_v^{-2ns+n-1}) p_\eta(s,1).$$

Note that by (17) we have more generally

$$P_{\alpha\eta}(s) = |\alpha|_{\mathrm{fin}}^{2s-1} |D|^{(2-2n)s+n-\frac{1}{2}} \prod_{v\in S_{\mathrm{fin}}} (1-q_v^{-2ns+n-1}) p_\eta(s,\alpha).$$

We choose $f = f_{\mathrm{fin}} \otimes f_\infty$, where $f_{\mathrm{fin}} = f_{\mathrm{fin},s}^{\mathfrak{a}}$ is chosen as in Proposition 7 and $f_\infty = \otimes_{v\in S_\infty} f_v^\circ$. By (26), replacing $\tilde{g}$ by $\boldsymbol{i}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ and integrating against $\psi(\alpha x)$, the left-hand side of (31) produces

$$C\, \boldsymbol{G}(s)^r\, \zeta_S(2ns-n+1)\, \mathcal{D}(s,\Psi,\alpha)\ |D|^{\frac{1}{2}-s} \prod_{v\in S_\infty} |\alpha_v|_v^{s-1/2}\, K_{2s-1}(4\pi\sqrt{|\alpha_v|_v}),$$

while the right-and side is

$$\begin{aligned} |D|^{(1-2n)s+n} \prod_{v\in S_{\mathrm{fin}}} \left(\frac{1-q_v^{-2ns+n}}{1-q_v^{2ns-n-1}}\right) C\, \boldsymbol{G}(1-s)^r\, \zeta_S(2n(1-s)-n+1) \\ \times \mathcal{D}(1-s,\Psi_{M(s)f,\alpha,1-s},\alpha) \prod_{v\in S_\infty} |\alpha_v|_v^{1/2-s}\ K_{1-2s}(4\pi\sqrt{|\alpha_v|_v}). \end{aligned}$$

The meromorphic continuation of $\mathcal{D}(s,\Psi,\alpha)$ now follows. We recall that $K_{2s-1} = K_{1-2s}$, so we may cancel the Bessel functions and the constant C, and make use of (27) to obtain

$$\begin{aligned} \boldsymbol{G}(s)^r\, \zeta_S(2ns-n+1)\, \mathcal{D}(s,\Psi,\alpha) &= \\ \prod_{v\in S_\infty} |\alpha_v|_v^{1-2s} |D|^{(2-2n)s+n-\frac{1}{2}} \prod_{v\in S_{\mathrm{fin}}} (1-q_v^{2ns-n-1})^{-1} \\ \boldsymbol{G}(1-s)^r\, \zeta_S(n-2ns+1) \sum_\eta p_\eta(s,\alpha)\ \mathcal{D}(1-s,\tilde{\Psi}_\eta,\alpha)\,. \end{aligned}$$

We restore the missing Euler factors at $v \in S_{\mathrm{fin}}$, moving those for $\zeta_F(2ns-n+1)$ to the right-hand side and simplify to obtain (5).

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