Schur Polynomials and the Yang-Baxter Equation

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Abstract

We describe a parametrized Yang-Baxter equation with nonabelian parameter group. That is, we show that there is an injective map $g \mapsto R(g)$ from $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ to $\mathrm{End}(V \otimes V)$ where V is a two-dimensional vector space such that if $g, h \in G$ then $R_{12}(g)R_{13}(gh)\,R_{23}(h) = R_{23}(h)\,R_{13}(gh)R_{12}(g)$. Here R_{ij} denotes R applied to the i, j components of $V \otimes V \otimes V$. The image of this map consists of matrices whose nonzero coefficients a_1, a_2, b_1, b_2, c_1, c_2 are the Boltzmann weights for the six-vertex model, constrained to satisfy $a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$. This is the exact center of the disordered regime. As an application, we give a new proof based on the Yang-Baxter equation of a result of Hamel and King representing a Schur polynomial times a deformation of the Weyl denominator as the partition function of a six-vertex model. The parameter group can be expanded (within the eight-vertex model) to a group having $\mathrm{GL}(2) \times \mathrm{GL}(1)$ as a subgroup of index two. In this expanded context we find a second representation of Schur polynomials times a different deformation of the Weyl denominator as a partition function. These structures give a *Yang-Baxter system* in the sense of Hlavatý.

Tokuyama [36] proved a deformation of the Weyl character formula for $\mathrm{GL}_n(\mathbb{C})$. A substantial generalization of Tokuyama's deformation was found by Hamel and King [11]. The formula of Hamel and King expresses the Schur polynomial times a deformation of the Weyl denominator as a sum over states of the *two-dimensional* *ice* or *six-vertex model* in statistical mechanics. It turns out that there are two fundamentally distinct ways of doing this.

In statistical physics, the *partition function* is the sum of certain *Boltzmann weights* over all states of the system. The six-vertex model is an example that is much studied in the literature. In our presentation of it the states are represented by labeling the edges of a finite rectangular lattice by $\pm$ signs, called *spins*. If the Boltzmann weights are invariant under sign reversal the system is called *field-free*, corresponding to the physical assumption of the absence of an external field. For field-free weights, the six-vertex model was solved by Lieb [23] and Sutherland [35], in the sense that the partition function can be exactly computed. A very interesting treatment based on the "star-triangle relation" or Yang-Baxter equation ([16], [25]) was given by Baxter [1] and [2], Chapter 9. The papers of Lieb, Sutherland and Baxter assume periodic boundary conditions, but non-periodic boundary conditions were treated by Korepin [17] and Izergin [15]. Much of the literature assumes that the model is field free, but Baxter asserts that the six-vertex model can be solved even in the presence of fields. We do not know whether this has been carried out using the method of [1] and [2].

We will exhibit two particular choices of Boltzmann weights and boundary conditions in the six-vertex model giving systems $\mathfrak{S}_\lambda^\Gamma$ and $\mathfrak{S}_\lambda^\Delta$ for every partition λ of length $\leqslant n$. We will study the system by the method of [1] and [2]. We will prove that the partition functions are

$$Z(\mathfrak{S}_\lambda^\Gamma) = \prod_{i<j}(t_i z_j + z_i) s_\lambda(z_1, \cdots, z_n), \qquad Z(\mathfrak{S}_\lambda^\Delta) = \prod_{i<j}(t_j z_j + z_i) s_\lambda(z_1, \cdots, z_n), \quad (1)$$

where t_i are deformation parameters and s_λ is the Schur polynomial (Macdonald [24]). The Boltzmann weights we use are not field-free. The Δ model is essentially that given by Hamel and King.

To justify these evaluations of the partition function define

$$s_\lambda^\Gamma(z_1, \cdots, z_n; t_1, \cdots, t_n) = \frac{Z(\mathfrak{S}_\lambda^\Gamma)}{\prod_{i<j}(t_i z_j + z_i)}. \qquad (2)$$

Then one seeks to show that s_λ^Γ is symmetric in the sense that it is unchanged if the same permutation is applied to both z_i and t_i. Once this is known, it is possible to show that it is a polynomial in the z_i and t_i, then that it is independent of the t_i; finally, taking $t_i = -1$ one may invoke the Weyl character formula and conclude that it is equal to the Schur polynomial.

In order to prove the symmetry property of s_λ^Γ we will use an instance of the *star-triangle relation*, which is (9.6.8) of Baxter [2]. We thus obtain a new proof of Tokuyama's formula and of Corollary 5.1 in Hamel and King [11], which is our Theorem 11. A second instance of the star-triangle relation solves the same problem for the analogously defined s_λ^Δ, and a third instance shows directly, without using the above evaluations, that $s_\lambda^\Gamma = s_\lambda^\Delta$.

The *star-triangle relation* is the identity

$$R_{12}M_{13}N_{23} = N_{23}M_{13}R_{12}, \tag{3}$$

where the "R-matrices" R, M, N are endomorphisms of $V \otimes V$ for some vector space V, and R_{ij} means apply R to the i-th and j-th components of $V \otimes V \otimes V$ and the identity map to the k-th component, where i, j, k are $1, 2, 3$ in some order. If the matrices R, M, N are equal, we call this the *Yang-Baxter equation*. On the other hand (3) could be generalized by taking $R \in \mathrm{End}(V_1 \otimes V_2)$, $M \in \mathrm{End}(V_1 \otimes V_3)$ and $N \in \mathrm{End}(V_2 \otimes V_3)$ for three different spaces. This point of view leads to a categorical interpretation of the Yang-Baxter equation, one manifestation of which is Drinfeld's universal R-matrix [5].

But we are interested in cases where $V_1 = V_2 = V_3 = V$ and the R-matrices can all be put into one space. Sometimes there is a group G with a map $R \longmapsto R(g)$ from G to $\mathrm{End}(V \otimes V)$, with an identity

$$R_{12}(g)R_{13}(gh)R_{23}(h) = R_{23}(h)R_{13}(gh)\,R_{12}(g) \tag{4}$$

when $g, h \in G$.

There are many such examples in the literature. In those that we are aware of the group G is abelian, say $G = \mathbb{R}$. Particularly, such examples occur within the field-free case of the six-vertex model. There is one such parametrized Yang-Baxter equation for each value of a parameter $\triangle$, defined below in (9). (This is not the Δ that appears in (1).)

One may ask whether the parameter subgroup G may be enlarged by including R-matrices outside the field-free case. If $\triangle \neq 0$ the group G may *not* be so enlarged. However we will show in Theorem 3 that if $\triangle = 0$, then the group G may be enlarged to $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$ by expanding the set of R-matrices to include non-field-free ones. In this *expanded* $\triangle = 0$ *regime* $R(g)$ is not field-free for general g. It is from this enlarged space of Boltzmann weights that we find the solutions to the Yang-Baxter equation that allow us to prove the evaluation of $Z(\mathfrak{S}_\lambda^\Gamma)$ in (1).

The group law on the expanded $\triangle = 0$ regime helps to explain the existence of ice models for Schur polynomials. It makes explicit the relationship between the R-matrices for Gamma ice and Gamma-Gamma ice, and once we have Theorem 3 one may verify the evaluations (1) with a minimum of computation.

The set of Boltzmann weights which we use in Gamma ice and Delta ice (Tables 1 and 2) may be motivated in different ways. They may be derived heuristically from the the requirement that the partition function in (1) when $\lambda = 0$ matches the deformed denominator. Together with the requirement that $\triangle = 0$, this condition essentially determines these weights.

Alternatively, considering the case of Gamma ice, the proof of Theorem 5 requires that (in the notation of Section 1) the weights a_1 and a_2 for Gamma-Gamma ice be in the ratio of $t_j z_i + z_j$ to $t_i z_j + z_i$. This is related to the fact that the deformed Weyl denominator is a product of factors of this type. When the group structure in Theorem 3 is understood, it may be seen that Gamma ice produces weights for Gamma-Gamma ice with this property.

Theorem 3 may be thought of as a parametrized Yang-Baxter equation with a nonabelian parameter group. One may ask in other cases where a set of R-matrices is closed under such a composition law whether they would form a group. We may tentatively say that they would. Thus in Section 2 we give a plausible argument to show that if there is a set of R-matrices such that for any S and T in that set there exists R such that $R_{12}S_{13}T_{23} = T_{23}S_{13}R_{12}$ then an associativity property is satisfied, so that (4) is satisfied. Of course our rigorous results do not depend on this plausible reasoning, but it seems useful to know that the associativity that we observe is not entirely accidental.

We will prove the Yang-Baxter equation for several types of ice. We will, as we have mentioned, give Boltzmann weights of two different types Γ and Δ. Moreover if $X, Y \in \{\Gamma, \Delta\}$ we will give an R-matrix R_{XY} which has the effect of interchanging a strand of X ice with a strand of Y ice; thus in (3), M is of type X and N is of type Y. We will prove that $R_{\Gamma\Gamma}$ and $R_{\Delta\Delta}$ both satisfy the Yang-Baxter equation, and we will prove similar relations that involve all four types of ice R_{XY} in various combinations.

Of the six types of ice that we will consider–Γ, Δ, $R_{\Gamma\Gamma}$, $R_{\Gamma\Delta}$, $R_{\Delta\Gamma}$ and $R_{\Delta\Delta}$, only Γ and $R_{\Gamma\Gamma}$ come from the space of R-matrices parametrized by $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$. It is possible that the remaining types that we consider may also fall within a larger family within the eight-vertex model which can be parametrized by some larger group. We will not consider this question in this paper.

In another direction, Hlavatý [12] has defined the notion of a *Yang-Baxter system*, which like our setup involves six types of ice. We will show that our construction is an example of a Yang-Baxter system. In the case where the t_i are equal, these Yang-Baxter systems are related to those previously found by Nichita and Parashar [29], [28].

Hlavatý's definition has two independent motivations. On the one hand, there is the work of Freidel and Maillet [9] on integrable systems, and on the other hand, there is work of Vladimirov [38] on quantum doubles. Briefly, Drinfeld [5] indicated that one expects to be able to construct a quasitriangular Hopf algebra from solutions of the Yang-Baxter equation. Faddeev, Reshetikhin and Takhtajan [6] gave details of such a construction. Further development was given by Majid [25], and a version for parametrized Yang-Baxter equation was given by Cotta-Ramusino, Lambe and Rinaldi [4]. Vladimirov's work seeks to clarify the relation of these constructions to another construction of Drinfeld [5], namely that of the *quantum double*. Snobl [28] is a development from Vladimirov's work.

Tokuyama's formula expresses what we have denoted $Z(\mathfrak{S}_\lambda^\Gamma)$ as a sum over strict Gelfand-Tsetlin patterns. These are triangular arrays of integers with descending rows that interleave (Section 4). The strict Gelfand-Tsetlin patterns with top row $\lambda+\rho$ are in bijection with states of the model. This connection between states of the ice model and strict Gelfand-Tsetlin patterns has one historical origin in the literature for alternating sign matrices. (An independent historical origin is in the Bethe Ansatz.) The bijection with the set of alternating sign matrices and strict Gelfand-Tsetlin patterns is in Mills, Robbins and Rumsey [27], while the connection with what are recognizably states of the six-vertex model is in Robbins and Rumsey [32]. This connection was used by Kuperberg [18] who gave a second proof (after the purely combinatorial one by Zeilberger [39]) of the conjecture of Mills, Robbins and Rumsey [27]. Kuperberg's paper follows Korepin [17] and Izergin [15] and makes use of the Yang-Baxter equation. It was observed by Okada [30] and Stroganov [34] that the number of $n \times n$ alternating sign matrices, that is, the value of Kuperberg's ice (with particular Boltzmann weights involving cube roots of unity) is a special value of the particular Schur function in $2n$ variables with $\lambda = (n, n, n-1, n-1, \cdots, 1, 1)$ divided by a power of 3. Moreover Stroganov gave a proof using the Yang-Baxter equation. See also Zinn-Justin [40] for a discussion. This occurrence of Schur polynomials in the six-vertex model is different from the one we discuss, since Baxter's parameter $\triangle$ is nonzero for these investigations.

Kuperberg [19] generalizes his work on alternating sign matrix enumeration to other symmetry types using the ice-type models and the Yang-Baxter equation, and Hamel and King [10] relate one of these classes to a symplectic analog of Tokuyama's theorem. Okada [31] enumerates many of the ice types of Kuperberg in terms of dimensions of highest weight representations of classical groups. In [30] he gives deformations of the Weyl character identity analogous to Tokuyama's result with $\lambda = 0$ for other classical groups.

Lascoux [21], [20] gave six-vertex model representations of Schubert and Grothendieck of Lascoux and Schützenberger [22] and related these to the Yang-Baxter equation. Fomin and Kirillov [7], [8] also gave theories of the Schubert and Grothendieck polynomials based on the Yang-Baxter equation. Tsilevich [37] gives an interpretation of Schur polynomials and Hall-Littlewood polynomials in terms of a quantum mechanical system. Zinn-Justin [40] gives an interpretation of Schur polynomials in terms of two-dimensional fermionic systems.

Peter McNamara [26] has clarified that the Lascoux papers are potentially related to ours at least in that the Boltzmann weights [21] belong to the expanded $\triangle = 0$ regime. Moreover, he is able to show based on Lascoux' work how to construct models of the factorial Schur functions of Biedenharn and Louck.

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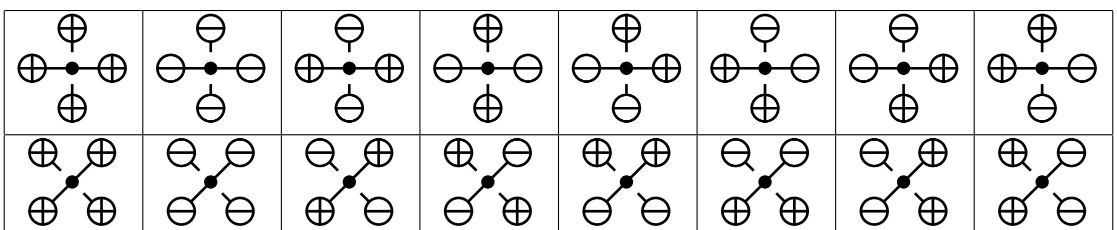
1 The Six-Vertex Model

We review the six-vertex model from statistical mechanics. Let us consider a lattice (or sometimes more general graph) in which the edges are labeled with "spins" $\pm$. Depending on the spins on its adjacent edges, each vertex will be assigned a *Boltzmann weight*.

The Boltzmann weight will be zero unless the number of adjacent edges labeled $-$ is even. Let us denote the possibly nonzero Boltzmann weights as follows:

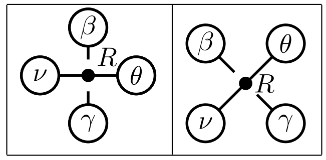


a_1	a_2	b_1	b_2	c_1	c_2	d_1	d_2

We will consider the vertices in two possible orientations, as shown. We will put these Boltzmann weights into a matrix as follows:

$$R = \begin{pmatrix} a_1 & & & d_1 \\ & b_1 & c_1 & \\ & c_2 & b_2 & \\ d_2 & & & a_2 \end{pmatrix} = \begin{pmatrix} a_1(R) & & & d_1(R) \\ & b_1(R) & c_1(R) & \\ & c_2(R) & b_2(R) & \\ d_2(R) & & & a_2(R) \end{pmatrix}. \tag{5}$$

If the edge spins are labeled $\nu, \beta, \gamma, \theta \in \{+, -\}$ as follows:



then we will denote by $R_{\nu\beta}^{\theta\gamma}$ the corresponding Boltzmann weight. Different vertices of the lattice may have different Boltzmann weights, so we label each vertex by the corresponding matrix. Thus $R_{++}^{++} = a_1(R)$, etc.

We think of R as the matrix of an endomorphism of $V \otimes V$, where V is a two-dimensional vector space with basis v_+ and v_-. Write

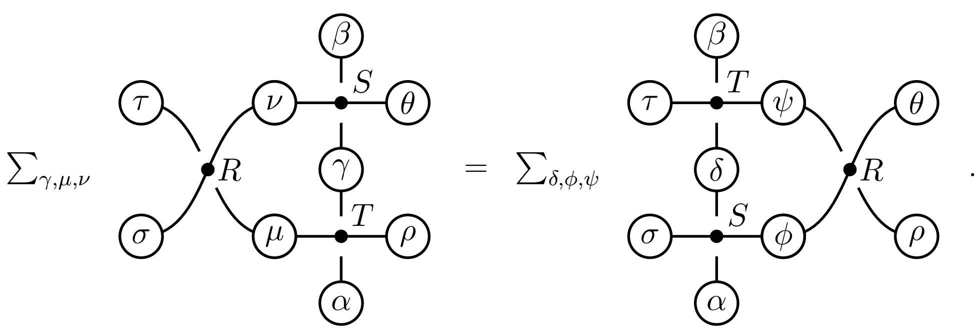
$$R(v_\nu \otimes v_\beta) = \sum_{\theta,\gamma} R_{\nu\beta}^{\theta\gamma} v_\theta \otimes v_\gamma. \tag{6}$$

Then with respect to the basis $v_+ \otimes v_+$, $v_+ \otimes v_-$, $v_- \otimes v_+$, $v_- \otimes v_-$ if $a_1 = a_1(R)$ etc. the matrix R is as in (5).

If ϕ is an endomorphism of $V \otimes V$ we will denote by ϕ_{12}, ϕ_{13} and ϕ_{23} endomorphisms of $V \otimes V \otimes V$ defined as follows. If $\phi = \phi' \otimes \phi''$ where $\phi', \phi'' \in \mathrm{End}(V)$ then $\phi_{12} = \phi' \otimes \phi'' \otimes 1$, $\phi_{13} = \phi' \otimes 1 \otimes \phi''$ and $\phi_{23} = 1 \otimes \phi' \otimes \phi''$. We extend this definition to all ϕ by linearity. Now if ϕ, ψ, χ are three endomorphisms of $V \otimes V$ we define the *Yang-Baxter commutator*

$$[\![\phi, \psi, \chi]\!] = \phi_{12}\psi_{13}\chi_{23} - \chi_{23}\psi_{13}\phi_{12}.$$

Lemma 1 *The vanishing of $[\![R, S, T]\!]$ is equivalent to the star-triangle identity*



(7)

for every fixed combination of spins $\sigma, \tau, \alpha, \beta, \rho, \theta$.

The term *star-triangle identity* was used by Baxter. The meaning of equation (7) will be clarified in the proof.

Proof For fixed $\sigma, \tau, \alpha, \beta, \rho, \theta, \mu, \nu, \gamma$ the value or Boltzmann weight of the left-hand side is just the product of the Boltzmann weights at the three vertices, that is, $R^{\nu\mu}_{\sigma\tau} S^{\theta\gamma}_{\nu\beta} T^{\rho\alpha}_{\mu\gamma}$, and similarly the right-hand side. Hence the meaning of (7) is that for fixed $\sigma, \tau, \alpha, \beta, \rho, \theta$

$$\sum_{\gamma,\mu,\nu} R^{\nu\mu}_{\sigma\tau} S^{\theta\gamma}_{\nu\beta} T^{\rho\alpha}_{\mu\gamma} = \sum_{\delta,\phi,\psi} T^{\psi\delta}_{\tau\beta} S^{\phi\alpha}_{\sigma\delta} R^{\theta\rho}_{\phi\psi}. \tag{8}$$

Now let us apply $[\![R, S, T]\!]$ to the vector $v_\sigma \otimes v_\tau \otimes v_\beta$. On the one hand by (6)

$$\begin{aligned} R_{12}S_{13}T_{23}(v_\sigma \otimes v_\tau \otimes v_\beta) &= R_{12}S_{13} \sum_{\psi,\delta} T^{\psi\delta}_{\tau\beta}(v_\sigma \otimes v_\psi \otimes v_\delta) \\ &= R_{12} \sum_{\psi,\delta,\phi,\alpha} S^{\psi\delta}_{\tau\beta} T^{\phi\alpha}_{\sigma\delta}(v_\phi \otimes v_\psi \otimes v_\alpha) \\ &= \sum_{\psi,\delta,\phi,\alpha,\theta,\rho} T^{\psi\delta}_{\tau\beta} S^{\phi\alpha}_{\sigma\delta} R^{\theta\rho}_{\phi\psi}(v_\theta \otimes v_\rho \otimes v_\alpha), \end{aligned}$$

and similarly

$$S_{23}T_{13}R_{12}(v_\sigma \otimes v_\tau \otimes v_\beta) = \sum_{\nu,\mu.\theta,\gamma,\rho,\alpha} R^{\nu\mu}_{\sigma\tau} S^{\theta\gamma}_{\nu\beta} T^{\rho\alpha}_{\mu\gamma}(v_\theta \otimes v_\rho \otimes v_\alpha).$$

We see that the vanishing of $[\![R, S, T]\!]$ is equivalent to (8). ❄

In this section we will be concerned with the *six-vertex model* in which the weights are chosen so that $d_1 = d_2 = 0$. Baxter [2] in Chapter 9 considered the question of when given S and T a matrix R can be found such that $[\![R, S, T]\!] = 0$. We will slightly generalize his analysis. He considered mainly the *field-free case* where $a_1(R) = a_2(R) = a(R)$, $b_1(R) = b_2(R) = b(R)$ and $c_1(R) = c_2(R) = c(R)$. The condition $c_1(R) = c_2(R) = c(R)$ is easily removed, but with no gain in generality. The other two conditions $a_1(R) = a_2(R) = a(R)$, $b_1(R) = b_2(R) = b(R)$ are more serious restrictions.

In the field-free case, let

$$\triangle(R) = \frac{a(R)^2 + b(R)^2 - c(R)^2}{2a(R)\,b(R)}, \qquad a_1(R) = a_2(R) = a(R), \quad \text{etc}. \tag{9}$$

Then Baxter showed that the condition on S and T for there to exist R such that $[\![R, S, T]\!] = 0$ is that $\triangle(S) = \triangle(T)$.

In the application to statistical physics, phase transitions occur when $\triangle = \pm 1$. If $|\triangle| > 1$ the system is "frozen" in the sense that there are correlations between distant vertices. By contrast $-1 < \triangle < 1$ is the disordered range where no such correlations occur.

Something interesting happens at $\triangle = 0$ in the middle of the disordered range. In order to see this it is better to work with the full six-vertex model without the condition that $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$.

Generalizing this result to the non-field-free case, we find that there are not one but two parameters

$$\begin{aligned} \triangle_1(R) &= \frac{a_1(R)a_2(R) + b_1(R)b_2(R) - c_1(R)c_2(R)}{2a_1(R)b_1(R)}, \\ \triangle_2(R) &= \frac{a_1(R)a_2(R) + b_1(R)b_2(R) - c_1(R)c_2(R)}{2a_2(R)b_2(R)}. \end{aligned}$$

to be considered.

Theorem 1 *Assume that $a_1(S)$, $a_2(S)$, $b_1(S)$, $b_2(S)$, $c_1(S)$, $c_2(S)$, $a_1(T)$, $a_2(T)$, $b_1(T)$, $b_2(T)$, $c_1(T)$ and $c_2(T)$ are nonzero. Then a necessary and sufficient condition for there to exist parameters $a_1(R)$, $a_2(R)$, $b_1(R)$, $b_2(R)$, $c_1(R)$, $c_2(R)$ such that $[\![R, S, T]\!] = 0$ with $c_1(R), c_2(R)$ nonzero is that $\triangle_1(S) = \triangle_1(T)$ and $\triangle_2(S) = \triangle_2(T)$.*

Proof Suppose that $\triangle_1(S) = \triangle_1(T)$ and $\triangle_2(S) = \triangle_2(T)$. Then we may take

$$\begin{aligned} a_1(R) &= \frac{b_2(S)a_1(T)b_1(T) - a_1(S)b_1(T)b_2(T) + a_1(S)c_1(T)c_2(T)}{a_1(T)} \\ &= \frac{a_1(S)b_1(S)a_2(T) - a_1(S)a_2(S)b_1(T) + c_1(S)c_2(S)b_1(T)}{b_1(S)}, \end{aligned} \tag{10}$$

$$\begin{aligned} a_2(R) &= \frac{b_1(S)a_2(T)b_2(T) - a_2(S)b_1(T)b_2(T) + a_2(S)c_1(T)c_2(T)}{a_2(T)} \\ &= \frac{a_2(S)b_2(S)a_1(T) - a_1(S)a_2(S)b_2(T) + c_1(S)c_2(S)b_2(T)}{b_2(S)} \end{aligned} \tag{11}$$

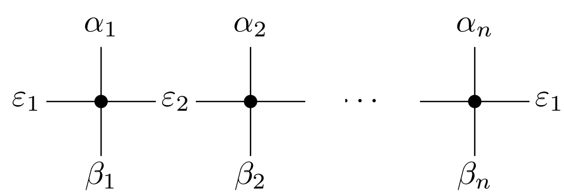
$$b_1(R) = b_1(S)a_2(T) - a_2(S)b_1(T), \qquad b_2(R) = b_2(S)a_1(T) - a_1(S)b_2(T), \tag{12}$$

$$c_1(R) = c_1(S)c_2(T), \qquad c_2(R) = c_2(S)c_1(T). \tag{13}$$

Using $\triangle_1(S) = \triangle_1(T)$ and $\triangle_2(S) = \triangle_2(T)$ it is easy to that the two expressions for $a_1(R)$ agree, and similarly for $a_2(R)$. One may check that $[\![R, S, T]\!] = 0$. On the other hand, it may be checked that the relations required by $[\![R,S,T]\!]=0$ are contradictory unless $\triangle_1(S)=\triangle_1(T)$. ❄

In the field-free case, these two relations reduce to a single one, $\triangle(S)=\triangle(T)$, and it is remarkable that $\triangle(R)$ has the same value: $\triangle(R)=\triangle(S)=\triangle(T)$.

To see what this means, we recall the *row-transfer matrices* that are the subject of Baxter's application of the star-triangle relation. Given Boltzmann weights $a_1(R), a_2(R), \cdots$, we associate a matrix $2^n \times 2^n$ matrix $V(R)$. The entries in this matrix are indexed by pairs $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\beta=(\beta_1,\cdots,\beta_n)$, where $\alpha_i,\beta_i\in\{\pm\}$. If $\varepsilon_1,\cdots,\varepsilon_n\in\{\pm\}$ we may consider the Boltzmann weight of the configuration:



Here $\varepsilon_{n+1}=\varepsilon_1$, so the boundary conditions are periodic. The coefficient $V(R)_{\alpha,\beta}$ is then the "partition function" for this one-row configuration, that is, the sum over possible states (assignments of the ε_i).

It follows from Baxter's argument that if R can be found such that $[\![R,S,T]\!]=0$ then $V(S)$ and $V(T)$ commute, and can be simultaneously diagonalized. We will not review Baxter's argument here, but variants of it with non-periodic boundary conditions will appear later in this paper.

In the field-free case when $[\![R,S,T]\!]=0$, $V(R)$ belongs to the same commuting family as $V(S)$ and $V(T)$. This gives a great simplification of the analysis in Chapter 9 of Baxter [2] over the analysis in Chapter 8 using different methods based on the Bethe Ansatz.

In the non-field-free case, however, the situation is different. If $\triangle_1(S)=\triangle_1(T)$ and $\triangle_2(S)=\triangle_2(T)$ then by Theorem 1 there exists R such that $[\![R,S,T]\!]=0$, and so one may use Baxter's method to prove the commutativity of $V(S)$ and $V(T)$. However $\triangle_1(R)$ and $\triangle_2(R)$ are not necessarily the same as $\triangle_1(S)=\triangle_1(T)$ and $\triangle_2(S)=\triangle_2(T)$ and so $V(R)$ may not commute with $V(S)$ and $V(T)$.

In addition to the field-free case, however, there is *another* case where $V(R)$ necessarily does commute with $V(S)$ and $V(T)$, and it is that case which we turn to next. This is the case where $a_1a_2+b_1b_2-c_1c_2=0$. In this case, with $a_1=a_1(R)$, etc., we define

$$\pi(R) = \pi \begin{pmatrix} a_1 & & & \\ & b_1 & c_1 & \\ & c_2 & b_2 & \\ & & & a_2 \end{pmatrix} = \begin{pmatrix} c_1 & & & \\ & a_1 & b_2 & \\ & -b_1 & a_2 & \\ & & & c_2 \end{pmatrix}. \tag{14}$$

Theorem 2 *Suppose that*

$$a_1(S)a_2(S) + b_1(S)b_2(S) - c_1(S)c_2(S) = a_1(T)a_2(T) + b_1(T)b_2(T) - c_1(T)c_2(T) = 0. \tag{15}$$

Then $[\![R, S, T]\!] = 0$ *where* $\pi(R) = \pi(S)\,\pi(T)^{-1}$*. We have*

$$a_1(R)\,a_2(R) + b_1(R)\,b_2(R) - c_1(R)\,c_2(R) = 0. \tag{16}$$

Proof The matrix R will not be the matrix in Theorem 1, but will rather be a constant multiple of it. We have

$$\pi(T)^{-1} = \frac{1}{D} \begin{pmatrix} c_2(T) & & & \\ & a_2(T) & -b_2(T) & \\ & b_1(T) & a_1(T) & \\ & & & c_1(T) \end{pmatrix}$$

where $D = a_1(T)a_2(T) + b_1(T)b_2(T) = c_1(T)c_2(T)$. With notation as in Theorem 1, using (15) equations (10) and (11) may be written

$$\begin{aligned} a_1(R) &= a_1(S)a_2(T) + b_2(S)b_1(T), \\ a_2(R) &= a_2(S)a_1(T) + b_1(S)b_2(T). \end{aligned}$$

Combined with (12) and (13) these imply that $\pi(R) = \pi(S)\,D\pi(T)^{-1}$. However we are free to multiply R by a constant without changing the validity of $[\![R, S, T]\!] = 0$, so we divide it by D. ❄

We started with S and T and produced R such that $[\![R, S, T]\!] = 0$ as a function of these because this is the construction that is motivated by Baxter's method of proving that the transfer matrices associated with S and T commute. However it is perhaps more elegant to start with R and T and produce S as a function of these. Thus let $\mathcal{R}$ be the set of endomorphisms R of $V \otimes V$ of the form (5) where $a_1a_2 + b_1b_2 = c_1c_2$. Let $\mathcal{R}^*$ be the subset consisting of such R such that $c_1c_2 \neq 0$.

Theorem 3 *There exists a composition law on* $\mathcal{R}^*$ *such that if* $R, T \in \mathcal{R}^*$*, and if* $S = R \circ T$ *is the composition then* $[\![R, S, T]\!] = 0$*. This composition law is determined by the condition that* $\pi(S) = \pi(R)\pi(T)$ *where* $\pi : \mathcal{R}^* \longrightarrow \mathrm{GL}(4, \mathbb{C})$ *is the map (14). Then* $\mathcal{R}^*$ *is a group, isomorphic to* $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$*.*

Proof This is a formal consequence of Theorem 2. ❄

2 Composition of R-matrices

Theorem 3, defining a group structure on a set of R-matrices, may be regarded as a non-abelian parametrized Yang-Baxter equation. The fact that the composition law on R-matrices that makes S the product of R and T when $[\![R, S, T]\!] = 0$ is associative seems worthy of appreciation. This section (which is not needed for the sequel) contains some heuristic reasoning to explain this associativity.

Let us assume that we are given a vector space V over a field F and a subset $\mathcal{R}$ of $\mathrm{End}(V \otimes V)$ which is homogeneous in the sense that if $0 \neq R \in \mathcal{R}$ then $\mathcal{R}$ contains the entire ray FR. Let $\mathbb{P}(\mathcal{R})$ be the set of such rays.

Let us assume that if R, T are nonzero elements of $\mathcal{R}$ then there is another $S \in \mathcal{R}$ that is unique up to scalar multiple such that $[\![R, S, T]\!] = 0$. We reiterate that there might be such an S that would be useable for applications but that it might not lie in the same space $\mathcal{R}$, and indeed this is the usual situation for the six vertex model with weights that are not field-free. But with this assumption, $(R, T) \mapsto S$ is a well-defined composition law on $\mathbb{P}(\mathcal{R})$. Let us denote this composition $S = R \circ T$. We will give a plausible argument that this composition law should be associative.

We begin with three nonzero elements R, S, T of $\mathcal{R}$. We will compare endomorphisms of $V \otimes V \otimes V \otimes V$. In addition to identities such as $R_{12}(R \circ S)_{13} S_{23} = S_{23}(R \circ S)_{13} R_{12}$ we will use identities such as $R_{13} T_{24} = T_{24} R_{13}$ which are true for arbitrary endomorphisms of $V \otimes V$. First, we have

$$\begin{aligned}
S_{23}(S \circ T)_{24}(R \circ S)_{13}(R \circ (S \circ T))_{14} T_{34} R_{12} &= \\
S_{23}(R \circ S)_{13}(S \circ T)_{24}(R \circ (S \circ T))_{14} R_{12} T_{34} &= \\
S_{23}(R \circ S)_{13} R_{12}(R \circ (S \circ T))_{14}(S \circ T)_{24} T_{34} &= \\
R_{12}(R \circ S)_{13} S_{23}(R \circ (S \circ T))_{14}(S \circ T)_{24} T_{34} &= \\
R_{12}(R \circ S)_{13}(R \circ (S \circ T))_{14} S_{23}(S \circ T)_{24} T_{34} &= \\
R_{12}(R \circ S)_{13}(R \circ (S \circ T))_{14} T_{34}(S \circ T)_{24} S_{23}. &
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
S_{23}(S\circ T)_{24}T_{34}(R\circ(S\circ T))_{14}(R\circ S)_{13}R_{12} &= \\
T_{34}(S\circ T)_{24}S_{23}(R\circ(S\circ T))_{14}(R\circ S)_{13}R_{12} &= \\
T_{34}(S\circ T)_{24}(R\circ(S\circ T))_{14}S_{23}(R\circ S)_{13}R_{12} &= \\
T_{34}(S\circ T)_{24}(R\circ(S\circ T))_{14}R_{12}(R\circ S)_{13}S_{23} &= \\
T_{34}R_{12}(R\circ(S\circ T))_{14}(S\circ T)_{24}(R\circ S)_{13}S_{23} &= \\
R_{12}T_{34}(R\circ(S\circ T))_{14}(R\circ S)_{13}(S\circ T)_{24}S_{23}.
\end{aligned}$$

Now consider an endomorphism X of $V\otimes V\otimes V$ that is constrained to satisfy

$$S_{23}(S\circ T)_{24}X_{134}R_{12}=R_{12}X_{134}(S\circ T)_{24}S_{23}.$$

This is a linear equation in the matrix coefficients of X in which the number of conditions exceeds the number of variable. It is reasonable to assume that if this has a nonzero solution that solution is determined up to constant multiple. Therefore (up to constant) we have

$$(R\circ S)_{13}(R\circ(S\circ T))_{14}T_{34}=T_{34}(R\circ(S\circ T))_{14}(R\circ S)_{13}.$$

Taking the determinant shows that the constant must be a root of unity. If $F=\mathbb{R}$ or $\mathbb{C}$ and $\mathbb{P}(\mathcal{R})$ is connected, then by continuity this constant must be 1. This means that $(R\circ(S\circ T))$ satisfies the definition of $(R\circ S)\circ T$, so at least plausibly, a composition law defined this way should be associative.

3 Gamma ice

Let $z_1,\cdots,z_n$ and $t_1,\cdots,t_n$ be complex numbers with $z_i\neq 0$. We will refer to the z_i as *spectral parameters* and the t_i as *deformation parameters* since these are the roles these variables will play when we turn to Tokuyama's theorem. Denote

$$\Gamma(i)=\begin{pmatrix}1&&&\\&t_i&z_i(t_i+1)&\\&1&z_i&\\&&&z_i\end{pmatrix},\qquad \pi_\Gamma(i)=\begin{pmatrix}z_i(t_i+1)&&&\\&1&z_i&\\&-t_i&z_i&\\&&&1\end{pmatrix}.$$

Let $\pi_{\Gamma\Gamma}(i,j)=\text{const}\times\pi_\Gamma(i)\pi_\Gamma(j)^{-1}$, where it is convenient to take the constant to be $z_j(t_j+1)$. It follows from Theorem 2 that

$$[\![R_{\Gamma\Gamma}(i,j),\Gamma(i),\Gamma(j)]\!]=0 \tag{17}$$

where $R_{\Gamma\Gamma}(i,j)$ is related to $\pi_{\Gamma\Gamma}(i,j)$ by the relation (14). Concretely,

$$R_{\Gamma\Gamma} = \begin{pmatrix} z_j + t_j z_i & & & \\ & t_i z_j - t_j z_i & z_i(t_i+1) & \\ & z_j(t_j+1) & z_i - z_j & \\ & & & z_i + t_i z_j \end{pmatrix}. \quad (18)$$

The Boltzmann weights are given in Table 1.

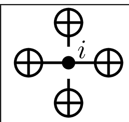
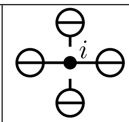
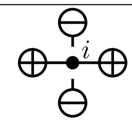
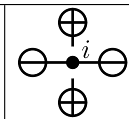
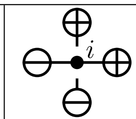
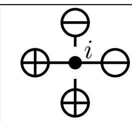
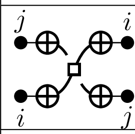
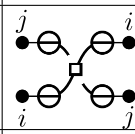
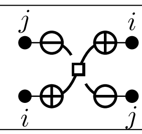
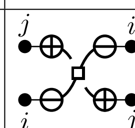
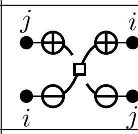
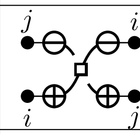
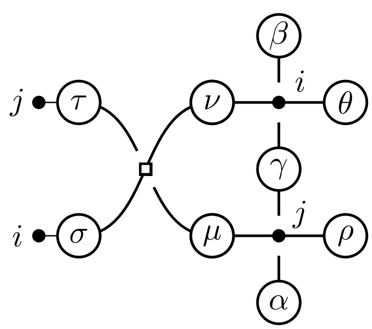
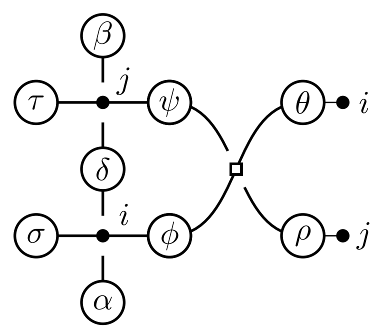
Gamma Ice						
Boltzmann weight	1	z_i	t_i	z_i	$z_i(t_i+1)$	1
Gamma-Gamma R-ice						
Boltzmann weight	$t_j z_i + z_j$	$t_i z_j + z_i$	$t_i z_j - t_j z_i$	$z_i - z_j$	$(t_i+1)z_i$	$(t_j+1)z_j$

Table 1: Boltzmann weights for Gamma ice and Gamma-Gamma ice.

Theorem 4 *The star-triangle identity*

$\sum_{\gamma,\mu,\nu}$ 

$=$ $\sum_{\delta,\phi,\psi}$ 

is valid with Boltzmann weights as in Table 1.

Proof This follows from Theorem 2 since $\pi_{\Gamma\Gamma}(i,j) = \text{const} \times \pi_\Gamma(i)\pi_\Gamma(j)^{-1}$. ❄

We will use Gamma ice to represent Schur polynomials, which are essentially the characters of finite-dimensional irreducible representations of $\mathrm{GL}_n(\mathbb{C})$. If $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}^n$ then we may regard μ as an element of the $\mathrm{GL}_n(\mathbb{C})$ weight lattice and call it a *weight*. If $\mu_1 \geqslant \cdots \geqslant \mu_n$ we say it is *dominant*, and if $\mu_1 > \cdots > \mu_n$ we say it is *strictly dominant*. If μ is dominant and $\mu_n \geqslant 0$, it is a *partition*.

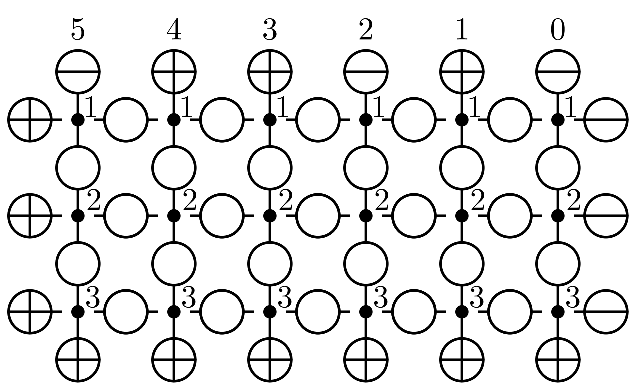
Note: The word "partition" occurs in two different senses in this paper. The partition function in statistical physics is different from partitions in the combinatorial sense. So for us a reference to a "partition" without "function" really means a partition. Another potentially ambiguous usage is that "weight" will sometimes refer to an element of the GL_n weight lattice, which we identify with $\mathbb{Z}^n$. Therefore if we mean Boltzmann weight, we will not omit Boltzmann.

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a fixed partition. We will denote $\rho = (n-1, n-2, \cdots, 0)$. We will consider a rectangular grid with n rows and $\lambda_1 + n$ columns. We will number the columns of the lattice in descending order from $\lambda_1 + n - 1$ to 0.

A *state* of the model will consist of an assignment of "spins" $\pm$ to every edge. We will also assign labels to the vertices themselves, which will be integers between 1 and n. For Gamma ice the vertices in the i-th row will have the label i. The spins of the boundary edges are prescribed as follows.

Boundary Conditions determined by λ. *On the left and bottom boundary edges, we put $+$; on the right edges we put $-$. On the top, we put $-$ at every column labeled $\lambda_i + n - i$ $(1 \leqslant i \leqslant n)$, that is, for the columns labeled with values in $\lambda + \rho$. Top edges not labeled by $\lambda_i + n - i$ for any i are given spin $+$.*

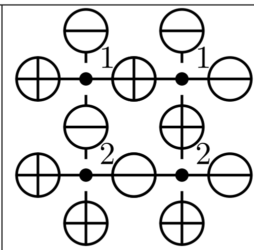
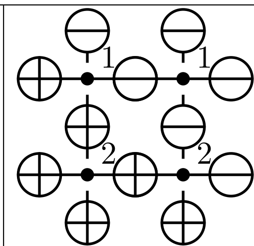
For example, suppose that $n = 3$ and $\lambda = (3, 1, 0)$, so that $\lambda + \rho = (5, 2, 0)$. Then the spins on the boundary are as follows.



(19)

The column labels are written at the top, and the vertex labels are written next to each vertex. The edge spins are marked inside circles. We have left the edge spins on the interior of the domain blank, since the boundary conditions only prescribe the spins we have written. The interior spins are not entirely arbitrary, since we require that at every vertex • the configuration of spins adjacent to the vertex be one of the six listed in Table 1 below under "Gamma ice."

Let $\mathfrak{S}_\lambda^\Gamma$ be the *Gamma ensemble determined by* λ, by which we mean the set of all such configurations, with the prescribed boundary conditions. If $x \in \mathfrak{S}_\lambda^\Gamma$, we assign a value $w(x)$ called the *Boltzmann weight*. Indeed, Table 1 assigns a Boltzmann weight to every vertex, and $w(x)$ is just the product over all the vertices of these Boltzmann weights. The *partition function* $Z(\mathfrak{S})$ of an ensemble $\mathfrak{S}$ is $\sum_{x\in\mathfrak{S}} w(x)$. As an example, suppose that $n = 2$ and $l = (0, 0)$ so $\lambda + \rho = (1, 0)$. In this case $\mathfrak{S}_\lambda^\Gamma$ has cardinality two, and $Z(\mathfrak{S}_\lambda^\Gamma) = t_1 z_2 + z_1$. The states are:

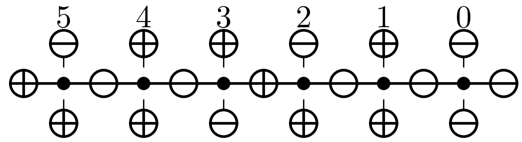
state		
Boltzmann weight	$t_1 z_2$	z_1

The partition function will be evaluated later in this paper using the star-triangle relation.

We will encounter more general partition functions involving different arrangements of vertices. We may consider a planar graph in which each vertex has four adjacent edges. Each edge along the boundary of the graph is assigned signs in some fixed way, and each vertex is assigned an R-matrix from a six-vertex model, or later, an eight-vertex model. The states of the system depend on the choice of spins to label the interior edges of the graph. Given such a state, the Boltzmann weight of the configuration may be computed by multiplying the weights over all vertices. These may then be summed over all possible states, and this is the partition function. For example, in the star-triangle identity (7) either side of the equation is the partition function for such a configuration involving exactly three vertices.

4 Gelfand-Tsetlin patterns

Let us momentarily consider a Gamma ice with just one layer of vertices, so there are two rows of edges, top and bottom. Let $\alpha_1, \cdots, \alpha_m$ be the locations (from left to right) of $-$ in the top row of edges with this labeling, and let $\beta_1, \cdots, \beta_{m'}$ be the locations of $-$ in the bottom row of edges. For example, in this ice:

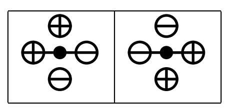


we have $m = 3$, $m' = 2$, $(\alpha_1, \alpha_2, \alpha_3) = (5, 2, 0)$ and $(\beta_1, \beta_2) = (3, 0)$. Since the columns are labeled in decreasing order, we have $\alpha_1 > \alpha_2 > \cdots$ and $\beta_1 > \beta_2 > \cdots$.

Lemma 2 *Suppose that the spin at the left edge is* $+$*. Then we have* $m = m'$ *or* $m' + 1$ *and* $\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \ldots$ *. If* $m = m'$ *then the spin at the right edge is* $+$*, while if* $m = m' + 1$ *it is* $-$*.*

We express the condition that $\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \ldots$ by saying that the sequences $\alpha_1, \alpha_2, \cdots$ and $\beta_1, \beta_2, \cdots$ *interleave*. This Lemma is essentially the *line-conservation* principle in Baxter [2], Section 8.3.

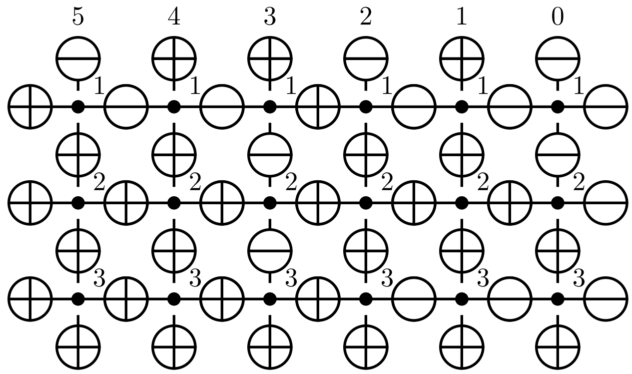
Proof The spins in the middle row are determined by those in the top and bottom rows and the left-most spin in the middle row, which is $+$, since the edges at each vertex have an even number of $+$ spins. If the rows do not interleave then one of the illegal configurations



will occur. Thus $\alpha_1 \geqslant \beta_1$ since if not, the vertex in the β_1 column would be surrounded by spins in the first illegal configuration. Now $\beta_1 \geqslant \alpha_2$ since otherwise the vertex in the α_2 column would be surrounded by spins in the first above illegal configuration, and so forth. The last statement is a consequence of the observation that the total number of spins must be even. ❄

We recall that a *Gelfand-Tsetlin pattern* is a triangular array of dominant weights, in which each row has length one less than the one above it, and the rows interleave. The pattern is called *strict* if the rows are strictly dominant.

It follows from Lemma 2 that taking the locations of $-$ in the rows of vertical edges gives a sequence of strictly dominant weights forming a strict Gelfand-Tsetlin pattern. For example, given the state



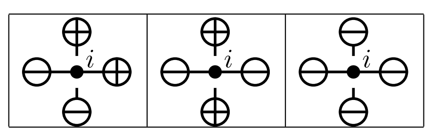
the pattern is

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} 5 & & 2 & & 0 \\ & 3 & & 0 & \\ & & 3 & & \end{array} \right\}. \tag{20}$$

It is not hard to see that this gives a bijection between strict Gelfand-Tsetlin patterns and states with boundary conditions determined by λ. Let us say that the *weight* of a state is $(\mu_1, \cdots, \mu_n)$ if the Boltzmann weight is the monomial $\boldsymbol{z}^\mu = \prod z_i^{\mu_i}$ times a polynomial in t_i. If $\mathfrak{T}$ is a Gelfand-Tsetlin pattern, let $d_k(\mathfrak{T})$ be the sum of the k-th row. We let $d_{n+1}(\mathfrak{T}) = 0$.

Lemma 3 *If $\mathfrak{T}$ is the Gelfand-Tsetlin pattern corresponding to a state of weight μ, then $\mu_k = d_k(\mathfrak{T}) - d_{k+1}(\mathfrak{T})$.*

Proof From Table 1, μ_k is the number of vertices in the k-th row that have an edge configuration of one of the three forms:



Let α_i and β_i be, respectively, the numbers of the columns where the top edge spin or the bottom edge spin of the vertex in the k-th row and i-column is $-$ (with columns numbered in descending order, as always). By Lemma 2 we have $\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{n+1-k}$. It is easy to see that the vertex in the i-column has one of the above configurations if and only if its column number i satisfies $\alpha_j > i \geqslant \beta_j$ for some j. Therefore the number of such i is $\sum \alpha_j - \sum \beta_j = d_k(\mathfrak{T}) - d_{k+1}(\mathfrak{T})$. ❄

5 Evaluation of Gamma Ice

In this section we will prove

Theorem 5 *Let* $\lambda = (\lambda_1, \cdots, \lambda_n)$ *be a partition. Then*

$$Z(\mathfrak{S}_\lambda^\Gamma) = \prod_{i<j} (t_i z_j + z_i) s_\lambda(z_1, \cdots, z_n).$$

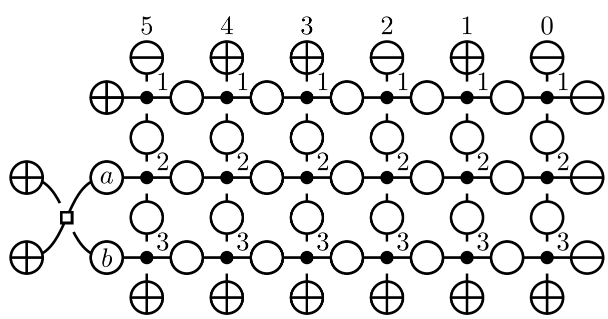
To begin with, define

$$s_\lambda^\Gamma(z_1, \cdots, z_n; t_1, \cdots, t_n) = \frac{Z(\mathfrak{S}_\lambda^\Gamma)}{\prod_{i<j}(t_i z_j + z_i)}. \tag{21}$$

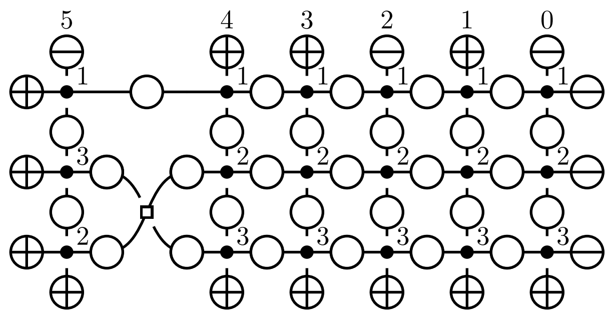
We will eventually show that s_λ^Γ is the Schur polynomial s_λ. But *a priori* it is not obvious from this definition that s_λ^Γ is symmetric, nor that it is a polynomial, nor that it is independent of t.

Lemma 4 *The expression* $(t_{k+1} z_k + z_{k+1}) Z(\mathfrak{S}_\lambda^\Gamma)$ *is invariant under the interchange of the spectral parameters:* $(z_k, t_k) \longleftrightarrow (z_{k+1}, t_{k+1})$.

Proof We modify the ice by adding a Gamma-Gamma R-vertex (that is, one of the vertices from the bottom row in Table 1) to the left of the k and $k+1$ rows. Thus (19) becomes (with $k = 2$ for illustrative purposes)

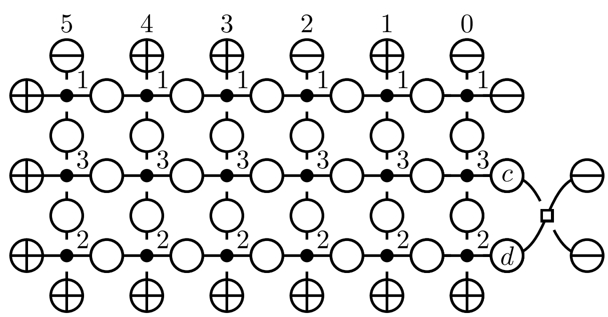


which is a new boundary value problem. The only legal values for a and b are $+$, so every state of this problem determines a unique state of the original problem, and the partition function for this state is the original partition function multiplied by the Boltzmann weight of the R-vertex, which is $t_{k+1}z_k + z_{k+1}$. Now we apply the star-triangle identity, and obtain equality with the the following configuration.



Thus if $\mathfrak{S}'$ denotes this ensemble the partition function $Z(\mathfrak{S}') = (t_{k+1}z_k+z_{k+1})Z(\mathfrak{S}^\Gamma_\lambda)$.

Repeatedly applying the star-triangle identity, we eventually obtain the configuration in which the R-vertex is moved entirely to the right.



Now there is only one legal configuration for the R-vertex, so $c = d = -$. The Boltzmann weight at the R-vertex is therefore $t_k z_{k+1} + z_k$. Note that (z_k, t_k) and (z_{k+1}, t_{k+1}) have been interchanged. This proves that $(t_{k+1}z_k + z_{k+1})Z(\mathfrak{S}^\Gamma_\lambda)$ is unchanged by switching (z_k, t_k) and (z_{k+1}, t_{k+1}). ❄

Proposition 1 s^Γ_λ *is a symmetric polynomial in* $z_1, \cdots, z_n$*, and is independent of the* t_i.

Proof Consider

$$\prod_{i<j}(t_j z_i + z_j)Z(\mathfrak{S}^\Gamma_\lambda). \tag{22}$$

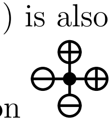
We will show that this is invariant under the interchange $k \leftrightarrow k+1$. This means that we interchange both z_k with z_{k+1} and t_k with t_{k+1}. Indeed, we may write (22) as $(t_{k+1}z_k + z_{k+1})Z(\mathfrak{S}^\Gamma_\lambda)$ times the product of all factors $t_j z_i + z_j$ with $i < j$ *except*

$(i,j) = (k,k+1)$. These factors are permuted by $k \leftrightarrow k+1$, so the statement follows from Lemma 4. Thus (22) is invariant under permutations of the indices, where it is understood that the same permutation is applied to the t_i as to the z_i. Now (22) equals $\prod_{i\neq j}(t_j z_i + z_j)\, s_\lambda^\Gamma(z_1, \cdots, z_n)$, so it follows that s_λ^Γ is also invariant under such permutations. Moreover, (22) is divisible by each $t_j z_i + z_j$ with $i < j$ in the unique factorization ring $\mathbb{C}[z_1, \cdots, z_n, t_1, \cdots, t_n]$. The symmetry property implies that it is also divisible by $t_i z_j + z_i$ with $i < j$, and since these are coprime to $\prod_{i<j}(t_j z_i + z_j)$ it follows that $Z(\mathfrak{S}_\lambda^\Gamma)$ is divisible by these. Therefore s_λ^Γ is a polynomial in $\mathbb{C}[z_1, \cdots, z_n, t_1, \cdots, t_n]$.

It remains to be seen that it is independent of t. In

$$s_\lambda^\Gamma = \frac{Z(\mathfrak{S}_\lambda^\Gamma)}{\prod_{i<j}(t_i z_j + z_i)},$$

we regard the numerator and the denominator as both being elements of $R[t_i]$ where $R = \mathbb{C}[z_1, \cdots, z_n, t_j (j \neq i)]$. From what we have shown, s_λ^Γ is a polynomial. We claim that both the numerator and denominator have the same degree $i-1$ in t_i. For the denominator, this is clear. For the numerator, the number of $-$ in the top row of vertical edge spins is n by the boundary conditions, and it follows from Lemma 2 that each successive row has one fewer $-$. This means that there are $i-1$ vertices labeled i such that the spin on the edge below it is $-$, and from Table 1, it follows that the number of Boltzmann weights equal to $z_i(t_i+1)$ or t_i in any particular state is $\leqslant i-1$. The degree of the numerator is thus $\leqslant i-1$ and since the degree of the denominator is $i-1$, and the quotient is a polynomial, both numerator and denominator must have degree $i-1$ in t_i. Thus the quotient has degree zero, and does not involve t_i.

We may now conclude the proof of Theorem 5 by showing that $s_\lambda^\Gamma = s_\lambda$. Since s_λ^Γ is independent of t_i, we may take all $t_i = -1$. Now in (21) the denominator becomes $\prod_{i<j}(z_i - z_j)$. Since this is skew-symmetric under permutations, the numerator $Z(\mathfrak{S}_\lambda^\Gamma)$ is also skew-symmetric. With $t_i = -1$ any state containing a vertex in configuration  has Boltzmann weight 0, so we are limited to states omitting this configuration. In view of the bijection between states and strict Gelfand-Tsetlin patterns, this means that the corresponding Gelfand-Tsetlin pattern $\mathfrak{T}$ has the property that every entry from any row but the first is equal to one of the two entries directly above it. It is easy to see that the weight μ of such a coefficient, described by Lemma 3, is a permutation σ of the top row of $\mathfrak{T}$, that is, of $\lambda + \rho$. These weights are all distinct since $\lambda + \rho$ is strongly dominant, i.e. without repeated entries. Since it is skew-symmetric, its value is $\mathrm{sgn}(\sigma)$ times a constant times $\prod z_j^{\mu_j} = z_j^{\rho_{\sigma(j)}+\lambda_{\sigma(j)}}$. To determine the constant, we may take the state whose Gelfand-Tsetlin pattern is

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} \lambda_1+\rho_1 & & \lambda_2+\rho_2 & & \cdots & & \lambda_n \\ & \lambda_2+\rho_2 & & & & \lambda_n & \\ & & \ddots & & \iddots & & \\ & & & \lambda_n & & & \end{array} \right\}.$$

This has weight $\prod z_j^{\lambda_j+\rho_j}$ and so

$$s_\lambda^\Gamma(z_1,\cdots,z_n) = \frac{\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\prod z_j^{\rho_{\sigma(j)}+\lambda_{\sigma(j)}}}{\prod_{i<j}(z_j-z_i)}$$

which equals $s_\lambda(z_1,\cdots,z_n)$ by the Weyl character formula.

6 Tokuyama's theorem

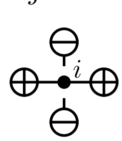
We recall some definitions from Tokuyama [36]. An entry of a Gelfand-Tsetlin pattern (not in the top row) is classified as *left-leaning* if it equals the entry above it and to the left. It is *right-leaning* if it equals the entry above it and to the right. It is *special* if it is neither left- nor right-leaning. Thus in (20), the 3 in the bottom row is left-leaning, the 0 in the second row is right-leaning and the 3 in the middle row is special. If $\mathfrak{T}$ is a Gelfand-Tsetlin pattern, let $l(\mathfrak{T})$ be the number of left-leaning entries. Let $d_k(\mathfrak{T})$ be the sum of the k-th row of $\mathfrak{T}$, and $d_{n+1}(\mathfrak{T})=0$.

Theorem 6 (Tokuyama) *We have*

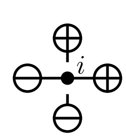
$$\sum_{\mathfrak{T}}\left(\prod_{k=1}^n z_k^{d_k(\mathfrak{T})-d_{k+1}(\mathfrak{T})}\right)t^{l(\mathfrak{T})}(t+1)^{s(\mathfrak{T})} = \prod_{i<j}(z_i+tz_j)s_\lambda(z_1,\cdots,z_n),$$

where the sum is over all strict Gelfand-Tsetlin patterns with top row $\lambda+\rho$.

Proof If $\mathfrak{T}$ corresponds to a state of the Gamma ice with boundary conditions determined by λ, then we will show that the Boltzmann weight of the state is the term on the left-hand side. From Lemma 3 the powers of z are correct. It is easy to see that if the k-th row of $\mathfrak{T}$ is left leaning (respectively special), and that value is j, then the entry in the j-column and the k-th row of the ice is



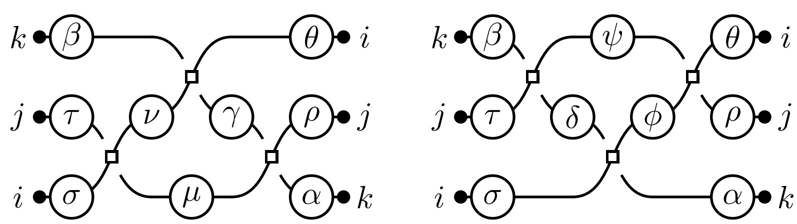
respectively



so from Table 1, it follows that the powers of t_i are also correct. The statement now follows from Theorem 5. ❄

7 The Yang-Baxter equation for Gamma-Gamma ice

We will prove a star-triangle relation that only involves Gamma-Gamma ice. Let us think of Gamma ice as being organized into strands of horizontal edges, with every Gamma vertex of the strand having the same label i. We may think of Gamma-Gamma ice as a tool that switches two strands. The following result states that this tool respects the braid relation. We have drawn this picture differently from that in Theorem 4 since this Yang-Baxter equation involves only horizontal edges, while that in Theorem 4 involves both horizontal and vertical edges.



With $\sigma, \tau, \beta, \alpha, \rho, \theta$ fixed, we may regard these two configurations as ensembles each involving three Gamma-Gamma vertices. The Yang-Baxter equation says that they have the same partition function.

Theorem 7 *The Yang-Baxter equation is true in the form*

$$\sum_{\mu,\nu,\gamma} R(j,k)^{\rho\alpha}_{\mu\gamma} R(i,k)^{\theta\gamma}_{\nu\beta} R(i,j)^{\nu\mu}_{\sigma\tau} = \sum_{\delta,\phi,\psi} R(j,k)^{\psi\delta}_{\tau\beta} R(i,k)^{\phi\alpha}_{\sigma\delta} R(i,j)^{\theta\rho}_{\phi\psi},$$

with $R = R_{\Gamma\Gamma}$.

Proof This follows from Theorem 3 since $\pi_{\Gamma\Gamma}(i,j) = \text{const} \times \pi_\Gamma(i)\pi_\Gamma(j)^{-1}$, so

$$\pi_{\Gamma\Gamma}(i,j)\pi_{\Gamma\Gamma}(j,k) = \text{const} \times \pi_{\Gamma\Gamma}(i,k).$$

❄

8 More Star-Triangle Relations

There are further star-triangle relations which go outside the six-vertex model. We find that the discussion in Section 1 can be extended the set of Boltzmann weights in the eight vertex model that has either $a_1a_2 + b_1b_2 - c_1c_2 = 0$ and $d_1 = d_2 = 0$ or $a_1a_2 + b_1b_2 - d_1d_2 = 0$ and $c_1 = c_2 = 0$. The parameter subgroup will have the $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ of Theorem 3 as a subgroup of index two. Let $\widehat{\mathcal{R}}^*$ be the set of R as in (5) with such weights, where it is assumed $a_1a_2 + b_1b_2 \neq 0$.

Theorem 8 *There exists a composition law on* $\hat{\mathcal{R}}^*$ *such that if* $R, T \in \hat{\mathcal{R}}^*$, *and if* $S = R \circ T$ *is the composition then* $[\![R, S, T]\!] = 0$. *This composition law is determined by the condition that* $\pi(S) = \pi(R)\pi(T)$ *where* $\pi : \hat{\mathcal{R}}^* \longrightarrow \mathrm{GL}(4,\mathbb{C})$ *is the map defined by (14) if* c_1, c_2 *are nonzero, and by*

$$\pi(R) = \pi \begin{pmatrix} a_1 & & & d_1 \\ & b_1 & & \\ & & b_2 & \\ d_2 & & & a_2 \end{pmatrix} = \begin{pmatrix} & & & d_1 \\ & ia_2 & -ib_1 & \\ & ib_2 & ia_1 & \\ d_2 & & & \end{pmatrix}$$

if d_1, d_2 *are nonzero.*

Here $i = \sqrt{-1}$.

Proof Let us call $R \in \hat{\mathcal{R}}^*$ of *Type* C if c_1, c_2 are nonzero (so $d_1 = d_2 = 0$) and of *Type* D in the other case. There are four cases to consider. One, where R and T are both of type C, is already in Theorem 3. In the other three cases, we compute $[\![R, S, T]\!] = 0$ with S as follows.

If R is of type C and T is of type D then S is of type D with

$$\begin{aligned}
a_1(S) &= a_2(R)a_1(T) + b_1(R)b_1(T), \\
a_2(S) &= a_1(R)a_2(T) + b_2(R)b_2(T), \\
b_1(S) &= -b_2(R)a_1(T) + a_1(R)b_1(T), \\
b_2(S) &= -b_1(R)a_2(T) + a_2(R)b_2(T), \\
d_1(S) &= c_1(R)d_1(T), \\
d_2(S) &= c_2(R)d_1(T).
\end{aligned}$$

If R is of type D and T is of type C then S is of type D with

$$\begin{aligned}
a_1(S) &= a_1(R)a_2(T) + b_2(R)b_2(T), \\
a_2(S) &= a_2(R)a_1(T) + b_1(R)b_1(T), \\
b_1(S) &= b_1(R)a_1(T) - a_2(R)b_2(T), \\
b_2(S) &= b_2(R)a_1(T) - a_1(R)b_1(T), \\
d_1(S) &= d_1(R)c_2(T), \\
d_2(S) &= d_2(R)c_1(T).
\end{aligned}$$

Finally, if R and T are of type D then S is of type C with

$$\begin{aligned}
a_1(S) &= -a_2(R)a_2(T) + b_1(R)b_2(T), \\
a_2(S) &= -a_1(R)a_1(T) - b_2(R)b_1(T), \\
b_1(S) &= b_2(R)a_2(T) + a_1(R)b_2(T), \\
b_2(S) &= b_1(R)a_1(T) + a_2(R)b_1(T), \\
c_1(S) &= d_1(R)d_2(T), \\
c_2(S) &= d_2(R)d_1(T).
\end{aligned}$$

These computations may be translated into the identity $\pi(S) = \pi(R)\,\pi(T)$. ❄

We will give some applications of this. The Boltzmann weights for a variety of other models are given in Table 2. While Gamma ice is of Type C in the terminology of the last proof, we also introduce Delta ice which is of Type D. Delta-Delta ice is of Type D and Gamma-Delta and Delta-Gamma ice are of Type C. We will distinguish between Gamma ice and Delta ice by using $\bullet$ to represent Gamma ice and $\circ$ to represent Delta ice, and variants of this convention will also distinguish the other four types of ice.

Thus in addition to (18) we have:

$$R_{\Delta\Delta}(z_i, t_i, z_j, t_j) = \begin{pmatrix} z_i t_i + z_j & & & \\ & z_i - z_j & z_j t_j + z_j & \\ & z_i t_i + z_i & z_j t_j - z_i t_i & \\ & & & z_j t_j + z_i \end{pmatrix},$$

$$R_{\Gamma\Delta}(z_i, t_i, z_j, t_j) = \begin{pmatrix} -z_i + t_i t_j z_j & & & z_i t_i + z_i \\ & z_j t_j + z_i & & \\ & & z_j t_i + z_i & \\ z_j t_j + z_j & & & z_i - z_j \end{pmatrix},$$

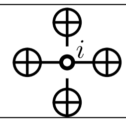
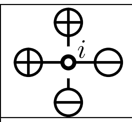
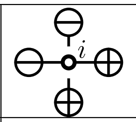
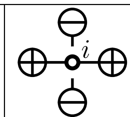
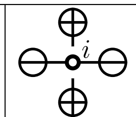
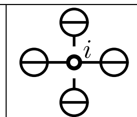
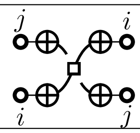
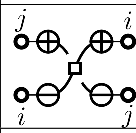
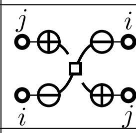
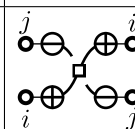
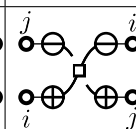
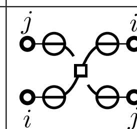
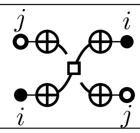
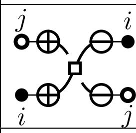
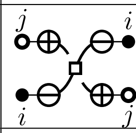
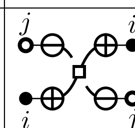
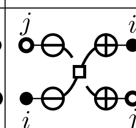
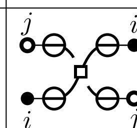
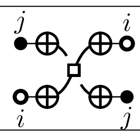
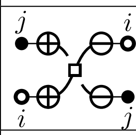
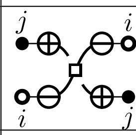
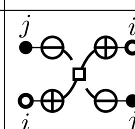
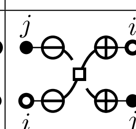
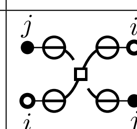
Delta Ice						
Boltzmann weight	z_i	$z_i(t_i+1)$	1	$z_i t_i$	1	1
Delta-Delta R-ice						
Boltzmann weight	$t_i z_i + z_j$	$z_j(t_j+1)$	$t_j z_j - t_i z_i$	$z_i - z_j$	$(t_i+1)z_i$	$z_i + t_j z_j$
Gamma-Delta R-ice						
Boltzmann weight	$t_i t_j z_j - z_i$	$(t_j+1)z_j$	$t_i z_j + z_i$	$t_j z_j + z_i$	$(t_i+1)z_i$	$z_i - z_j$
Delta-Gamma R-ice						
Boltzmann weight	$z_i - z_j$	$(t_i+1)z_i$	$t_j z_i + z_j$	$t_i z_i + z_j$	$(t_j+1)z_j$	$-t_i t_j z_i + z_j$

Table 2: Boltzmann weights for various types of ice with spectral parameters (z_i, t_i) and (z_j, t_j). (See Table 1 for Gamma and Gamma-Gamma ice.)

$$R_{\Delta\Gamma}(z_i,t_i,z_j,t_j) = \begin{pmatrix} z_i - z_j & & & z_j t_j + z_j \\ & z_i t_i + z_j & & \\ & & z_i t_j + z_j & \\ z_i t_i + z_i & & & z_j - t_i t_j z_i \end{pmatrix}.$$

We will denote by $\Gamma(z_i,t_i)$ what was previously denoted $\Gamma(i)$. We have also $\Delta(z_i,t_i)$:

$$\Gamma(z_i,t_i) = \begin{pmatrix} 1 & & & \\ & t_i & (t_i+1)z_i & \\ & 1 & z_i & \\ & & & z_i \end{pmatrix}, \qquad \Delta(z_i,t_i) = \begin{pmatrix} z_i & & & 1 \\ & z_i t_i & & \\ & & 1 & \\ z_i(t_i+1) & & & 1 \end{pmatrix}.$$

Theorem 9 *If $X, Y \in \{\Gamma, \Delta\}$ then*

$$[\![R_{XY}(z_i,t_i,z_j,t_j)\,, X(z_i,t_i), Y(z_j,t_j)]\!] = 0. \tag{23}$$

Proof In each of the four cases

$$\pi(R_{XY}(z_i, t_i, z_j, t_j))\pi(Y(z_j, t_j)) = z_j(t_j + 1) \times \pi(X(z_i, t_i)).$$

The result then follows from Theorem 8. ❄

Now we turn to generalizations of the Yang-Baxter equation. For every choice of z and t and $X \in \{\Gamma, \Delta\}$, let $V^X(z, t)$ be a two-dimensional vector space with basis $v_+^X(z, t)$ and $v_-^Y(z, t)$. Then $R^{XY}(z_1, t_1, z_2, t_2)$ defines an endomorphism of $V^X(z_1, t_1) \otimes V^Y(z_2, t_2)$ by

$$R(v_\sigma \otimes v_\tau) = \sum_{\mu\nu} R_{\sigma\tau}^{\nu\mu} v_\nu \otimes v_\mu, \qquad R = R^{XY}(z_1, t_1, z_2, t_2).$$

Theorem 10 *If $X, Y, Z \in \{\Gamma, \Delta\}$ then we have*

$$[\![R_{XY}(z_1, t_1, z_2, t_2), R_{XZ}(z_1, t_1, z_3, t_3), R_{YZ}(z_2, t_2, z_3, t_3)]\!] = 0.$$

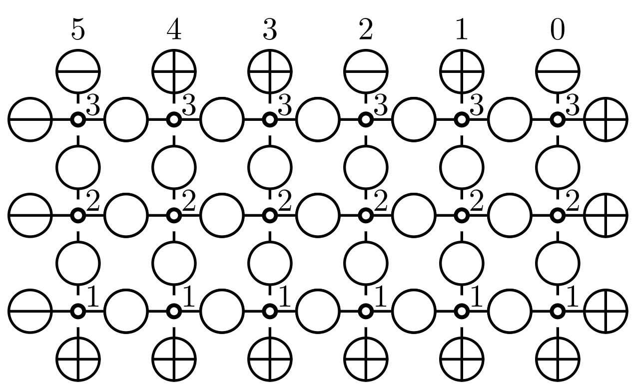
Moreover

$$[\![R_{XY}(z_2, t_1, z_1, t_2), R_{XZ}(z_3, t_1, z_1, t_3), R_{YZ}(z_3, t_2, z_2, t_3)]\!] = 0.$$

Proof This follows from Theorem 8. ❄

We now describe the boundary conditions for Delta ice in the ensemble $\mathfrak{S}_\lambda^\Delta$ that appears in the second identity in (1). The columns are labeled, as with the Gamma ice, in decreasing order. However we label the vertices in decreasing row order, so the labels of the vertices of the top row are n, and so forth.

The Delta ice boundary conditions are as follows. We again fix a partition λ. On the left boundary edges, we put $-$; on the right and bottom edges we put $+$. On the top, we put $-$ at every column labeled $\lambda_i + n - i$ $(1 \leqslant i \leqslant n)$, that is, for the columns labeled with values in $\lambda + \rho$. Top edges not labeled by $\lambda_i + n - i$ for any i are given spin $+$. Thus if $\lambda = (3, 1, 0)$, here is the Delta ice. (To indicate that this is Delta ice, the vertices are marked $\circ$.)



Theorem 11 *The partition function is*

$$Z(\mathfrak{S}_\lambda^\Delta)(z_1,\cdots,z_n;t_1,\cdots,t_n)=\prod_{i<j}(t_jz_j+z_i)s_\lambda(z_1,\cdots,z_n).$$

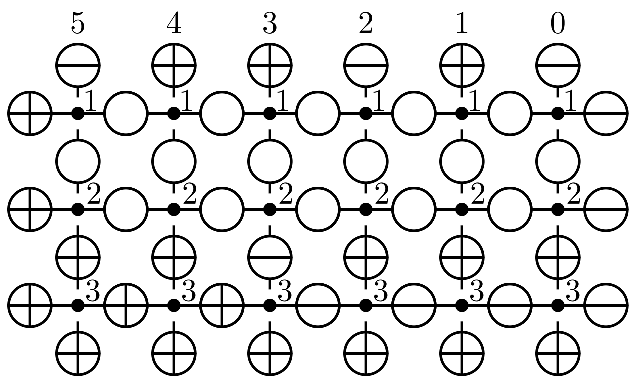
Proof This is proved analogously to Theorem 5, using the case $X=Y=\Delta$ of Theorem 9. We leave the details of the proof to the reader. ❄

Theorem 9 may be used to show that

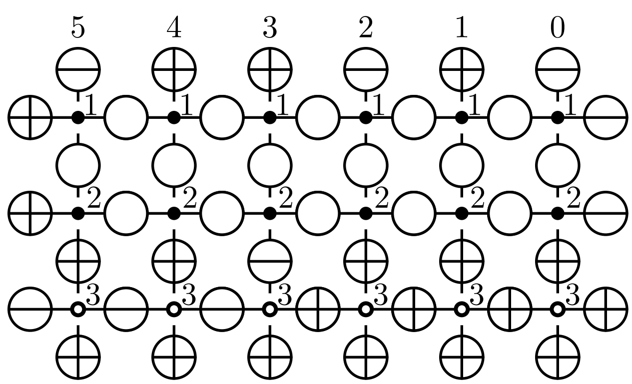
$$\prod_{i<j}(t_jz_j+z_i)Z(\mathfrak{S}_\lambda^\Gamma)=Z(\mathfrak{S}_\lambda^\Delta)\prod_{i<j}(t_iz_j+z_i) \tag{24}$$

directly without invoking Theorems 5 and 11. This fact is closely related to Statement B in Brubaker, Bump and Friedberg [3], and the following argument may be used to give an alternative proof of that result in the special case where the degree (denoted n in [3]) equals 1.

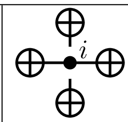
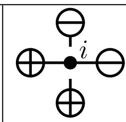
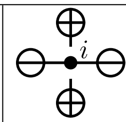
Begin with an element x of $\mathfrak{S}_\lambda^\Gamma$, say (for example with $\lambda=(3,1,0)$):



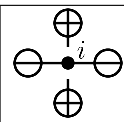
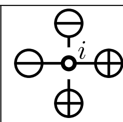
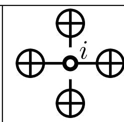
(The unlabeled edges can be filled in arbitrarily.) We wish to transform this into an element of an ensemble that has a row of Delta ice so that we may use the mixed star-triangle relation. We simply change the signs of all the entries on the edges in the 3 row:



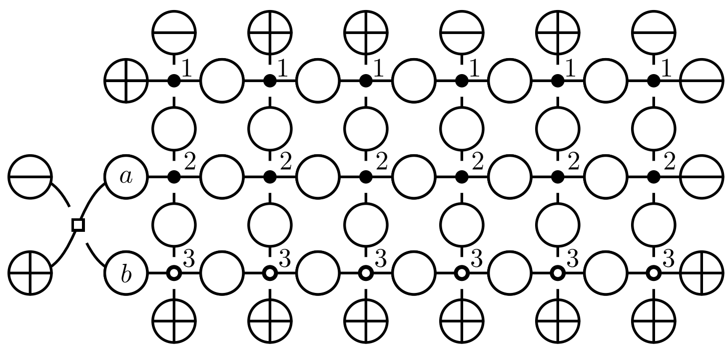
Let x' be this element of the mixed ensemble $\mathfrak{S}'$. We observe that the Boltzmann weights satisfy $w(x) = w(x')$. Indeed, in the bottom row only the following types of Gamma ice can appear:

Gamma Ice			
	1	1	z_i

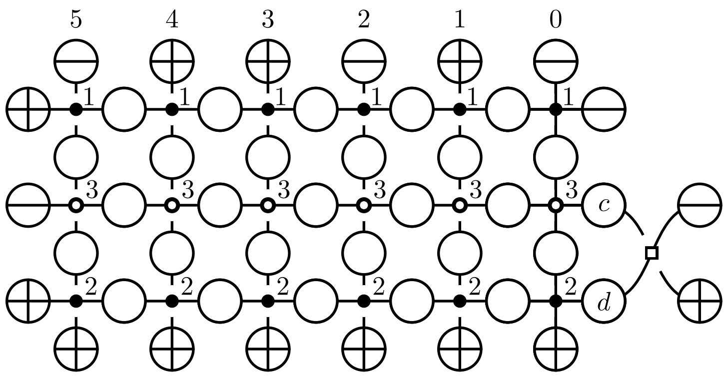
These change to:

Delta Ice			
	1	1	z_i

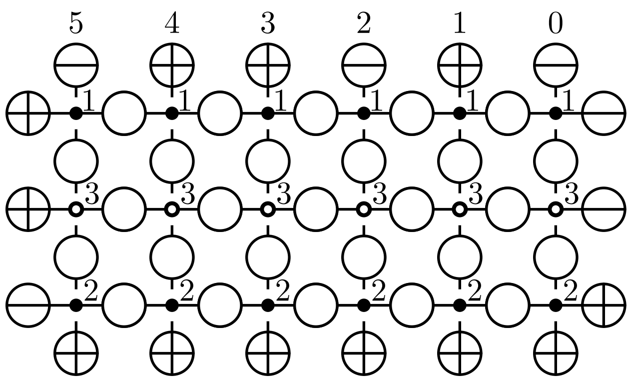
Observe that the weights are unchanged. *Note that this would not work in any row but the last* because it is essential that there be no $-$ on the bottom edge spins. Now we add a Gamma-Delta R-vertex.



If $\mathfrak{S}''$ is this ensemble, we claim that $Z(\mathfrak{S}'') = (t_3z_3 + z_2)Z(\mathfrak{S}') = (t_3z_3 + z_2)Z(\mathfrak{S}^{\Gamma}_{\lambda})$. Indeed, from Table 2, the values of a and b must be $+, -$ respectively and so the value of the R-vertex is $t_3z_3 + z_2$ for every element of the ensemble. Now using the star-triangle relation, we obtain $Z(\mathfrak{S}'') = Z(\mathfrak{S}''')$ where $\mathfrak{S}'''$ is the ensemble:



Here we must have $c, d = +, -$and so $(t_3 z_3 + z_2) Z(\mathfrak{S}_\lambda^\Gamma) = Z(\mathfrak{S}''') = (t_2 z_3 + z_2) Z(\mathfrak{S}^{(\mathrm{iv})})$ where $\mathfrak{S}^{(\mathrm{iv})}$ is the ensemble:



We repeat the process, first moving the Delta layer up to the top, then introducing another Delta layer at the bottom, etc., until we have the ensemble $\mathfrak{S}_\lambda^\Delta$, obtaining (24).

9 Yang-Baxter Systems

The results of this section are further applications of Theorem 8.

An important property of the R-matrices $R_{XY}(z_i, t_i, z_j, t_j)$ is that they are *projectively triangular*. That is,

$$R_{XY}(z_i, t_i, z_j, t_j)^{-1} = c_{XY}(z_i, t_i, z_j, t_j) P \, R_{YX}(z_j, t_j, z_i, t_i) \, P \tag{25}$$

where $c_{XY}(z_i, t_i, z_j, t_j)$ is a scalar and

$$P = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}.$$

The constant c_{XY} may be eliminated by multiplying R_{XY} by suitable scalar - for example in the case $X = Y = \Gamma$ if $R'_{\Gamma\Gamma}(z_i, t_i, z_j, t_j) = (z_j t_i + z_i)^{-1} R_{\Gamma\Gamma}(z_i, t_i, z_j, t_j)$ then R' satisfies (25) without the c_{XY}, at the cost of introducing denominators.

Yang-Baxter systems occur with varying degrees of generality in connection with different problems. One type occurs in the work of Vladimirov [38] on quantum doubles; another type occurs in Hlavaty [13] on quantized braided groups. The most general formulation [14], [12] involves four types of matrices which correspond to our R_{XY}, $X, Y \in \{\Gamma, \Delta\}$.

The axioms for a parametrized (or "colored") Yang-Baxter system in the most general definition require four types of matrices, A, B, C, D, depending on the parameter z and subject to the properties

$$\begin{array}{rclcrcl} [\![A, A, A]\!] &=& 0, & \qquad & [\![D, D, D]\!] &=& 0, \\ [\![A, C, C]\!] &=& 0, & & [\![D, B, B]\!] &=& 0, \\ [\![A, B^\ddagger, B^\ddagger]\!] &=& 0, & & [\![D, C^\ddagger, C^\ddagger]\!] &=& 0, \\ [\![A, C, B^\ddagger]\!] &=& 0, & & [\![D, B, C^\ddagger]\!] &=& 0, \end{array} \tag{26}$$

where we now denote

$$[\![X, Y, Z]\!] = X_{12}(z_1, z_2) Y_{13}(z_1, z_3) Z_{23}(z_2, z_3) - Z_{23}(z_2, z_3) Y_{13}(z_1, z_3) X_{12}(z_1, z_2)$$

and $X^\ddagger(z_1, z_2) = PX(z_2, z_1)P$. We have two spectral parameters z and t, so we interpret

$$X^\ddagger(z_1, t_1, z_2, t_2) = PX(z_2, t_2, z_1, t_1)P.$$

Theorem 12 *Let $X, Y \in \{\Gamma, \Delta\}$. Then*

$$A = R_{XX}, \qquad C = B^\ddagger = R_{XY}, \qquad D = R_{YY}{}^\ddagger$$

is a Yang-Baxter system satisfying (26).

Proof We leave the verification to the reader. ❄

Note that by projective triangularity we may replace B by $R_{YX}{}^{-1}$, which is a scalar multiple of $R_{XY}{}^\ddagger$. Thus if $X = \Gamma, Y = \Delta$ we have the Yang-Baxter system

$$A = R_{\Gamma\Gamma}, \qquad B = R_{\Delta\Gamma}{}^{-1}, \qquad C = R_{\Gamma\Delta}, \qquad D = R_{\Delta\Delta}{}^\ddagger,$$

which uses each of the four braided ice in Table 2 exactly once. It is probably most interesting to take $X \neq Y$, but worth noting that we can also make a Yang-Baxter system involving only $R_{\Gamma\Gamma}$ (or $R_{\Delta\Delta}$) playing all four roles. And we also obtain a Yang-Baxter system as follows by interchanging the z_i (but not the t_i) in the spectral parameters.

Theorem 13 *Another set of four Yang-Baxter systems may be obtained by taking*

$$A = \hat{R}_{XX}, \qquad C = B^{\ddagger} = \hat{R}_{XY}, \qquad D = \hat{R}^{\ddagger}_{YY},$$

where

$$\hat{R}_{XY}(z_1, t_1, z_2, t_2) = R_{XY}(z_2, t_1, z_1, t_2).$$

Proof We leave this to the reader. ❄

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