NOTES ON REPRESENTATIONS OF $GL(r)$ OVER A FINITE FIELD

by Daniel Bump

1. Induced representations of finite groups. Let G be a finite group, and H a subgroup. Let V be a finite-dimensional H-module. The induced module V^G is by definition the space of all functions $F : G \to V$ which have the property that $F(hg) = h.F(g)$ for all $h \in H$. This is a G-module, with group action $(gF)(g') = F(g'g)$. On the other hand, if U is a G-module, let U_H denote the H-module obtained by restricting the action of G on U to the subgroup H. Thus the underlying space of U_H is the same as that of U. If V is a G-module, we will occasionally denote by $\pi_V : G \to GL(V)$ the representation of G associated with V. Thus if $g \in G$, $v \in V$, $g.v$ and $\pi_V(g)(v)$ are synonymous.

The *Frobenius reciprocity law* amounts to a natural isomorphism

$$\mathrm{Hom}_G(U, V^G) \cong \mathrm{Hom}_H(U_H, V). \tag{1.1}$$

This is the correspondence between $\phi \in \mathrm{Hom}_G(U, V^G)$ and $\phi' \in \mathrm{Hom}_H(U_H, V)$, where $\phi'(w) = \phi(w)(1)$, and conversely, $\phi(w)(g) = \phi'(gw)$. There is also a natural isomorphism

$$\mathrm{Hom}_G(V^G, U) \cong \mathrm{Hom}_H(V, U_H). \tag{1.2}$$

This may be described as follows. Given an element $\phi \in \mathrm{Hom}_H(V, U_H)$, we may associate an element $\phi' \in \mathrm{Hom}_G(V^G, U)$, defined by

$$\phi'(f) = \sum_{\gamma \in G/H} \gamma\,\phi(f(\gamma^{-1})).$$

Thus induction and restriction are adjoint functors between the categories of H-modules and G-modules. It is also worth noting that they are *exact* functors.

Another important property of induction is transitivity. Thus if $H \subseteq K \subseteq G$, where H and K are subgroups of G, and if V is an H-module, then

$$(V^K)^G \cong V^G. \tag{1.3}$$

The isomorphism is as follows. Suppose that $F \in (V^K)^G$. Thus $F : G \to V^K$. We associate with this the element f of V^G defined by $f(g) = F(g)(1)$. It is easily checked that this $F \to f$ is an isomorphism $(V^K)^G \to V^G$.

The problem of classifying intertwining operators between induced representations was considered by Mackey [Mac] who proved the following Theorem.

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Theorem 1.1. *Suppose that G is a finite group, that H_1 and H_2 are subgroups of G, and that V_1, V_2 are H_1- and H_2-modules, respectively. Then $\mathrm{Hom}_G(V_1^G, V_2^G)$ is naturally isomorphic to the space of all functions $\Delta : G \to \mathrm{Hom}_{\mathbf{C}}(V_1, V_2)$ which satisfy*

$$\Delta(h_2\, g\, h_1) = \pi_{V_2}(h_2) \circ \Delta(g) \circ \pi_{V_1}(h_1). \tag{1.4}$$

We may exhibit the correspondence explicitly as follows. Firstly, let us define a collection f_{g,v_1} of elements of V_1^G indexed by $g \in G$, $v_1 \in V_1$. Indeed, we define

$$f_{g,v_1}(g') = \begin{cases} g'g^{-1}\, v_1 & \text{if } g'g^{-1} \in H_1; \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that if g, $g' \in G$, $h_1 \in H_1$, $v_1 \in V_1$, then

$$f_{h_1 g, h_1 v_1} = f_{g,v_1}, \qquad g'\, f_{gg',v_1} = f_{g,v_1},$$

and if $F \in V_1^G$,

$$F = \sum_{\gamma \in H_1 \backslash G} f_{\gamma, F(\gamma)}.$$

From these relations, the existence of a correspondence as in the theorem is easily deduced, where if $L \in \mathrm{Hom}_G(V_1^G, V_2^G)$ corresponds to the function $\Delta : V_1 \to V_2$, then

$$\Delta(g)\, v_1 = (L f_{g^{-1},v_1})(1),$$

and, for $F \in V_1^G$,

$$(LF)(g) = \sum_{\gamma \in H_1 \backslash G} \Delta(\gamma^{-1})\, F(\gamma\, g). \tag{1.5}$$

This completes the proof of Theorem 1.1. ∎

An intertwining operator $L : V_1^G \to V_2^G$ therefore determines a function Δ on G. We say that L is *supported* on a double coset $H_2 \backslash g / H_1$ if the function Δ vanishes off this double coset. The situation is particularly simple if V_1 and V_2 are one dimensional. If χ is a character of a subgroup H of G, we will denote by χ^G the representation V^G, where V is a one-dimensional representation of H affording the character χ. Also, if $g \in G$, we will denote by ${}^g\chi$ the character ${}^g\chi(h) = \chi(g^{-1}hg)$ of the group gHg^{-1}.

Corollary 1.2. *If χ_1 and χ_2 are characters of the subgroups H_1 and H_2 of G, the dimension of $\mathrm{Hom}_G(\chi_1^G, \chi_2^G)$ is equal to the number of double cosets in $H_2 \backslash g / H_1$ which support intertwining operators $\chi_1^G \to \chi_2^G$. Moreover, a double coset $H_2 \backslash g / H_1$ supports an intertwining operator if and only if the characters χ_2 and ${}^g\chi_1$ agree on the group $H_1 \cap g^{-1} H_2 g$.* ∎

The composition of intertwining operators corresponds to the convolution of the functions Δ satisfying (1.4). More precisely,

Corollary 1.3. *Let V_1, V_2 and V_3 be modules of the subgroups H_1, H_2 and H_3 respectively, and suppose that $L_1 \in \mathrm{Hom}_G(V_1^G, V_2^G)$ and $L_2 \in \mathrm{Hom}_G(V_2^G, V_3^G)$ correspond by Theorem 1.1 to functions $\Delta_1 : G \to \mathrm{Hom}_{\mathbf{C}}(V_1, V_2)$ and $\Delta_2 : G \to \mathrm{Hom}_{\mathbf{C}}(V_2, V_3)$. Then the composition $L_2 L_1$ corresponds to the convolution*

$$\Delta(g) = \sum_{\gamma \in H_2 \backslash G} \Delta_2(g\gamma^{-1}) \circ \Delta_1(\gamma). \tag{1.6}$$

This is easily checked. ∎

An important special case:

Corollary 1.4. *If V is a module of the subgroup H of G, the endomorphism ring $\mathrm{End}_G(V^G)$ is isomorphic to the convolution algebra of functions $\Delta : G \to \mathrm{End}_{\mathbf{C}}(V)$ which satisfy $\Delta(h_2 g h_1) = \pi_V(h_2) \circ \Delta(g) \circ \pi_V(h_1)$ for h_1, $h_2 \in H$.* ∎

2. The Bruhat Decomposition. Let $F = \mathbf{F}_q$ be a finite field with q elements, and let $G = GL(r, F)$. By an *ordered partition* of r, we mean a sequence $J = \{j_1, \ldots, j_k\}$ of positive integers whose sum is r. If J is such a ordered partition, then $P = P_J$ will denote the subgroup of elements of G of the form

$$\begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1k} \\ 0 & G_{22} & \cdots & G_{2k} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & G_{kk} \end{pmatrix},$$

where G_{uv} is a $j_u \times j_v$ block. The subgroups of the form P_J are called the *standard parabolic subgroups* of G. More generally, a *parabolic subgroup* of G is any subgroup conjugate to a standard parabolic subgroup. The term *maximal parabolic subgroup* refers to a subgroup which is maximal among the *proper* parabolic subgroups of G. Thus P_J is maximal if the cardinality of J is two.

We have $P_J = M_J N_J$ where $M = M_J$ is the subgroup characterized by $G_{uv} = 0$ if $u \neq v$, and $N = N_J$ is the subgroup characterized by $G_{uu} = I_{j_u}$ (the $j_u \times j_u$ identity matrix). Evidently $M \cong GL(j_1, F) \times \cdots \times GL(j_k, F)$. The subgroup M is called the *Levi factor* of P, and the subgroup N is called the *unipotent radical*.

We wish to extend the definitions of parabolic subgroups to subgroups of groups of the form $G = G_1 \times \cdots \times G_k$, where $G_j \cong GL(j_k, F)$. By a *parabolic subgroup* of such a group G, we mean a subgroup of the form $P = P_1 \times \cdots \times P_k$, where P_j is a parabolic subgroup of G_j. We will call such a P a *maximal* parabolic subgroup if exactly one of the P_j is a proper subgroup, and that subgroup is a maximal parabolic subgroup. If M_j and N_j are the Levi factors and unipotent radicals of the P_j, then $M = M_1 \times \cdots \times M_k$ and $N = N_1 \times \cdots \times N_k$ will be called the Levi factor and unipotent radical respectively of P.

A *permutation matrix* is by definition a square matrix which has exactly one nonzero entry in each row and column, each nonzero entry being equal to one. We will also use the term *subpermutation matrix* to denote a matrix, not necessarily square, which has *at most* one nonzero entry in each row and column, each nonzero entry being equal to one. Thus a permutation matrix is a subpermutation matrix, and each minor in a subpermutation matrix is equal to one. Let W be the subgroup of G consisting of permutation matrices. If M is the Levi factor of a standard parabolic subgroup, let $W_M = W \cap M$. If $M = M_J$, we will also denote $W_M = W_J$. If M is the Levi factor of P, then in fact $W_M = W \cap P$.

Proposition 2.1. *Let M and M' be the Levi factors of standard parabolic subgroups P and P', respectively, of G. The inclusion of W in G induces a bijection between the double cosets $W_{M'}\backslash W/W_M$ and $P'\backslash G/P$.*

First let us prove that the natural map $W \to P'\backslash G/P$ is *surjective.* Indeed, we will show that if $g = (g_{uv}) \in G$, then there exists $b' \in B$ such that $b'g$ has the form wb, for $b \in B$. Since $B \subseteq P$, P' this will show that $w \in P'\backslash g/P$. We will recursively define a sequence of integers $\lambda(r)$, $\lambda(r-1), \cdots, \lambda(1)$ as follows. $\lambda(r)$ is to be the first positive integer such that $g_{r,\lambda(r)} \neq 0$. Assuming $\lambda(r)$, $\lambda(r-1), \cdots, \lambda(u+1)$ to be defined, we will let $\lambda(u)$ be the first positive integer such that the minor

$$\det \begin{pmatrix} g_{u,\lambda(r)} & g_{u,\lambda(r-1)} & \cdots & g_{u,\lambda(u)} \\ g_{u+1,\lambda(r)} & g_{u+1,\lambda(r-1)} & \cdots & g_{u+1,\lambda(u)} \\ \vdots & & & \vdots \\ g_{r,\lambda(r)} & g_{r,\lambda(r-1)} & \cdots & g_{r,\lambda(u)} \end{pmatrix} \neq 0.$$

Now the columns of b' are to be specified as follows. The last column of b' is to be determined by the requirement that the $\lambda(r)$-th column of $b'g$ is to have 1 in the r, $\lambda(r)$ position, and zeros above. Assuming that the last $u-1$ columns of b' have been specified, we specify the u-th column from the right of b' by requiring that the $r-u, \lambda(r-u)$ entry of $b'g$ equal 1, while if $v < r-u$, then the $v, \lambda(r-u)$ entry of $b'g$ equals zero. Now let w be the permutation matrix with has ones in the $u, \lambda(u)$ positions, zeros elsewhere. Then $b = w^{-1}b'g$ is upper triangular. This shows that the natural map $W \to P'\backslash G/P$ is surjective.

Now let us show that the induced map $W_{M'}\backslash W/W_M \to P'\backslash G/P$ is injective. Suppose that $P = P_J$, $P' = P_{J'}$, $J = \{j_1, \cdots, j_k\}$, $J' = \{j'_1, \cdots, j'_l\}$. Suppose that w, $w' \in W$, p, $p' \in P'$ such that $p'wp = w'$. We must show that w and w' lie in the same double coset of $W_{M'}\backslash W/W_M$. Let us write

$$w = \begin{pmatrix} W_{11} & \cdots & W_{1k} \\ \vdots & & \vdots \\ W_{l1} & \cdots & W_{lk}, \end{pmatrix} \qquad w' = \begin{pmatrix} W'_{11} & \cdots & W'_{1k} \\ \vdots & & \vdots \\ W'_{l1} & \cdots & W'_{lk} \end{pmatrix},$$

$$p = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1l} \\ 0 & P_{22} & \cdots & P_{2l} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & P_{ll} \end{pmatrix}, \qquad p' = \begin{pmatrix} P'_{11} & P'_{12} & \cdots & P'_{1k} \\ 0 & P'_{22} & \cdots & P'_{2k} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & P'_{kk} \end{pmatrix},$$

where each matrix W_{uv} or W'_{uv} is a $j'_u \times j_v$ block, P_{uv} is a $j_u \times j_v$ block, and P'_{uv} is a $j'_u \times j'_v$ block. Let us also denote

$$\widetilde{W}_{tv} = \begin{pmatrix} W_{t1} & \cdots & W_{tv} \\ \vdots & & \vdots \\ W_{lv} & \cdots & W_{lv} \end{pmatrix}, \qquad \widetilde{W}'_{tv} = \begin{pmatrix} W'_{t1} & \cdots & W'_{tv} \\ \vdots & & \vdots \\ W'_{lv} & \cdots & W'_{lv} \end{pmatrix},$$

$$\widetilde{P}'_{t}, = \begin{pmatrix} P'_{tt} & \cdots & P'_{tl} \\ \vdots & & \vdots \\ 0 & \cdots & P'_{ll} \end{pmatrix}, \qquad \widetilde{P}_{t}, = \begin{pmatrix} P_{11} & \cdots & P_{1v} \\ \vdots & & \vdots \\ 0 & \cdots & P_{vv} \end{pmatrix}.$$

Evidently $\widetilde{W}'_{uv} = \widetilde{P}'_u \widetilde{W}_{tv} \widetilde{P}_v$, so W'_{tv} and W_{tv} have the same rank. Now the rank of a subpermutation matrix is simply equal to the number of nonzero entries, and so

$$\operatorname{rank}(W_{tv}) = \operatorname{rank}(\widetilde{W}_{tv}) - \operatorname{rank}(\widetilde{W}_{t-1,v}) - \operatorname{rank}(\widetilde{W}_{t,v+1}) + \operatorname{rank}(\widetilde{W}_{t-1,v+1}).$$

Thus $\operatorname{rank}(\widetilde{W}_{uv}) = \operatorname{rank}(\widetilde{W}'_{uv})$. Since this is true for every u, v it is apparent that we can find elements multiply w on the left by elements of $w_M \in W_M$ and $w_{M'} \in W_{M'}$ such that $w_M w w_{M'} = w'$. ■

Corollary 2.2 (The Bruhat Decomposition). *We have*

$$G = \bigcup_{w \in W} BwB \qquad \text{(disjoint)}.$$

This follows by taking $P = B$. ■

3. Parabolic induction and the "Philosophy of Cusp forms". Let $J = \{j_1, \cdots, j_k\}$ be an ordered partition of r, and let V_u, $u = 1, \cdots, k$ be modules for $GL(j_u, F)$. Then $V = V_1 \otimes \cdots \otimes V_k$ is a module for M_J. We may extend the action of M_J on V to all of P, by allowing N_J to act trivially. Then let $\mathcal{I}(V) = \mathcal{I}_{M,G}(V)$ denote the G-module obtained by inducing V from P. The module $\mathcal{I}(V)$ is be said to be formed from V by *parabolic induction.*

In order to have an analog of Frobenius reciprocity for parabolic induction, it is necessary to define a functor from the category of G-modules to the category of M-modules, the so-called "Jacquet functor." Thus, let W be a G-module. We define a module $\mathcal{J}(W) = \mathcal{J}_{G,M}(W)$ to be the set of all elements u such that $n.u = u$ for all $n \in N$. Since N is normalized by M, $\mathcal{J}(W)$ is an M-submodule of W. It is called the *Jacquet module.* Other names for the Jacquet functor from the category of G-modules to the category of M-modules which have occurred in the literature are "localization functor," "truncation," and "functor of coinvariants."

Lemma 3.1. *Let U be a G-module, U_0 the additive subgroup generated by all elements of the form $u - nu$ with $u \in U$, $n \in N$. Then U_0 is an M-submodule of U and we have a direct sum decomposition*

$$U = \mathcal{J}(U) \oplus U_0.$$

Since M normalizes N, U_0 is an M-submodule of U. We show first that $\mathcal{J}(U) \cap U_0 = \{0\}$. Indeed, suppose that $u_0 = \sum_i (u_i - n_i\, u_i)$ is an element of U_0. If this element is also in $\mathcal{J}(U)$, then

$$u_0 = \frac{1}{|N|} \sum_{n \in N} n\, u_0 = \frac{1}{|N|} \sum_i \left[\sum_{n \in N} n u_i - \sum_{n \in N} n n_i u_i \right] = 0.$$

On the other hand, to show $U = \mathcal{J}(U) + U_0$, let $u \in U$. Then

$$u = \frac{1}{|N|} \sum_{n \in N} nu + \frac{1}{|N|} \sum_{n \in N} (u - nu),$$

where the first element on the right is in $\mathcal{J}(U)$, while the second is in U_0. ■

Proposition 3.2. *Let U be a G-module, M and N the Levi factor and unipotent radical, respectively, of a proper standard parabolic subgroup P of G. Then a necessary and suffient condition for $\mathcal{J}_{G,M}(U) \neq 0$ is that there exists a nonzero linear functional T on U such that $T(n\,u) = T(u)$ for all $n \in N$, $u \in U$.*

Indeed, a necessary and sufficient condition for a given linear functional T to have the property that $T(n\,u) = T(u)$ for all $n \in N$, $u \in U$ is that its kernel contain the submodule U_0 of Lemma 3.1. Thus there will exist a nonzero such functional if and only if $U_0 \neq U$, i.e. if and only if $\mathcal{J}(U) \neq 0$. ■

Proposition 3.3. *Suppose that V is an M-module and U a G-module. We have a natural isomorphism*

$$\mathrm{Hom}_G(U, \mathcal{I}(V)) \cong \mathrm{Hom}_M(\mathcal{J}(U), V). \tag{3.1}$$

Indeed, by Frobenius reciprocity (1.1), we have an isomorphism

$$\mathrm{Hom}_G(U, \mathcal{I}(V)) \cong \mathrm{Hom}_P(U_P, V).$$

Recall that the action of M on V is extended by definition to an action of P by allowing N to act trivially. Then it is clear that a given M-module homomorphism $\phi : U \to V$ is a P-module homomorphism if and only if $\phi(U_0) = 0$. Thus $\mathrm{Hom}_P(U_P, V) \cong \mathrm{Hom}_M(U_M/U_0, V)$. By Lemma 3.1, $U_M/U_0 \cong \mathcal{J}(U)$. ■

Proposition 3.3 shows that the Jacquet construction and parabolic induction are adjoint functors.

There is also a transitivity property of parabolic induction, analogous to (1.3).

Proposition 3.4. *Let M is the Levi factor of a parabolic subgroup P of G, and let Q be a parabolic subgroup of M with Levi factor M_0. Then there exists a parabolic subgroup $P_0 \subseteq P$ of G such that the Levi factor of P_0 is also M_0. Thus if V is an M_0-module, then both $\mathcal{I}_{M_0,M}(V)$ and $\mathcal{I}_{M,G}(V)$ are defined as M- and G-modules, respectively. We have*

$$\mathcal{I}_{M,G}(\mathcal{I}_{M_0,M}(V)) \cong \mathcal{I}_{M_0,G}(V). \tag{3.2}$$

We leave the proof to the reader. ■

It is also very easy to show that:

Proposition 3.5. *The Jacquet and parabolic induction functors are exact.* ■

An important strategy in classifying the irreducible representations of reductive groups over finite or local fields consists in trying to build up the representations from lower rank groups by parabolic induction. This strategy was called the *Philosophy of Cusp Forms* by Harish-Chandra, who found motivation in the work of Selberg and Langlands on the spectral theory of reductive groups. An irreducible representation which does not occur in $\mathcal{I}_{M,G}(V)$ for any representation V of the Levi factor of a proper parabolic subgroup is called *cuspidal.*

Proposition 3.6. *Any irreducible representation of G occurs in the composition series of some represention of the form $\mathcal{I}_{M,G}(V)$, where V is a cuspidal representation of the Levi factor M of a parabolic subgroup P.*

Indeed, let P be minimal among the parabolic subgroups such that the given representation of G occurs as a composition factor of $\mathcal{I}_{M,G}(V)$ for some representation

V of the Levi factor M of P. (There always exist such parabolics P, since we may take P to be G itself.) If V is not cuspidal, it occurs in the composition series of $\mathcal{I}_{M_0,M}(V_0)$ for some Levi factor M_0 of a proper parabolic subgroup of M. Now in Proposition 3.3 the given representation of G also occurs in $\mathcal{I}_{M_0,G}(V_0)$, contradicting the assumed minimality of P. ■

According to the Philosophy of Cusp Forms, the cuspidal representations should be regarded as the basic building blocks from which other representations are constructed by the process of parabolic induction. Proposition 3.6 shows that every irreducible representation of G may be realized as a subrepresentation of $\mathcal{I}_{M,G}(V)$ where V is a cuspidal representation of the Levi component P of a parabolic subgroup. Moreover, there is also a sense in which this realization is unique: although P is not unique, its Levi factor M and the representation V are determined up to isomorphism. More precisely, we have

Theorem 3.7. *Let V and V' be cuspidal representations of the Levi factors M and M' of standard parabolic subgroups P and P' respectively of G. Then either $\mathcal{I}_{M,G}(V)$ and $\mathcal{I}_{M',G}(V')$ have no composition factor in common, or there is a Weyl group element w in G such that $wMw^{-1} = M'$, and a vector space isomorphism $\phi : V \to V'$ such that*

$$\phi(m\,v) = wmw^{-1}\,\phi(v) \qquad \text{for } m \in M,\ v \in V. \tag{3.3}$$

In the latter case, the modules $\mathcal{I}_{M,G}(V)$ and $\mathcal{I}_{M',G}(V')$ have the same composition factors.

Remark. The significance of (3.3) is that if $wMw^{-1} = M'$, then M and M' are isomorphic, and so V' may be regarded as an M-module. Thus (3.3) shows that ϕ is an isomorphism of V and V' as M-modules. In other words, for the induced representations to have a common composition factor, not only do M and M' have to be conjugate, but V and V' must be isomorphic as M-modules.

We will defer the proof of this Theorem until the next section. Of course the complete reducibility of representations of a finite group G implies that two G-modules have the same composition factors if and only if they are isomorphic. We have stated the theorem this way because this is the correct formulation over a local field. Over a local field, one encounters induced representations which may have the same composition factors, but still fail to be isomorphic.

There are two problems to be solved according to the Philosophy of Cusp forms: firstly, the construction of the cuspidal representations; and secondly, the decomposition of the representations obtained by parabolic induction from the cuspidal ones. We will examine the second problem in the following sections.

It follows from transitivity of induction that it is sufficient for a given irreducible representation to be cuspidal that the representation does not occur in $\mathcal{I}_{M,G}(V)$ for any *maximal* parabolic subgroup P. Indeed, suppose that the representation is *not* cuspidal. Then it occurs in $\mathcal{I}_{M_0,G}(V)$ for some proper parabolic subgroup P_0 of G, and some representation V of the Levi factor M_0 of P_0. If P is a maximal parabolic subgroup containing P_0, and if M is the Levi factor of P, then by (3.2) it also occurs in $\mathcal{I}_{M,G}(\mathcal{I}_{M_0,M}(V))$. Thus if an irreducible representation occurs in a representation induced from a proper parabolic subgroup, it may be assumed that the parabolic is maximal.

It follows from Proposition 3.3 that a a necessary and sufficient condition for the representation U to be cuspidal is that $\mathcal{J}_{G,M}(U) = 0$ for every Levi factor M of a maximal parabolic subgroup.

4. Intertwining operators for induced representations. In this section we shall analyze the intertwining operators between two induced representations. We shall also prove Theorem 3.3. First let us establish

Proposition 4.1. *Let $J = \{j_1, \cdots, j_k\}$, $J' = \{j'_1, \cdots, j'_l\}$ be two ordered partitions of r, and let $P = P_J$, $P' = P_{J'}$. Let $M \cong GL(j_1, F) \times \cdots \times GL(j_k, F)$ and $M' \cong GL(j'_1, F) \times \cdots \times GL(j'_l, F)$ be their respective Levi factors. Let V_u, (resp. V'_u) be given cuspidal $GL(j_u, F)$-modules (resp. $GL(j'_u, F)$-modules). Let $V = V_1 \otimes \cdots \otimes V_k$, $V' = V'_1 \otimes \cdots \otimes V_l$, and let $d = \dim_{\mathbf{C}} \operatorname{Hom}_G(\mathcal{I}_{M,G}(V), \mathcal{I}_{M',G}(V'))$. Then $d = 0$ unless $k = l$, in which case, d is equal to the number of permutations σ of $\{1, \cdots, k\}$ such that $j_{\sigma(u)} = j'_u$, and such that $V_{\sigma(u)} \cong V'_u$ as $GL(j_{\sigma(u)}, F)$-modules for each $u = 1, \cdots, k$.*

To prove Proposition 4.1, assume first that we are given nonzero intertwining operator in $\operatorname{Hom}_G(\mathcal{I}_{M,G}(V), \mathcal{I}_{M',G}(V'))$, which is supported on a single double coset of $P' \backslash G / P$. We will associate with this intertwining operator a bijection $\sigma : \{1, \cdots, k\} \to \{1, \cdots, l\}$ which has the required properties. Then we will show that the correspondence between double cosets which support intertwining operators and such σ is a bijection, and that no coset can support more than one intertwining operator. This will show that the dimension d is equal to the number of such σ.

Given the intertwining operator, let $\Delta : G \to \operatorname{Hom}_{\mathbf{C}}(V, V')$ be the function associated in Theorem 3.1. Thus

$$\Delta(p'gp).v = p'\,\Delta(g)\,p.v \qquad \text{for } p \in P,\ p' \in P',\ v \in V. \tag{4.1}$$

Recall that we are assuming that Δ is supported on a single double coset $P'wP$, where by the Bruhat decomposition we may take the representative $w \in W$. Let $\phi = \Delta(w) : V \to V'$.

Now let us show that $wMw^{-1} = M'$.

First we show that $M' \subseteq wMw^{-1}$. Suppose on the contrary that $M' \not\subseteq wMw^{-1}$. Let N_P be the unipotent radical of P. Then $Q = M' \cap wPw^{-1}$ is a proper (not necessarily standard) parabolic subgroup of M', whose unipotent radical $N_Q = M' \cap wN_Pw^{-1}$ is contained in the unipotent radical of wPw^{-1}. Now let $v \in V$, and $n \in N_Q$. Applying (4.1) with $g = w$, $p = w^{-1}n^{-1}w$, $p' = n$, we see that $n.\phi(w^{-1}n^{-1}w.v) = \phi(v)$. Now since $w^{-1}n^{-1}w$ is contained in the unipotent radical of P, $w^{-1}n^{-1}w.v = v$, and so if $n \in N_Q$ we have $n.\phi(v) = \phi(v)$. Thus $\phi(v) = 0$, since V' is cuspidal. This shows that ϕ is the zero map, which is a contradiction. Therefore $M' \subseteq wMw^{-1}$. The proof of the opposite inequality $M' \supseteq wMw^{-1}$ is similar.

Now the isomorphism $m \to wmw^{-1}$ of M onto M' makes V' into an M-module. (4.1) implies that if $m \in M$, $v \in V$, then

$$wmw^{-1}\,\phi(v) = \phi(mv). \tag{4.2}$$

This implies that if V, V' are regarded as M-modules, they are isomorphic, since ϕ is an isomorphism. By Schur's Lemma, there can be only one isomorphism (up to constant multiple) between irreducible M-modules, and since by (4.1) Δ is determined by ϕ, it follows that there can be at most one intertwining operator supported on each double coset. On the other hand, if w is given such that $M' = wMw^{-1}$, and if the induced M-module structure on V' makes V and V' isomorphic, then denoting by ϕ such an isomorphism, so that (4.2) is satisfied, it is clear that (4.1) is also satisfied, since the unipotent matrices in both P and P' act trivially on V and V'.

Thus by Proposition 2.1, the dimension d of the space $\mathrm{Hom}_G(\mathcal{I}_{M,G}(V), \mathcal{I}_{M',G}(V'))$ of intertwining operators is exactly equal to the number of $w \in W_{M'}\backslash W/W_M$ such that $wMw^{-1} = M'$, and such that the induced M-module structure on V' makes $V \cong V'$. It is clear that this is equal to the number of permutations σ of $\{1, \cdots, k\}$ such that $j_{\sigma(i)} = j'_i$, and such that $V_{\sigma(i)} \cong V'_i$ as $GL(j_{\sigma(i)}, F)$-modules for each $i = 1, \cdots, k$. ■

Corollary 4.2. *Let $P \subseteq G$ be the standard parabolic subgroup P_J where J is the ordered partition $\{j_1, \cdots, j_k\}$ of r. Let $M \cong GL(j_1, F) \times \cdots \times GL(j_k, F)$ be the Levi factor of P, and let V_u be given cuspidal $GL(j_u, F)$-modules. Let $V = V_1 \otimes \cdots \otimes V_k$. Then $\mathcal{I}(V)$ is reducible if and only there exist distinct u, v such that $j_u = j_v$, and $V_u \cong V_v$ as $GL(j_u, F)$-modules.*

This follows from Proposition 4.1 by taking $P' = P$. ■

We now give the proof of Theorem 3.7. The only part which is not contained in Proposition 4.1 is the final assertion that if there exists a Weyl group element w in G such that wMw^{-1}, and ϕ such that (3.3) is satisfied, then the induced modules have the same composition factors.

The problem may be stated as follows. Let $J = \{j_1, \cdots, j_k\}$ and $J' = \{j'_1, \cdots, j'_k\}$ be two sets of positive integers whose sum is r, and assume that J' is obtained from J by permuting the indices j_1. Thus there exists a bijection $\sigma : \{1, \cdots, k\} \to \{1, \cdots, k\}$ such that $j_{\sigma(i)} = j'_i$. Let V_i be a cuspidal $GL(j_i, F)$-module for $i = 1, \cdots, k$, and let $V'_i = V_{\sigma(i)}$. Let $P = P_J$, $P' = P_{J'}$, $M = M_P$, and $M' = M_{P'}$. Then $V = V_1 \otimes \cdots \otimes V_k$ and $V' = V'_1 \otimes \cdots \otimes V'_k$ are M- and M'-modules respectively. What is to be shown is that $\mathcal{I}_{M,G}(V)$ and $\mathcal{I}_{M',G}(V')$ have the same composition factors. Clearly it is sufficient to show this when σ simply interchanges two adjacent components. Moreover in that case, by transitivity and exactness of parabolic induction (Propositions 3.4 and 3.5) it is sufficient to show this when $k = 2$. Now if $V \cong V'$ this is obvious. On the other hand, if $V \ncong V'$ then by Corollary 4.2, $\mathcal{I}_{M',G}(V')$ is irreducible, and a nonzero intertwining map $\mathcal{I}_{M,G}(V) \to \mathcal{I}_{M',G}(V')$ exists by Proposition 4.1. Consequently $\mathcal{I}_{M,G}(V) \cong \mathcal{I}_{M',G}(V')$. This completes the proof of Theorem 3.7. ■

5. The Kirillov Representation. Let $G = G_r = GL(r, F)$ as before, and let P_r be the subspace consisting of elements having bottom row $(0, \ldots, 0, 1)$. Let $N = N_r$ be the subgroup of unipotent upper triangular matrices, and let U_r be the subgroup consisting of matrices which have only zeros above the diagonal, except for the entries in the last column. Thus $U_r \cong F^{r-1}$. If $k \leq r$, we will denote by G_k the subgroup of G, isomorphic to $GL(k, F)$, consisting of matrices of the form

$$\begin{pmatrix} * & 0 \\ 0 & I_{r-k} \end{pmatrix},$$

where '$*$' denotes an arbitrary $k \times k$ block, and I_{r-k} denotes the $r-k \times r-k$ identity matrix. Identifying $GL(k,F)$ with this subgroup of G_r, the subgroups P_k, N_k and U_k are then contained as subgroups of G_r. Thus $P_k = G_{k-1}.U_k$ (semidirect product).

Let $\psi = \psi_F$ be a fixed nontrivial character of the additive group of F. For $k \leq r$, let $\theta_k : N_k \to \mathbf{C}^\times$ be the character of N_k defined by

$$\theta_k\left(\begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1k} \\ & 1 & x_{23} & \cdots & x_{2k} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}\right) = \psi(x_{12} + x_{23} + \cdots + x_{k-1,k}).$$

We will denote θ_r as simply θ. Then let $\mathcal{K} = \mathcal{K}_r$ be the module of P_r induced from the character θ of N_r. By definition $\mathcal{K}$ is a space of functions $P_r \to \mathbf{C}$. However, each function in $\mathcal{K}$ is determined by its value on G_{r-1}, and it is most convenient to regard $\mathcal{K}$ as a space of functions on G_{r-1}. Specifically,

$$\mathcal{K} = \{f : G_{r-1} \to \mathbf{C} | f(ng) = \theta_{r-1}(ng)\, f(g) \text{ for } n \in N_{r-1}\},$$

and the group action is defined by

$$\left(\begin{pmatrix} h & u \\ & 1 \end{pmatrix} f\right)(g) = \theta(gu)\, f(gh) \qquad \text{for } g,\, h \in G_{r-1},\, u \in U_r,\, f \in \mathcal{K}.$$

$\mathcal{K}$ is called the *Kirillov representation* of the group P_r.

Theorem 5.1. *The Kirillov representation of P_r is irreducible.*

To prove this, it is sufficient to show that $\mathrm{Hom}_{P_r}(\mathcal{K},\mathcal{K})$ is one dimensional. Let there be given a double coset in $N_r \backslash P_r / N_r$ which supports an intertwining operator $\mathcal{K} \to \mathcal{K}$. Let $\Delta : P_r \to \mathbf{C}$ be the function associated with the given intertwining operator. We may take a coset representative h which lies in G_{r-1}. We will prove that $h = I_{r-1}$. Theorem 5.1 will then follow from Corollary 1.2.

Suppose by induction that we have shown that $h \in G_k$, where $1 \leq k \leq r-1$. We will show then that $h \in G_{k-1}$. (If $k = 1$, this is to be interpreted as the assertion that $h = 1$.) Let

$$\begin{pmatrix} I_{k-1} & u \\ & 1 \end{pmatrix} \in U_k,$$

where u is a column vector in F^{k-1}. We have

$$\begin{pmatrix} h & \\ & I_{r-k} \end{pmatrix} = \begin{pmatrix} I_{k-1} & h.u & \\ & 1 & \\ & & I_{r-k+1} \end{pmatrix} \begin{pmatrix} h & \\ & I_{r-k} \end{pmatrix} \begin{pmatrix} I_{r-1} & u & \\ & 1 & \\ & & I_{r-k+1} \end{pmatrix}^{-1},$$

and the bi-invariance property (1.4) of Δ implies that

$$\theta_k\left(\begin{pmatrix} I_{k-1} & h.u & \\ & 1 & \\ & & I_{r-k} \end{pmatrix}\right) = \theta_k\left(\begin{pmatrix} I_{k-1} & u & \\ & 1 & \\ & & I_{r-k} \end{pmatrix}\right).$$

Since this is true for all u, it follows that $h \in G_{k-1}$. ∎

6. Generic representations. Now let $\mathcal{G}$ be the representation of G induced from the representation θ of N which was introduced in Section 5. Our objective is to prove the following famous theorem of Gelfand and Graev [GG]. The corresponding theorems over local fields and adele groups are due to Shalika [Sh]. These results are often referred to as "multiplicity one" theorems.

Theorem 6.1. *The representation $\mathcal{G}$ of G is multiplicity-free.*

Let

$$w_0 = w_0(r) = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}. \tag{6.1}$$

be the "longest" element of the Weyl group. If $J = \{j_1, \cdots, j_k\}$ is a ordered partition of r, we will also denote

$$w(J) = \begin{pmatrix} w_0(j_1) & & \\ & \ddots & \\ & & w_0(j_k) \end{pmatrix},$$

where $w_0(j)$ is defined by (6.1).

We will need some elementary facts about the action of the Weyl group on the root system. Let Φ be the set of all roots of G relative to the Cartan subgroup A, and let Ω be the set of all simple positive roots. If $\alpha \in \Omega$, $s_\alpha \in W$ will denote the simple reflection such that $s_\alpha(\alpha) = -\alpha$. If S is any subset of Ω, there is a ordered partition $J = \{j_1, \cdots, j_k\}$ of r such that the subgroup of W generated by the s_α such that $\alpha \in S$ is W_M, where $M = M_J$. Thus there is a bijection between the set of subsets of Ω and the ordered partitions of r. The root system $\Omega(M)$ of M relative to A (which is the disjoint union of the root systems for $GL(j_1), \cdots, GL(j_k)$) is naturally included in Ω.

Lemma 6.2. *Let M be the Levi factor of a standard parabolic subgroup of G. If $\alpha, \beta_1, \cdots, \beta_l \in \Omega$ such that $\alpha \notin \Omega(M)$, $\beta_1, \cdots \beta_l \in \Omega(M)$, and if $\alpha + \beta_1 + \ldots + \beta_l$ is a root, then $\alpha + \beta_1 + \ldots + \beta_l > 0$ if and only if $\alpha > 0$.* ∎

Lemma 6.3. *Let S be a subset of Ω, and let J be the ordered partition of r such that the subgroup of W generated by the s_α such that $\alpha \in S$ is W_M, where $M = M_J$. Then if $\alpha \in S$, $w(J)(\alpha) < 0$, and $-w(J)(\alpha) \in S$. On the other hand, if $\alpha \in \Omega$ but $\alpha \notin S$, then $w(J)(\alpha) = \alpha + \beta$, where β is the sum of roots in $\Omega(M)$, and in this case $w(J)(\alpha) > 0$.* ∎

We omit the proofs of Lemmas 6.2 and 6.3, which are not hard to check.

Lemma 6.4. *If $w \in W$ and $a \in A$ such that $\theta(n) = \theta(wa\,n\,(wa)^{-1})$ whenever n and $wa\,n\,(wa)^{-1}$ are both in N, then there exists a ordered partition J of r such that $w = w_0\,w(J)$, and a is in the center of M_J.*

To prove this, we apply Lemma 6.3 with S be the set of $\alpha \in \Omega$ such that $w\alpha$ is a positive root. Let us show first that if $\alpha \in S$, then $w\alpha$ is a simple root. Let $x_\alpha : F \to G$ be the standard one-parameter subgroup of G, so that if $a \in A$, $\xi \in F$, then $ax_\alpha(\xi)a^{-1} = x_\alpha(\alpha(a)\xi)$. Let X_α be the image of x_α. Let $\xi \in F$, and let

$n = x_\alpha(\xi) \in X_\alpha$. Then n and $(wa)\,n\,(wa)^{-1}$ are both in N, since $(wa)\,n\,(wa)^{-1} = x_{w\alpha}(\alpha(a)\,\xi)$. Since $\theta|_{X_\alpha}$ is nontrivial, the hypothesis of the Lemma implies that $\theta|_{X_{w\alpha}}$ is also nontrivial. Thus $w\alpha$ is a simple root.

Furthermore, the hypothesis of the Lemma implies that $\psi(\alpha(a)\,\xi) = \psi(\xi)$, and consequently $\alpha(a) = 1$ for all $\alpha \in S$. This implies that a is in the center of W_J.

Let us show now that wMw^{-1} is the Levi factor of a standard parabolic subgroup. Indeed, M is generated by the set of one parameter subgroups

$$\{X_\alpha | \alpha \in \Phi \text{ is a linear combination of roots in } S\},$$

so wMw^{-1} is generated by the set of one parameter subgroups

$$\{X_\alpha | \alpha \in \Phi \text{ is a linear combination of roots in } wS\}.$$

As we have just shown that wS is a subset of Ω, this wMw^{-1} is the Levi factor of a standard parabolic subgroup.

Now we show that if $\alpha \in \Omega$, then $(w\,w(J))(\alpha) < 0$. Firstly, if $\alpha \notin S$, then by Lemma 6.3, $(w\,w(J))(\alpha) = w(\alpha) + w(\beta)$, where β is the sum of roots in $\Omega(M)$. Now we apply Lemma 6.2. Note that $w(\alpha) \notin \Omega(wMw^{-1})$, while $w(\beta)$ is the sum of roots in $\Omega(wMw^{-1})$, so by Lemma 6.2, $(w\,w(J))(\alpha)$ is negative since $w(\alpha)$ is negative by the definition of S. On the other hand, if $\alpha \in S$, then by Lemma 6.2 $-w(J)(\alpha) \in S$, and so $w(-w(J)(\alpha)) > 0$ by the definition of S. Thus $(w\,w(J))(\alpha) < 0$ in this case also.

Since $w\,w(J)$ takes every simple positive root to a negative root, $w\,w(J) = w_0$, and so $w = w_0\,w(J)$. This completes the proof of Lemma 6.2. ∎

We turn now to the proof of Theorem 6.1. The strategy is to prove that the algebra of endomorphisms of $\mathcal{G}$ is abelian. This implies that $\mathcal{G}$ is multiplicity free, since if $\mathcal{G}$ contains k copies of some irreducible representation, then the endomorphism ring of $\mathcal{G}$ contains a copy of the ring of $k \times k$ matrices over $\mathbf{C}$.

The proof depends on the existence of the anti-automorphism $\iota(g) = w_0\,{}^t g\,w_0$ of G. Evidently $\iota(gg') = \iota(g)\,\iota(g')$. Furthermore, ι stabilizes N, and its character θ.

By Corollary 1.4, the endomorphism ring of $\mathcal{G}$ is isomorphic to the convolution algebra of functions Δ satisfying

$$\Delta(n_1\,g\,n_2) = \theta(n_1)\,\Delta(g)\,\theta(n_2) \tag{6.2}$$

for n_1, $n_2 \in N$, $g \in G$. Evidently ι induces an anti-involution on this ring. We will argue that any such function Δ is stabilized by ι. This will prove that the ring is abelian, since then $\Delta_1 * \Delta_2 = {}^\iota(\Delta_1 * \Delta_2) = {}^\iota\Delta_2 * {}^\iota\Delta_1 = \Delta_2 * \Delta_1$.

Let us therefore consider a function Δ satisfying (6.2), which is supported on a single double coset in $N\backslash G/N$. It follows from the Bruhat decomposition that we may choose a coset representative in the form wa where $w \in W$, $a \in A$. Then (6.2) amounts to the assertion that the hypotheses of Lemma 6.2 are satisfied. We may therefore find J such that $w = w_0\,w(J)$, and a is in the center of M_J. Thus $\iota(wa) = w_0\,{}^t(w_0\,w(J)\,a)\,w_0 = w_0\,a\,w(J)\,w_0\,w_0 = w_0\,w(J)a = wa$. This shows that ι stabilizes every double coset of $N\backslash G/N$ which supports a function Δ satisfying (6.2). Therefore the convolution algebra is ι-stable, as required. ∎

Let V be an irreducible G-module. If there exists a nonzero G-homomorphism $V \to \mathcal{G}$, then we call V *generic*. In this case, the image of V in $\mathcal{G}$ is called the *Whittaker model* of V. By Theorem 6.1, it is the unique space $\mathcal{W}_V$ of functions f on G with the property that $f(ng) = \theta(n)\, f(g)$ for $n \in N$, stable under right translation by G such that the G-module action by right translation on $\mathcal{W}_V$ affords a representation isomorphic to V.

Let $\mathbf{C}_\theta$ denote a one-dimensional N-module affording the character θ, so that $\mathcal{G} = \mathbf{C}_\theta^G$. By Frobenius reciprocity (1.1), the existence of a G-module homomorphism $V \to \mathcal{G}$ is equivalent to the existence of an N-module homomorphism $V \to \mathbf{C}_\theta$. Thus V is generic if and only if there exists a linear functional T on V such that $T(n.v) = \theta(n)\, T(v)$ for all $n \in N$, and $v \in V$. If such a functional exists, it is unique up to scalar multiple, since by (1.1) and Theorem 6.1, the dimension of the space of such functionals is

$$\dim \mathrm{Hom}_N(V, \mathbf{C}_\theta) = \dim \mathrm{Hom}_G(V, \mathcal{G}) \leq 1.$$

Let $\mathbf{C}_\theta$ denote a one-dimensional N-module affording the character θ, so that $\mathcal{G} = \mathbf{C}_\theta^G$. Whether or not V is irreducible, we will call a linear functional T on V such that $T(n.v) = \theta(n)\, T(v)$ for all $n \in N$, and $v \in V$ *Whittaker functional.* Thus a Whittaker functional is essentially an N-module homomorphism $G \to \mathbf{C}_\theta$. By Frobenius reciprocity, the existence of a G-module homomorphism $V \to \mathcal{G}$ is equivalent to the existence of a Whittaker functional. Thus V admits a Whittaker functional if and only if it has an irreducible component which is generic. If V is irreducible, then by (1.1) and Theorem 6.1, the dimension of such functionals is

$$\dim \mathrm{Hom}_N(V, \mathbf{C}_\theta) = \dim \mathrm{Hom}_G(V, \mathcal{G}) \leq 1.$$

7. Cuspidal representations are generic. We will prove

Proposition 7.1. *Let V be a cuspidal G-module, T_0 be a nonzero functional on V. Then there exists a nonzero Whittaker functional T in the linear span of the functionals $v \mapsto T_0(pv)$ $(p \in P_r)$. Moreover, if $v_0 \in V$ such that $T_0(v_0) \neq 0$, then there exists $g \in G_{r-1}$ such that $T(gv_0) \neq 0$.*

We will follow the notations introduced in Section 5. Furthermore, if $1 \leq k \leq r$, let $N^k = U_{k+1} \cdot U_{k+2} \cdot \ldots \cdot U_r$, so that $N = N_k\, N^k$. Let us assume by induction that

There exists a nonzero functional T_k in the linear span of the functionals $v \mapsto T_0(pv)$ $(p \in P_r)$ such that $T_k(nv) = \theta(n)\, T_k(v)$ for $n \in N^{r-k}$. Moreover, there exists $g_k \in G_{r-1}$ such that $T_k(g_k v_0) \neq 0$.

Since N^r is reduced to the identity, the induction hypothesis is satisfied when $k = 0$. We will show that if it is satisfied for $k < r - 1$, then it is satisfied for $k + 1$.

Let S_k be the space of all linear functionals T in the linear span of the functionals $v \mapsto T_k(p\,v)$ $(p \in P_{r-k})$. Observe that if $T \in S_k$, then $T(nv) = \theta(n)\, T(v)$ for all $n \in N^{r-k}$, because if $p \in P_{r-k}$ and $n \in N^{r-k}$, then $pnp^{-1} \in N^{r-k}$, and $\theta(pnp^{-1}) = \theta(n)$. Thus we have a (right) action of P_{r-k} on S_k, defined by $T^p(v) = T(pv)$.

The subgroup U_{r-k} of P_{r-k} is abelian, and so its action on S_k may be decomposed into one-dimensional eigenspaces. Let T be an nonzero element of S_k be such that $T(nv) = \chi(n)\, T(v)$ for $n \in U_{r-k}$, where χ is a character of U_{r-k}. Since T_k is a linear combination of such eigenfunctions, we may assume that $T(g_k v_0) \neq 0$.

Note that χ cannot be the trivial character because V is cuspidal: for if J is the ordered partition $\{r-k-1, k+1\}$ of r, and if χ is zero, then if $\chi = 1$ we have $T(nv) = T(v)$ for all n in the unipotent radical of P_J, because such n can be factored as $n_1 n_2$ where $n_1 \in U_{r-k}$ and $n_2 \in N^{r-k}$, and n_2 satisfies $\theta(n_2) = 1$. By Proposition 3.2, this contradicts the cuspidality of V.

Now since $\chi \neq 1$, there exists $g \in G_{r-k-1}$ such that $\theta(n) = \chi(gng^{-1})$ for all $n \in U_{r-k}$. Then $T_{k+1} = T^g$ satisfies $T^g(nv) = \theta(n)\, T^g(v)$ for all $n \in U_{r-k}$, and indeed for all $n \in U^{r-k-1} = U_{r-k}\, U^{r-k}$. Also, we may take $g_{k+1} = g^{-1} g_k$, so that $T_{k+1}(g_{k+1} v_0) = T(g_k v_0) \neq 0$. This completes the induction.

Now T_{r-1} is clearly a nonzero Whittaker functional, and $T_{r-1}(g_{r-1} v_0) \neq 0$. ■

Theorem 7.2. *Cuspidal representations are generic.*

This is an immediate consequence of Proposition 7.1. ■

Theorem 7.3. *Let V be a cuspidal G-module. Then as a P_r-module, V is isomorphic to the Kirillov representation.*

To see this, observe first by Theorems 6.1 and 7.2, $\mathrm{Hom}_G(V, \mathcal{G})$ is one-dimensional. Thus Frobenius reciprocity (1.1),

$$\dim \mathrm{Hom}_{P_r}(V_{P_r}, \mathcal{K}) = \dim \mathrm{Hom}_G(V, \mathcal{G}) = 1.$$

Thus there exists a unique nontrivial P_r-homomorphism $\phi : V \to \mathcal{K}$, and by Theorem 5.1, this is surjective. We must show that it is injective. Let V_0 be the kernel of ϕ, which is a P_r-module. Then $\mathcal{K}$ does not occur in V_0 as a composition factor. If V_0 is not reduced to the identity, let v_0 be a nonzero vector. It follows from Proposition 7.1 that there exists a Whittaker functional T on V and $g \in G_{r-1}$ such that $T(gv_0) \neq 0$. Since $g \in P_r$, and since V_0 is a P_r-module, $gv_0 \in V_0$, and so the restriction of T to V_0 is not identically zero. Thus if $\mathbf{C}_\theta$ denotes a one dimensional N-module affording the character θ, $\dim \mathrm{Hom}_N(V_0, \mathbf{C}_\theta) > 0$. By Frobenius reciprocity, this is equal to the dimension of $\mathrm{Hom}_{P_r}(V_0, \mathcal{K})$. This is a contradiction, since V_0 does have $\mathcal{K}$ as a composition factor. ■

Thus a cuspidal representation V has a unique P_r-embedding in $\mathcal{K}$, which is an isomorphism. This realization of V as a space of functions on G_{r-1} is called the *Kirillov model* of V. Kirillov models were introduced on $GL(2)$ by Kirillov [K], and used extensively by Jacquet and Langlands [JL]. For $r > 2$, Kirillov models were introduced by Gelfand and Kazhdan [GK].

Corollary 7.4. *If V is a cuspidal G-module, then*

$$\dim(V) = (q^{r-1} - 1)(q^{r-2} - 1) \cdot \ldots \cdot (q - 1).$$

Indeed, this is the dimension of $\mathcal{K}$. ■

8. A further "Multiplicity One" Theorem. The theorem in this section complements Theorem 6.1.

Theorem 8.1. *Let P be a standard parabolic subgroup of G, and let V_0 be a cuspidal representation of the Levi factor M of G, and let $V = \mathcal{I}_{M,G}(V_0)$. Then V has a unique Whittaker model.*

Thus V has a unique generic composition factor, and if V is irreducible, V is itself generic.

Let $\mathbf{C}_\theta$ denote the complex numbers given the structure of a one-dimensional N-module affording the character θ. We must calculate the dimension of $\mathrm{Hom}_N(V,\theta)$, which by Frobenius reciprocity is the same as the dimension of $\mathrm{Hom}_G(V,\mathcal{G})$. By Theorem 1.1, this equals the dimension of the space of functions $\Delta : G \to \mathrm{Hom}_C(V_0, \mathbf{C}_\theta)$ which satisfy $\Delta(ngp) = \pi_{\mathbf{C}_\theta}(n) \circ \Delta(g) \circ \pi_{V_0}(p)$ if $n \in N$, $p \in P$. Thus

$$\Delta(ngp).v = \theta(n)\Delta(g).p(v) \tag{8.1}$$

for $n \in N$, $p \in P$, $v \in V_0$. We will show that the space of such functions is one-dimensional.

We will use the notations of Section 6 for the root system.

Let NwP be a double coset on which Δ does not vanish. By Proposition 1.2, we may choose $w \in W$, and we may choose w modulo right multiplication by elements of W_M. Let S be the set of all $\alpha \in \Omega$ such that $w^{-1}\alpha \in \Phi(M)$. Then $w^{-1}S$ is a set of linearly independent roots in $\Phi(M)$, and so there exists $w_1 \in W_M$ such that $(ww_1)^{-1}\alpha < 0$ for all $\alpha \in S$. Since $NwP = Nww_1P$, we may replace w by ww_1, i.e. we may choose the coset representative w so that $w^{-1}\alpha < 0$ for all $\alpha \in \Omega$ such that $w^{-1}\alpha \in \Phi(M)$.

We show now that this implies that $w = w_0$. It is sufficient to show that $w^{-1}\alpha < 0$ for all $\alpha \in \Omega$. Since we already know this when $w^{-1}\alpha \in \Phi(M)$, we may assume $w^{-1}\alpha \notin \Phi(M)$. Suppose on the contrary that $w^{-1}\alpha > 0$. Let $n \in X_\alpha$ such that $\theta(n) \neq 1$. Such n exists since α is a simple root. Now $w^{-1}nw \in X_{w^{-1}\alpha}$. Since $w^{-1}\alpha$ is a positive root which is not in $\Phi(M)$, $w^{-1}nw$ lies in the unipotent radical of P, and therefore $w^{-1}nw.v = v$ for all $v \in V_0$. Now by (8.1),

$$\Delta(w).v = \Delta(w.w^{-1}nw).v = \Delta(nw).v = \theta(n)\,\Delta(w).v,$$

so $\Delta(w)$ is simply the zero map. This contradiction shows that $w = w_0$.

We have shown that the only double coset which could support an intertwining operator is Nw_0P. Now let us show that this particular double coset supports exactly one such intertwining operator. Let $P = P_J$, and let $w(J)$ be as in Section 6. Then $w_0w(J)$ lies in the coset Nw_0P, and Δ is determined by the functional $T = \Delta(w_0w(J))$ of V_0, since we must have

$$\Delta(nw_0p) = \theta(n)\,T \circ \pi_{V_0}(w(J)p). \tag{8.2}$$

We will show that there is, up to constant multiple, a unique functional T on V_0 such that we may define Δ by (8.2). Indeed, for this definition to be consistent, it is necessary and sufficient that

$$\theta(n)\,T \circ \pi_{V_0}(w(J)p) = T \circ \pi_{V_0}(w(J)) \tag{8.3}$$

whenever $n \in N$, $p \in P$ such that $nw_0p = w_0$. If $nw_0p = w_0$, then $p = w_0^{-1}n^{-1}w_0$ is lower trianguler, hence is an element of $w(J)^{-1}N_Jw(J)$. Let us write $p = w(J)^{-1}n_1w(J)$, where $n_1 \in N_J$. Note that $\theta(n_1) = \theta(n)^{-1}$, so (8.3) is equivalent to

$$T \circ \pi_{V_0}(n_1) = \theta(n_1)T.$$

Thus T must be a Whittaker functional on V_0, and the space of such is one dimensional by Theorems 7.2 and 6.1. We see that the space $\mathrm{Hom}_N(V,\theta)$ is one dimensional. ■