Generalized Frobenius-Schur Numbers

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Let G be a finite group. Suppose that $\tau : G \longrightarrow G$ is an automorphism such that $\tau^r = 1$. The *norm map* $N : G \longrightarrow G$ is defined by

$$N(g) = g \cdot {}^{\tau}g \cdot {}^{\tau^2}g \cdot \ldots \cdot {}^{\tau^{r-1}}g. \tag{1}$$

Let χ be a character of G. We will study a class of character sums including

$$\epsilon_\tau(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(N(g)). \tag{2}$$

Theorem 9 gives a concrete interpretation of this sum (and its generalizations). Let (π, V) be a representation of G affording χ. Let $E_\tau(\pi)$ be the vector space of r-linear forms $T : V \times \cdots \times V \longrightarrow \mathbb{C}$ satisfying

$$T(\pi(g)v_1, \pi({}^{\tau}g)v_2, \cdots, \pi({}^{\tau^{r-1}}g)v_r) = T(v_1, \cdots, v_r).$$

Let $c_r : E_\tau(\pi) \longrightarrow E_\tau(\pi)$ be the linear transformation which replaces T by

$$(c_r T)(v_1, v_2, \cdots, v_r) = T(v_r, v_1, \cdots, v_{r-1}).$$

We will show in Theorem 9 that $\epsilon_\tau(\chi)$ is the trace of c_r. As a consequence (Theorem 10), we see that $\epsilon_\tau(\chi)$ is an algebraic integer, in fact, an element of $\mathbb{Z}[e^{2\pi i/r}]$.

In many cases $\epsilon_\tau(\chi) \in \mathbb{Z}$, so it may be worth giving an example where it is *not* a rational integer. Let

$$G = \left\langle x, y, z | x^3 = y^3 = z^3 = 1, zx = xz, zy = yz, xyx^{-1}y^{-1} = z \right\rangle,$$

a nonabelian group of order 27. Let $\tau : G \longrightarrow G$ be the outer automorphism of order $r = 3$ such that $\tau(x) = x$, $\tau(y) = xy$. Let χ be the character of degree 3 such that $\chi(z) = 3\rho$, where ρ is a primitive cube root of unity. We find that $\epsilon_\tau(\chi) = 1 + 2\rho$.

Let us give an application. If $g \in G$ let $M(g)$ be the number of solutions to the equation $N(x) = g$ with $x \in G$. Philip Hall [14], generalizing a theorem of Frobenius, proved that $M(g)$ is divisible by the greatest common divisor of r and the order of the centralizer of g. An interesting integer valued function on the group—is it perhaps a generalized character?

The answer is no, but we will now prove that M lies in the $\mathbb{Z}[e^{2\pi i/r}]$-algebra generated by the irreducible characters. Indeed, this is immediate from Theorem 10 and the identity

$$M(g) = \sum \epsilon_\tau(\chi)\overline{\chi(g)}$$

where the summation is over all irreducible characters χ of G. (The proof is a straightforward adaptation of Proposition 1 below.)

If $r = 2$, and τ is trivial, then $\epsilon_\tau(\chi)$ is $(1/|G|)\sum \chi(g^2)$, and such sums were considered by Frobenius and Schur [10]. If $r = 2$ and τ is not assumed to be trivial, these sums were considered by Kawanaka and Matsuyama [18]. The work of Kawanaka and Matsuyama has many interesting associations, for example to the notion of a *generalized involution model* which we discuss.

We will consider a couple of different generalizations of (2) involving another character η. In Theorem 9 the character η is a character of G, assumed τ-invariant. In Theorem 12, the character η is not a character of G, but of its subgroup τ-fixed points. These sums allow us to give a reinterpretation of a famous theorem of Shintani on the lifting of characters for $\mathrm{GL}(n)$ over a finite field.

This paper touches on different works of Frobenius and Schur. An invaluable guide to their work is Curtis [6].

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1 Theorems of Frobenius-Schur and Kawanaka-Matsuyama.

Let (π, V) be an irreducible complex representation of a compact group G. We will denote by $\chi = \chi_\pi$ the character of π. If π is self-contragredient, then there exists a nondegenerate G-equivariant bilinear form B on V, unique up to scalar multiple and for some constant $\epsilon = \pm 1$ we must have

$$B(v, w) = \epsilon \, B(w, v).$$

We will denote $\epsilon = \epsilon_1(\pi)$ or $\epsilon_1(\chi)$. If π is not self-contragredient, then we define $\epsilon_1(\pi) = 0$. Frobenius and Schur [10] proved

$$\int_G \chi(g^2) \, dg = \epsilon_1(\pi).$$

These *Frobenius-Schur numbers* have a concrete interpretation. If $\epsilon_1(\pi) = 1$, then $\pi(G)$ is conjugate to a subgroup of the orthogonal group $O(n)$, where $n = \dim(V)$. If $\epsilon_1(\pi) = -1$, then $\pi(G)$ is conjugate to a subgroup of the symplectic group $\mathrm{Sp}(n)$, and n is even. We therefore say that π is *orthogonal* or *symplectic* if $\epsilon_1(\pi) = 1$ or -1, respectively.

An important generalization of this result was obtained by Kawanaka and Matsuyama [18]. Let τ be an automorphism of G satisfying $\tau^2 = 1$. Define $\varepsilon_\tau(\pi)$ as follows. If there exists a nondegenerate bilinear form B on V such that

$$B(\pi(g)v, \pi({}^\tau g)w) = B(v, w) \tag{3}$$

then by the irreducibility of V, B is unique. The form $B(w, v)$ has the same property, so there exists a constant ϵ such that

$$B(v, w) = \epsilon B(w, v).$$

Moreover $B(v, w) = \epsilon^2 B(v, w)$ so $\epsilon = \pm 1$. In this case we denote $\epsilon = \epsilon_\tau(\pi) = \epsilon_\tau(\chi)$. If no nondegenerate bilinear form exists satisfying (3), let $\epsilon_\tau(\pi) = 0$. Kawanaka and Matsuyama proved that

$$\epsilon_\tau(\pi) = \int \chi(g \cdot {}^\tau g) \, dg. \tag{4}$$

We are normalizing the Haar measure so the volume of G is one.

We will reformulate this result of Kawanaka and Matsuyama in preparation for a more substantial generalization to multilinear forms. We begin by extending the definition of $\epsilon_\tau(g)$ to representations that are not necessarily irreducible. Let (π, V) be an arbitrary representation, not necessarily irreducible, and let χ be its character. Let $E_\tau(\pi)$ be the space of bilinear forms B on V satisfying (3). Define an endomorphism $c : E_\tau(\pi) \longrightarrow E_\tau(\pi)$ by

$$cB(v, w) = B(w, v). \tag{5}$$

Then we define $\epsilon_\tau(\pi)$ to be the trace of c. Clearly this coincides with the previous definition when π is irreducible.

Theorem 1 *If τ is an automorphism of the compact group G such that $\tau^2 = 1$, and (π, V) is a finite dimensional representation, then (4) holds.*

Rather than deduce this from the result of Kawanaka and Matsuyama, we will instead prove a more general result in Theorem 9 below.

If π is irreducible, Theorem 1 is the theorem of Kawanaka and Matsuyama. One might expect that the general case follows from the special case in a simple way. Indeed the right side of (4) is linear as a function of χ. So one might be tempted to assume that the space of bilinear forms satisfying (3) decomposes into a direct sum of the corresponding spaces for the irreducible constituents of χ and that the trace of c is therefore the sum of the traces over these irreducible subspaces.

That this expectation is not true may be seen by an example. Let G be the cyclic group of order 3, and let χ_1, χ_2 be the two nontrivial characters of G, both of order 3. We take $\tau = 1$. Then $\epsilon_1(\chi_i) = 0$, and there are no bilinear forms satisfying (3) for either χ_i. On the other hand, let $\chi = \chi_1 + \chi_2$. Since the right side of (4) is linear as a function of χ, we must still have $\epsilon_1(\chi) = 0$. Now, however, there are *two* invariant bilinear forms. The trace of c remains zero, but the space of bilinear forms does not decompose into subspaces corresponding to the irreducible constituents.

2 Combinatorial interpretation of the "Model"

If G is a finite group, we will denote by $\mathcal{M}$ a representation which contains every irreducible representation of G exactly once. Let $\chi_{\mathcal{M}}$ be its character. The values of $\chi_{\mathcal{M}}$ are algebraic integers, and since $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ permutes the constituents of $\mathcal{M}$, they are rational integers. There is a strong tendency for these values to be nonnegative, so strong that one might wonder whether this is always true.

Solomon [23] proved that the *row* sums of the character table are nonnegative. This result is reproduced in Feit [8] p. 34, who remarks that Solomon and Thompson had pointed out that the *column* sums of the character table (which are the values of $\chi_{\mathcal{M}}$) can be negative. Still, we had to look fairly hard to find an example. The Mathieu group M_{11} has the property that $\chi_{\mathcal{M}}$ takes negative values for two conjugacy classes.

Given that the values of $\chi_{\mathcal{M}}$ are, for many groups, nonnegative integers, it is natural to ask whether $\chi_{\mathcal{M}}$ has a combinatorial interpretation. Is, perhaps, $\chi_{\mathcal{M}}(a)$ the cardinality of some set naturally depending on $\mathcal{M}$? We now prepare to prove Theorem 3, where such an interpretation will be given, for certain groups.

Proposition 1 *Let G be a finite group, and let $a \in G$. Then $\sum_\pi \overline{\chi_\pi(a)}\, \epsilon_\tau(\pi)$ is the number of solutions g to the equation $g \cdot {}^\tau g = a$. (The sum is over all irreducible representations π.)*

Proof Substituting the definition of $\epsilon_1(\pi)$ and interchanging the order of summation gives

$$\sum_g \left[\frac{1}{|G|} \sum_\pi \overline{\chi_\pi(a)}\, \chi_\pi(g \cdot {}^\tau g) \right].$$

By Schur orthogonality, the expression in brackets is zero unless $g\,{}^\tau g$ is a conjugate of a, in which it is the reciprocal of the cardinality of the conjugacy class of a. The result follows by simple counting provided we know that the cardinality of the set $S(a)$ of solutions to $g \cdot {}^\tau g = a$ is constant on conjugacy classes. If $b = hah^{-1}$, then $g \longrightarrow hg\,{}^\tau h^{-1}$ is a bijection $S(a) \longrightarrow S(b)$, as required. $\square$

Proposition 2 *A necessary and sufficient condition that $\epsilon_\tau(\pi) = \pm 1$ for all irreducible π is that every $g \in G$ is conjugate to ${}^\tau g^{-1}$.*

Proof The condition $\epsilon_\tau(\pi) = \pm 1$ amounts to the existence of a bilinear form B on the space V of π such that $B(\pi(g)\, v_1,\, v_2) = B(v_1, \pi({}^\tau g^{-1})v_2)$. Therefore $\pi(g)$ and $\pi({}^\tau g^{-1})$ are adjoints with respect to B and have the same trace. Thus we see that all irreducible characters take the same value on g and ${}^\tau g^{-1}$, so they are conjugate. $\square$

The following fact is well-known in the special case where $\tau = 1$. The case where $z = 1$ is in Howlett and Zworestine [15]. Motivated by examples of Prasad [21], Steinberg [25] and Gow [12], and Proposition 3 below, we generalize this by using a central element z (which must be of order two). If (π, V) is an irreducible representation of G, let $\omega_\pi : Z(G) \longrightarrow \mathbb{C}^\times$ be the central character of π, so $\pi(z)v = \omega_\pi(z)v$ for all $v \in V$.

Theorem 2 *Let G be a finite group, and let τ be an automorphism satisfying $\tau^2 = 1$. Let $z \in Z(G)$ such that $z^2 = 1$. The following are equivalent:*
(i) Every irreducible representation of G satisfies $\epsilon_\tau(\pi) = \omega_\pi(z)$;
(ii) The dimension of $\mathcal{M}$ is equal to the number of $g \in G$ such that $g \cdot {}^\tau g = z$.

For example, if $G = S_n$ these conditions are satisfied with $\tau = 1$ and $z = 1$. Both assertions have concrete interpretations: for (i), the irreducible representations of S_n can be constructed over $\mathbb{Q}$, *a fortiori* over $\mathbb{R}$, and are therefore orthogonal. For (ii), the Robinson–Schensted correspondence can be used to show that the number of $g \in S_n$ such that $g^2 = 1$ is equal to the number of standard tableaux, which in turn is equal to the sum of the degrees of the irreducible representations of G.

Proof Using Proposition 1 and the identity $\chi_\pi(az)\omega_\pi(z) = \chi_\pi(a)$, (i) implies (ii).

On the other hand, assuming (ii), take $a = z$. Hypothesis (ii) means that $\sum_\pi [\omega_\pi(z)\epsilon_\tau(\pi)]\chi_\pi(1) = \sum_\pi \chi_\pi(z)\epsilon_\tau(\pi) = \sum_\pi \chi_\pi(1)$. Since each $|\omega_\pi(z)\epsilon_\tau(\pi)| \leqslant 1$, and since the $\chi_\pi(1) > 0$, this implies that the numbers $\omega_\pi(z)\epsilon_\tau(\pi)$ have absolute value 1, and the same complex argument. When π is the trivial representation, $\omega_\pi(z)\epsilon_\tau(\pi) = 1$, so this is true for all π. □

Theorem 3 *If the equivalent conditions of Theorem 2 are satisfied, then $\chi_{\mathcal{M}}(a)$ is the number of solutions to the equation $g \cdot {}^\tau g = az$.*

Thus if the equivalent conditions of Theorem 2 are satisfied, we have a suitable combinatorial interpretation of $\chi_{\mathcal{M}}$.

Proof This is immediate from Proposition 1. □

Let us recall results of Gow [13] and Klyachko [19]. Independently, Gow and Klyachko proved two theorems, and proved the equivalence of them. Let G be $\mathrm{GL}(n, \mathbb{F}_q)$, and let (π, V) be an irreducible representation.

Theorem 4 (Gow, Klyachko) *Let (π, V) be an irreducible representation of $\mathrm{GL}(n, \mathbb{F}_q)$, q odd. There exists a symmetric bilinear form B on V such that*

$$B(\pi(g)v, \pi({}^tg^{-1})w) = B(v, w). \tag{6}$$

Theorem 5 (Gow, Klyachko) *The sum of the dimensions of the irreducible representations of $G = \mathrm{GL}(n, \mathbb{F}_q)$ (q odd) is equal to the number of symmetric matrices in G.*

These papers were written before Kawanaka and Matsuyama [18], which clarifies the equivalence of these two facts. The first theorem asserts that if ${}^{\tau}g = {}^{t}g^{-1}$ then $\epsilon_\tau(\pi) = 1$ for all irreducible representations. As Howlett and Zworestine noted, Theorem 2 shows the equivalence in a transparent way.

Macdonald [20] removed the assumption that q is odd. Fulman and Guralnick [11] in a remarkable paper give explicit formulas for $\chi_{\mathcal{M}}$ at *all* conjugacy classes of $G = \mathrm{GL}(n, \mathbb{F}_q)$, and striking interpretations of these results.

When $n = 2$, in addition to the involution τ, there is another involution θ of $\mathrm{GL}(2, \mathbb{F}_q)$ which can be used to illustrate the need for a central element z in Theorem 2. If $g \in \mathrm{GL}(2, \mathbb{F}_q)$, let ${}^{\theta}g = \det(g)^{-1} \cdot g$.

Proposition 3 *If* (π, V) *is an irreducible representation of* $\mathrm{GL}(2, \mathbb{F}_q)$*, then* $\epsilon_\theta(\pi) = \omega_\pi(-I)$.

Proof We note the identity

$$ {}^{\tau}g = \eta \cdot {}^{\theta}g \cdot \eta^{-1}, \qquad \eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. $$

By Theorem 4, there exists a symmetric bilinear form B on V satisfying (6). Define another bilinear form T on V by $T(v, w) = B(v, \pi(\eta)w)$. Then $T(\pi(g)v, \pi({}^{\theta}g)w) = T(v, w)$. Since B is symmetric, and since ${}^{\tau}\eta = \eta$,

$$ T(w, v) = B(w, \pi(\eta)v) = B(\pi(\eta)v, w) = B(v, \pi(\eta^{-1})w). $$

As $\eta^{-1} = -\eta$, this equals $T(v, \pi(-I)w) = \omega_\pi(-I)T(v, w)$ and the result follows. $\square$

As another example, let $G = \mathrm{SL}(2, \mathbb{F}_q)$. If $q \equiv 1 \bmod 4$, then every irreducible representation is self-contragredient, that is $\epsilon_1(\pi) = \pm 1$, for it is easily seen that every element is conjugate to its inverse. This is easily checked for the semisimple conjugacy classes, and for the unipotent ones, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is conjugate to its inverse by $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, where $\alpha^2 = -1$. Moreover, it follows from Gow [12] Theorem 1 that $\epsilon_1(\pi) = \omega_\pi(-I)$. If $q \equiv 3 \bmod 4$, however, it is not true that every irreducible representation is self-contragredient. It is still true that *most* conjugacy classes of $\mathrm{SL}(2, \mathbb{F}_q)$ are conjugate to their inverses, the two unipotent classes being the only exceptions. And it is still true that *most* irreducible representations are self-dual, in fact all but four are, two each of degrees $\frac{1}{2}(q+1)$ and $\frac{1}{2}(q-1)$ being the only exceptions. And

Gow's result still asserts that when $\epsilon_1(\pi) \neq 0$, it equals $\omega_\pi(-I)$. These four exceptions are a defect which can be eliminated by introducing an involution. Let τ denote conjugation by

$$\zeta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_q).$$

This has the advantage that ${}^\tau g \sim g^{-1}$ for all g, even the unipotent elements, so that $\epsilon_\tau(\pi) = \pm 1$ for all π.

Proposition 4 *If $G = \mathrm{SL}(2, \mathbb{F}_q)$ and τ is conjugation by ζ, then $\epsilon_\tau(\pi) = 1$ for all irreducible representations (π, V).*

Proof It may be computed that $\dim \mathcal{M} = q(q+1)$ for this group. On the other hand, the solutions to $g \cdot {}^\tau g = I$ are the matrices

$$\left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} | a^2 + bc = 1 \right\},$$

and it is easy to count that there are $q(q+1)$ of these. The result follows by Theorem 2. □

Propositions 3 and 4 both have generalizations to the symplectic group, due to Vinroot [26]. Let

$$J = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}, \qquad \zeta = \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}.$$

and for any field F let

$$\mathrm{Sp}(2n, F) = \{g \in \mathrm{GL}(2n, F) | g\, J\, {}^t g = J\},$$
$$\mathrm{GSp}(2n, F) = \{g \in \mathrm{GL}(2n, F) | g\, J\, {}^t g = \mu(g) J\},$$

so that $\mu : \mathrm{GSp}(2n, F) \longrightarrow F^\times$ is a character. We have automorphisms θ and τ of $\mathrm{GSp}(2n, \mathbb{F}_q)$ and $\mathrm{Sp}(2n, \mathbb{F}_q)$, respectively, defined by

$${}^\theta g = \mu(g)^{-1} g, \qquad {}^\tau g = \zeta g \zeta^{-1}.$$

Theorem 6 (Vinroot) *(i) If (π, V) is an irreducible representation of $\mathrm{GSp}(2n, \mathbb{F}_q)$, then $\epsilon_\theta(\pi) = \omega_\pi(-I_{2n})$.*
(ii) If (π, V) is an irreducible representation of $\mathrm{Sp}(2n, \mathbb{F}_q)$, then $\epsilon_\tau(\pi) = 1$.

If $n = 1$, $\mathrm{GSp}(2n) = \mathrm{GL}(2)$ and $\mathrm{Sp}(2n) = \mathrm{SL}(2)$, so we have already proved this in this case. If $n > 1$ and $q \equiv 1 \bmod 4$, Vinroot's theorem may be deduced from results of Gow [12]. If $n > 1$ and $q \equiv 3 \bmod 4$, however, Vinroot's results improve Gow's results.

3 Generalized involution models

The term "model" is used in two different ways, which we now explain. A *model* of a (typically irreducible) representation π is an embedding of π in a multiplicity free induced representation, typically induced from a one-dimensional representation of a subgroup of G. The project of Bernstein, Gelfand and Gelfand [3] is to find several subgroups $H_1, \cdots, H_n$ of G and characters ψ_i (typically one dimensional) of H_i such that

$$\bigoplus_i \mathrm{Ind}_{H_i}^G(\psi_i) \cong \mathcal{M}. \tag{7}$$

Of course this means that $\mathrm{Ind}_{H_i}^G(\psi_i)$ is multiplicity free, so we obtain a model for every irreducible representation of G. Following Bernstein, Gelfand and Gelfand, such data are sometimes called a *model* for G.

Let us assume that G satisfies the equivalent conditions of Theorem 2. Then we are presented with a set of candidates for the groups H_i. Indeed, let G act on $X = \{x \in G | x \cdot {}^\tau x = z\}$ by *twisted conjugacy*: $g : x \longrightarrow g\, x\, {}^\tau g^{-1}$. Let $H_1, \cdots, H_n$ be the stabilizers of a set of orbit representatives for this action. Each orbit has $[G : H_i]$ elements, so $\sum_i [G : H_i] = |X| = \dim \mathcal{M}$. This numerical equality is a precondition for (7).

If $\tau = 1$ and $z = 1$, then the H_i are precisely the centralizers of involutions, and such a model is called an *involution model*. (For the purpose of this definition, the identity element of G is considered an involution.) A good example is for S_n, where the model was constructed by Inglis, Richardson and Saxl [16], and independently (details unpublished) by Klyachko. Baddeley [1] shows that most but not all Weyl groups have involution models.

If $\tau \neq 1$, it is natural to call a model for G arising from a set of stabilizers for the action of G on X by twisted conjugacy a *generalized involution model*. We will see that such a model may not exist, even if the conditions of Theorem 2 are satisfied. In all the examples that we will consider, $z = 1$.

If $G = \mathrm{GL}(n, \mathbb{F}_q)$, Klyachko gave a model for G in connection with Theorem 5. A relationship can be seen between Klyachko's model and the model of Inglis, Richardson and Saxl. To explain this point, Klyachko's models can be understood as follows. Let $2r < n$. In the parabolic subgroup with Levi factor $\mathrm{GL}(2r) \times \mathrm{GL}(n-2r)$, let H_r be the group

$$\left\{ \begin{pmatrix} h & * \\ 0 & k \end{pmatrix} | h \in \mathrm{Sp}(2r), k \in \mathrm{GL}(n-2r) \right\}.$$

We consider the representation $1 \otimes \Gamma_{n-2r}$, where Γ_{n-2r} is the Gelfand-Graev representation of $\mathrm{GL}(n-2r)$. Inducing to $\mathrm{GL}(n, \mathbb{F}_q)$ and summing over r gives $\mathcal{M}$. The key observation is that Weyl group of H_r is $W(C_r) \times S_{n-2r}$, where $W(C_r)$ is the Weyl group of $\mathrm{Sp}(2r)$ realized as a subgroup of S_{2r}, and these groups are precisely the centralizers of involutions in S_n used in the involution model.

Inglis and Saxl [17], being aware of this connection, and gave a variant of Klyachko's model in which this relationship between Klachko's $\mathrm{GL}(n)$ model and the models for the symmetric group is made clarified. But despite this relationship with the involution model for S_n, Klyachko's model is itself *not* a generalized involution model.

We show that $\mathrm{GL}(2, \mathbb{F}_q)$ does not have a generalized involution model with ${}^{\tau}g = {}^{t}g^{-1}$. In this case the subgroups H_1 and H_2 are the orthogonal groups of the two equivalence classes of binary quadratic forms. Their orders are $|H_1| = 2(q-1)$ and $|H_2| = 2(q+1)$.

Theorem 7 *In this setting, there do not exist characters ψ_i of the H_i such that (7) is satisfied.*

Proof Let W be the cyclic subgroup of $G = \mathrm{GL}(2, \mathbb{F}_q)$ generated by $-I$. Thus W has order two and is contained in the center of G. Then $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$, where $\mathcal{M}^+$ is the sum of irreducible representations on which W acts trivially, and $\mathcal{M}^-$ is the sum of the irreducible representation on which it acts nontrivially. We have

$$\dim(\mathcal{M}^+) = \frac{1}{2}(q^2+1)(q-1), \qquad \dim(\mathcal{M}^-) = \frac{1}{2}(q-1)^2(q+1).$$

Assume that ψ_1 and ψ_2 are characters of H_1 and H_2 such that (7) is satisfied. If $\psi_i(-I) = 1$, then every irreducible character in the induced module ψ_i^G is in $\mathcal{M}^+$, and if $\psi_i(-1) = -1$, then every irreducible character in ψ_i^G is in $\mathcal{M}^-$. We must have $\psi_1^G = \mathcal{M}^-$ and $\psi_2^G = \mathcal{M}^+$ or $\psi_1^G = \mathcal{M}^+$ and $\psi_2^G = \mathcal{M}^-$. However the indices of H_1 and H_2 in G are

$$[G : H_1] = \frac{1}{2}q(q^2-1), \qquad [G : H_2] = \frac{1}{2}q(q-1)^2.$$

It is therefore impossible. □

On the other hand, a variant of this setup *does* produce a model for $\mathrm{PGL}(2, \mathbb{F}_q)$, where q is odd. The following construction is similar to one proposed by Soto-Andrade [24]. However Soto-Andrade makes exceptions for the one-dimensional representations, and we do not. In $\mathrm{PGL}(2,\mathbb{F}_q)$, let H_1 be the normalizer of the diagonal torus, let H_2 be the normalizer of an anisotropic torus H_2°, and let $H_3 = G$. We take ψ_1 to be the trivial character of H_1, ψ_2 to be the quadratic character of H_2 whose kernel is H_2°, and let ψ_3 be the unique nontrivial one-dimensional character of $H_3 = G$.

Theorem 8 *For $G = \mathrm{PGL}(2,\mathbb{F}_q)$, we have $\mathcal{M} = \psi_1^G \oplus \psi_2^G \oplus \psi_3^G$. The groups H_i are the centralizers of the three conjugacy classes of involutions, so this model is an involution model.*

Proof We have $\sum_i [G : H_i] = q^2 + 1 = \dim(\mathcal{M})$, so it is enough to show that $\psi_1^G \oplus \psi_2^G \oplus \psi_3^G$ is multiplicity-free. The restriction of ψ_3 to H_1 and H_2 differs from ψ_1 and ψ_2 so does not occur in $\psi_1^G \oplus \psi_2^G$. It is sufficient to show that ψ_1^G and ψ_2^G are multiplicity-free and disjoint.

Lemma 1 (Gelfand) *Let G be a group and let $\iota : G \longrightarrow G$ be an anti-commutative involution. Let H be a subgroup of G and ψ a character of H such that H and ψ are stabilized by ι. Assume that every double coset HgH is ι-invariant, and has a representative g such that ${}^\iota g = h_1 g h_2$ where $\psi(h_1)\psi(h_2) = 1$. Then ψ^G is multiplicity-free.*

Proof By the geometric form of Mackey's theorem the endomorphism ring of ψ^G consists of the convolution ring of functions Δ on G satisfying $\Delta(hgh') = \psi(h)\,\Delta(g)\,\psi(h')$ for $h,\ h' \in H$ (see Bump [4], Proposition 4.1.2). Defining ${}^\iota\Delta(g) = \Delta({}^\iota g)$, since ι is anticommutative, ${}^\iota(\Delta_1 * \Delta_2) = {}^\iota\Delta_2 * {}^\iota\Delta_1$. The assumption that every double coset is ι-invariant means that ${}^\iota\Delta = \Delta$, since with g as in the statement of the Lemma, we have $\Delta({}^\iota g) = \Delta(g)$. Therefore $\mathrm{End}_G(\psi^G)$ is commutative. It follows that ψ^G is multiplicity free. $\square$

Returning to the proof of the theorem, we may apply the lemma with $H = H_1$, $\psi = \psi_1$ and ι the transpose involution. We see that ψ_1^G is multiplicity free.

We may take H_2° to be

$$\left\{ \begin{pmatrix} a & b \\ -\beta b & a \end{pmatrix} | a^2 + \beta\, b^2 \neq 0 \right\},$$

with $-\beta$ a fixed nonsquare in $\mathbb{F}_q^\times$. We may then apply the Gelfand Lemma with

$${}^\iota g = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & 1 \end{pmatrix} {}^t g \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}.$$

We may take the coset representatives for $H_2\backslash G/H_2$ to be diagonal. We see that ψ_2^G is multiplicity free.

Finally we must show that ψ_1^G and ψ_2^G are disjoint. By Mackey's theorem, it is sufficient to show that for every double coset H_1gH_2 there exists $\gamma \in H_1$ such that $g^{-1}\gamma g \in H_2$ and $\psi_1(\gamma) \neq \psi_2(g^{-1}\gamma g)$. If this is true for one coset representative g it is true for every representative. We may take the representative g to be a unipotent matrix $g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. Now choose $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right) \in H_1$, where $m = \frac{\beta}{1+x^2\beta}$. Then $\psi_1(\gamma) = 1$ while $\psi_2(g^{-1}\gamma g) = -1$. $\square$

The alternating groups A_n give interesting examples and counterexamples for generalized involution models. The involution τ will be trivial for some n, and for others, it will be conjugation by the transposition $(12) \in S_n - A_n$.

Proposition 5 *In A_n $(n > 2)$ every element is conjugate to its inverse if $n = 5,\ 6,\ 10$ or 14. If $n = 3,\ 4,\ 7,\ 8$ or 12, then every element g is conjugate to $(12)g(12)$. These are the only cases where there exists an involution τ of A_n such that g is conjugate to ${}^{\tau}g^{-1}$ for all $g \in A_n$.*

The first statement, about A_n when n is conjugate to its inverse is also proved in Berggren [2].

Proof For any $g \in A_n$, g is conjugate to its inverse in S_n. If the centralizer $C_{S_n}(g)$ is not contained in A_n then it follows that $g \sim g^{-1}$ in A_n, too. It is easy to see that if $C_{S_n}(g) \subseteq A_n$ then the cycle type of g is a partition of n into distinct odd integers. If we write n as a sum of k distinct odd integers, then g is conjugate to $(12)g^{-1}(12)$ if $\frac{1}{2}(n-k)$ is even, and g is conjugate to g^{-1} if $\frac{1}{2}(n-k)$ is odd. The partitions of n into distinct odd integers are:

n	partition	k	$\frac{1}{2}(n-k)$
3	3	1	1
4	3+1	2	1
5	5	1	2
6	5+1	2	2
7	7	1	3
8	7+1 or 5+3	2	3
9	9 or 5+3+1	1 or 3	4 or 3
10	9+1 or 7+3	2	4
11	11 or 7+3+1	1 or 3	5 or 4
12	11+1 or 9+3 or 7+5	2	5
13	13 or 9+3+1 or 7+5+1	1 or 3	6 or 5
14	13+1 or 11+3 or 9+5	2	6
15	15 or 11+3+1 or 9+5+1 or 7+5+3	1 or 3	7 or 6
16	15+1 or 7+5+3+1 etc.	2 or 4	7 or 6

Table 1: Partitions into odd and unequal parts.

If distinct partitions with $\frac{1}{2}(n-k)$ of both parities can be found, clearly no τ exists. This is true for odd $n \geqslant 9$ and even $n \geqslant 16$. The result is now evident. $\square$

Now let us investigate which of these have a generalized involution model. Let τ be the identity $A_n \longrightarrow A_n$ if $n = 5, 6, 10$ or 14, and τ is conjugation by (12) if $n = 3, 4, 7, 8$ or 12. Let X be the set of solutions to $g\,{}^{\tau}g = 1$. If $\tau = 1$, this is the set of involutions (i.e. elements satisfying $g^2 = 1$) in A_n. On the other hand, if τ is conjugation by (12), then $g \in X$ if and only if $g(12) = h$ is an involution in $S_n - A_n$. The map $g \longrightarrow g(12)$ also helps us to understand the action of $G = A_n$ on X. The action on X is twisted conjugacy, but the corresponding action on involutions in $S_n - A_n$ is ordinary conjugacy. Thus the stabilizers H_i in (7) will be the centralizers of a set of representatives of the conjugacy classes of involutions in A_n when τ is trivial; and when τ is not trivial, they will be the centralizers in A_n of the A_n-conjugacy classes of involutions in $S_n - A_n$.

A_3 has a generalized involution model: there is one conjugacy class of involutions in $S_3 - A_3$, and the stabilizer is the group $\{1\}$. Inducing its unique irreducible character gives $\mathcal{M}$.

A_4 also has a generalized involution model with τ nontrivial. There is one A_4 conjugacy class of involutions in $S_4 - A_4$, represented by (12). The centralizer in A_4 is the cyclic group $H = \langle (12)(34) \rangle$ of order 2. Inducing the trivial character of H gives $\mathcal{M}$.

A_5 has a generalized involution model with $\tau = 1$. There are two conjugacy classes of involutions in A_5, representatives being 1 and $(12)(34)$. The centralizers are $H_1 = A_5$ and the abelian subgroup H_2 of order 4 generated by $(12)(34)$ and $(13)(24)$. Inducing any nontrivial character ψ_2 of H_2 gives a 15 dimensional representation of A_5 containing every nontrivial irreducible representation precisely once, so we take ψ_1 to be the trivial representation of H_1.

A_6 has a generalized involution model with $\tau = 1$. There are two conjugacy classes of involutions in A_6, representatives being 1 and $(12)(34)$. The centralizers are $H_1 = A_6$ and the dihedral group H_2 of order 8 with generators $u = (1324)(56)$ and $t = (12)(56)$, satisfying $tut^{-1} = u^{-1}$, $t^2 = u^4 = 1$. The group H_2 has four one-dimensional characters, two of which take value -1 on u. Choosing ψ_2 to be either of these, and ψ_1 to be the trivial character gives a generalized involution model.

A_7 has a generalized involution model with $\tau \neq 1$. There are two A_7-conjugacy classes of involutions in $S_7 - A_7$. The first has centralizer $H_1 = C_{A_7}((67))$. We may construct an isomorphism $S_5 \longrightarrow H_1$ by mapping $\sigma \in S_5$ (acting on $\{1, 2, 3, 4, 5\}$) to itself if σ is even, and to $\sigma(67)$ if σ is odd. We take ψ_1 to be the trivial character of H_1. The second centralizer is $H_2 = C_{A_7}((12)(34)(56))$. It is a group of order 24 which may be identified with S_4. To construct an isomorphism $H_2 \longrightarrow S_4$ note that H_2 has four 3-Sylow subgroups, $\langle (135)(246) \rangle$, $\langle (136)(245) \rangle$, $\langle (145)(236) \rangle$ and $\langle (146)(235) \rangle$. Acting on these by conjugation gives a faithful permutation representation and an isomorphism $H_2 \longrightarrow S_4$. We take ψ_2 to be the nontrivial one-dimensional character of H_2. Labeling the irreducible characters as in the Atlas of Finite Groups [5], we find that $\psi_1^G = \chi_1 + \chi_2 + \chi_5$, while $\psi_2^G = \chi_3 + \chi_4 + \chi_6 + \chi_7 + \chi_8 + \chi_9$, and so we have a generalized involution model in this case, too.

A_8 has *no* generalized involution model with τ conjugation by (12). Indeed, there are two conjugacy classes of involutions in $S_8 - A_8$, one of which is $(12)(34)(56)$. Its centralizer is a group of order 48 which admits four characters of degree one. No matter which of these is chosen, the induced representation of A_8 is not multiplicity free, so no model can be constructed using this subgroup.

We do not know whether generalized involution models exist in the remaining case of A_{10}, A_{12} and A_{14}.

4 Higher order automorphisms

Let r be a positive integer, and let τ be an automorphism of the compact group G such that $\tau^r = 1$. As before, *twisted conjugacy* is the action of G on itself given by $g : x \longrightarrow g \cdot x \cdot {}^\tau g^{-1}$. A clue to the meaning of twisted conjugacy is given by the following result.

Proposition 6 *Suppose that G is finite, and τ is an automorphism of G. The number of twisted conjugacy classes of G with respect to τ equals the number of τ-invariant irreducible representations of G.*

This is implied by Proposition 1.2 of Digne [7], which is stated without proof; it was most likely known to Shintani. We give a simple counting argument.

Proof There are two permutation actions of $\langle\tau\rangle$ which we may consider: the action on the conjugacy classes of G, and the action on the irreducible representations of G. Both permutation actions are realized on the center of $\mathbb{C}[G]$, the first by taking the basis of the center comprised of the conjugacy class sums, the second by taking the basis of central idempotents parametrized by the irreducible representations. Thus these actions are equivalent, and the number of τ invariant representations equals the number of τ-invariant conjugacy classes.

We must therefore show that the number of τ-invariant conjugacy classes equals the number of twisted conjugacy classes. We will count the number N of solutions (x, g) to the equation $g^{-1}xg = {}^\tau x$ in two different ways.

First, N equals

$$\sum_{x \sim {}^\tau x} \#\{g \in G | g^{-1}xg = {}^\tau x\} = \sum_{x \sim {}^\tau x} |C_G(x)|$$

because given x which is conjugate to ${}^\tau x$, we may count the number of g conjugating x to ${}^\tau x$ by fixing one such g, then noting that any other such g must differ from that one by an element of the centralizer $|C_G(x)|$. Because the number of elements of the conjugacy class of x is $|G|/|C_G(x)|$, we see that N is $|G|$ times the number of τ-invariant conjugacy classes.

On the other hand, N equals

$$\sum_{g \in G} \#\{x \in G | xg^\tau x^{-1} = g\}.$$

The index in G of $\#\{x \in G | xg^\tau x^{-1} = g\}$ is the order of the twisted conjugacy class of g, so this is $|G|$ times the number of twisted conjugacy classes. $\square$

Returning to the more general case of a compact group, let (π, V) be a complex representation of G with character η. We do not assume that π is irreducible. Let (σ, W) be another representation of G. We assume that σ is irreducible, and moreover that ${}^\tau\sigma \cong \sigma$. This implies that there is a linear transformation $J : W \longrightarrow W$ such that $J \circ \sigma(g) = \sigma({}^\tau g) \circ J$.

Since we are assuming σ is irreducible, Schur's Lemma implies that J^r acts by a scalar on W. We may therefore choose a normalization of J so that J^r is the identity transformation of W.

Let $E_\tau(\pi, \sigma)$ be the vector space of r-linear maps $T : V \times \cdots \times V \longrightarrow W$ which satisfy

$$T(\pi(g)v_1, \pi({}^\tau g)v_2, \cdots, \pi({}^{\tau^{r-1}}g)v_r) = \sigma(g)T(v_1, \cdots, v_r). \tag{8}$$

We define a linear transformation $c_r : E_\tau(\pi, \sigma) \longrightarrow E_\tau(\pi, \sigma)$ by

$$(c_r T)(v_1, \cdots, v_r) = J\, T(v_r, v_1, \cdots, v_{r-1}). \tag{9}$$

Let $\epsilon_\tau(\pi, \sigma)$ be the trace of c_r on $E_\tau(\pi, \sigma)$.

We will denote by $N : G \longrightarrow G$ the map

$$N(g) = g \cdot {}^\tau g \cdot {}^{\tau^2} g \cdots {}^{\tau^{r-1}} g.$$

Theorem 9 *We have*

$$\epsilon_\tau(\pi, \sigma) = \int_G \overline{\eta(N(g))}\ \mathrm{tr}(J \circ \sigma(g)) dg. \tag{10}$$

Proof We will make use of the group $\hat{G}$ which is the semidirect product of G by a cyclic group $\langle t \rangle$ with generator t of order t, acting on G by conjugation according to the automorphism τ: $tgt^{-1} = {}^\tau g$ for $g \in G$. Let $U = V \otimes \cdots \otimes V$. We will give it a $\hat{G}$ module structure which was noted by Shintani [22], Lemma 1-4. Specifically, let G act by the representation

$$\Pi(g) = \pi(g) \otimes \pi({}^\tau g) \otimes \ldots \otimes \pi({}^{\tau^{r-1}} g),$$

which we extend to $\hat{G}$ by letting

$$\Pi(t)(v_1 \otimes \ldots \otimes v_r) = v_2 \otimes \ldots \otimes v_r \otimes v_1.$$

The content of Shintani's Lemma 1-4 is that if $g \in G$ then

$$\operatorname{tr} \Pi(tg) = \eta(N(g)). \tag{11}$$

Let us verify this. This follows immediately once we check that if $A_1, \cdots, A_r$ are endomorphisms of V, then the trace of the endomorphism

$$v_1 \otimes \cdots \otimes v_r \longrightarrow A_2 v_2 \otimes \cdots \otimes A_r v_r \otimes A_1 v_1 \tag{12}$$

is $\operatorname{tr}(A_1 A_2 \cdots A_r)$. If $x_1, \cdots, x_n$ is a basis of V, and if $A_i x_j = \sum_k a_{ikj} x_k$, then (12) takes $x_{i_1} \otimes \cdots \otimes x_{i_r}$ to

$$\sum_{j_1, \cdots, j_r} a_{2j_2i_2} a_{3j_3i_3} \cdots a_{rj_ri_r} a_{1j_1i_1} x_{j_2} \otimes \cdots \otimes x_{j_r} \otimes x_{j_1}.$$

The trace is the sum of the contributions with $j_2 = i_1, j_3 = i_2$, etc., that is,

$$\sum_{i_1, \cdots, i_r} a_{2i_1i_2} a_{3i_2i_3} \cdots a_{ri_{r-1}i_r} a_{1i_ri_1} = \operatorname{tr}(A_2 A_3 \cdots A_1) = \operatorname{tr}(A_1 A_2 \cdots A_r).$$

This proves (11).

We may also extend the representation σ of G on W to $\hat{G}$ by letting $\sigma(t) = J$.

Now let T be an r-linear form on V satisfying (8). By the universal property of the tensor product, T induces a map $U \longrightarrow W$ which is G-equivariant. Thus we may identify the space $E_\tau(\pi, \sigma)$ of such linear maps with $\operatorname{Hom}_G(U, W)$. We give $\operatorname{Hom}_G(U, W)$ a $\hat{G}$-module structure by the representation $\lambda(\hat{g}) f = \sigma(\hat{g}) \circ f \circ \Pi(\hat{g})^{-1}$. It is clear from the definitions that we have a commutative diagram:

$$\begin{array}{ccc} E_\tau(\pi, \sigma) & \cong & \operatorname{Hom}_G(U, W) \\ \downarrow c_r & & \downarrow \lambda(t) \\ E_\tau(\pi, \sigma) & \cong & \operatorname{Hom}_G(U, W) \end{array}$$

so $\epsilon_\tau(\pi, \sigma) = \operatorname{tr} \lambda(t)$ and this is what we must compute.

Let ζ be a primitive r-th root of unity. Let $\rho : \hat{G} \longrightarrow \mathbb{C}^\times$ be the linear character which is trivial on G, and which maps $t \longrightarrow \zeta$. We can decompose

$$\operatorname{Hom}_G(U, W) = \bigoplus_{k=0}^{r-1} \operatorname{Hom}_{\hat{G}}(U, W \otimes \rho^k),$$

and $\mathrm{Hom}_{\hat{G}}(U, W \otimes \rho^k)$ is the space of $f \in \mathrm{Hom}_G(U, W)$ for which $\lambda(t) f = \zeta^{-k} f$. Therefore

$$\epsilon_\tau(\pi, \sigma) = \mathrm{tr}\,\lambda(t) = \sum_{i=0}^{r-1} \zeta^{-k} \dim \mathrm{Hom}_{\hat{G}}(U, W \otimes \rho^k).$$

Since the dimension of $\mathrm{Hom}_{\hat{G}}(U, W \otimes \rho^k)$ is the inner product of the characters, this equals

$$\sum_{i=0}^{r-1} \zeta^k \int_{\hat{G}} \overline{\mathrm{tr}\,\Pi(\hat{g})}\,\mathrm{tr}\;\sigma(\hat{g})\,\rho(\hat{g})^k\,d\hat{g}.$$

Of course $\sum_k \zeta^{-k} \rho^k$ is r times the characteristic function of the coset $t\,G$ in $\hat{G}$. Since we are normalizing both groups to have Haar volume one, the restriction of the normalized Haar measure $d\hat{g}$ on $\hat{G}$ to G is $\frac{1}{r} dg$. Thus

$$\epsilon_\tau(\pi, \sigma) = \int_G \overline{\mathrm{tr}\;\Pi(tg)}\,\mathrm{tr}\;\sigma(tg)\,dg$$

and the theorem is proved. □

Theorem 10 *The character sum (10) is an algebraic integer, in fact an element of* $\mathbb{Z}[e^{2\pi i/r}]$.

Proof The linear endomorphism c_r of $E_\tau(\pi, \sigma)$ has order r, so its eigenvalues are r-th roots of unity. Therefore its trace is in $\mathbb{Z}[e^{2\pi i/r}]$. □

5 A formula of Frobenius

The purpose of this section is to briefly indicate a connection between Theorem 9 and Frobenius-Schur duality, and the representation theory of the symmetric group.

Let $V = \mathbb{C}^n$. The vector space $\otimes^r V$ has commuting actions of $U(n)$ and S_r, a left action of $U(n)$:

$$g \cdot (v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r, \qquad g \in U(n),$$

and a right action of S_r:

$$(v_1 \otimes \cdots \otimes v_r) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}, \qquad r \in S_r.$$

If (θ, M_θ) is a representation of S_r then $V_\theta = (\otimes^r V) \otimes_{\mathbb{C}[S_r]} M_\theta$ is a representation of $U(n)$.

It can be checked that the vector space of endomorphisms of $\otimes^r V$ which commute with the linear transformations of S_r is spanned by the linear transformations of $U(n)$. It follows that V_θ, if nonzero is irreducible, and (using the fact that the M_θ are self contragredient)

$$\otimes^r V \cong \bigoplus_\theta V_\theta \otimes M_\theta, \tag{13}$$

where θ runs through the irreducible representations of S_r such that V_θ is nonzero. (This will be all irreducible representations of S_r if $n \geqslant r$.) Let $\eta_\theta : U(n) \longrightarrow \mathbb{C}$ be the character of V_θ, and let χ_θ be the character of θ. This fact is known as Frobenius-Schur duality.

Now let λ be a partition of r. Thus $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_l)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ are positive integers and $\sum \lambda_i = r$. Let c_λ be a representation of the conjugacy class of S_r of cycle type λ. For example if $\lambda = (3, 2, 2)$ then we can take $c_\lambda = (123)(45)(67)$. A fundamental formula, known to Frobenius, asserts that if $g \in U(n)$, then

$$\prod \mathrm{tr}(g^{\lambda_i}) = \sum_\theta \chi_\theta(c_\lambda)\eta_\theta(g). \tag{14}$$

See Frobenius [9], (1.) of Section 4. Frobenius' result is stated in terms of symmetric polynomials, but it can be translated into a statement about characters of irreducible representations of unitary groups. His m is our n, and the quotient

$$\left(\sum [\lambda_1, \cdots, \lambda_m] x^{\lambda_1} \cdots x^{\lambda_m}\right) / \Delta(x_1, \cdots, x_m),$$

where the summation is over permutations of a fixed $(\lambda_1, \cdots, \lambda_m)$, is a Schur polynomial. Its values on the eigenvalues of $g \in U(m)$ is the character of an irreducible representation of $U(m)$, corresponding to our η_θ.

We will use Theorem 9 to prove (14) in the special case where $\lambda = (r)$, so c_λ is a r-cycle. The general case (14) can be deduced from the special case using considerations about induction from $S_{\lambda_1} \times S_{\lambda_2} \times \cdots$ to S_r, but we will omit this discussion. It may be instructive to ask whether (14) suggests generalizations of Theorem 9, though we have not pursued this question. We show therefore that

$$\mathrm{tr}(g^r) = \sum_\theta \chi_\theta(c_r)\eta_\theta(g),$$

where c_r is an r-cycle. Indeed, those characters of irreducible representations of $U(n)$ which are homogeneous polynomials of degree r are precisely the η_θ. Since $\mathrm{tr}(g^r)$ is a class function on $U(n)$ and a homogeneous polynomial of degree r, there are coefficients A_θ such that

$$\mathrm{tr}(g^r) = A_\theta \, \eta_\theta(g),$$

and by orthogonality,

$$A_\theta = \int \mathrm{tr}(g^r) \, \overline{\eta_\theta(g)} \, dg.$$

By Theorem 9, this is the trace of c_r on the space of r-linear forms

$$V \times \cdots \times V \longrightarrow V_\theta$$

which are $U(n)$-equivariant, or equivalently on $\mathrm{Hom}_{U(n)}(\otimes^k V, V_\theta)$ which by (13) is isomorphic to M_θ as an S_k-module. Therefore $A_\theta = \chi_\theta(c_r)$. (We have used the fact that A_θ is real.)

6 Base change for $\mathrm{GL}(n, \mathbb{F}_q)$

In this section we will consider some character sums which are similar to the ones in the previous section. We recall some results of Shintani [22] on base change for $\mathrm{GL}(n)$. Shintani found, first for representations of $\mathrm{GL}(n)$, and later for automorphic representations associated with holomorphic modular forms, character identities which led to the modern theory of base change for automorphic representations, by Langlands and by Arthur and Clozel.

We will give an apparently new formulation of Shintani's base change correspondence in terms of certain character sums. These sums resemble those of Theorem 9, but differ in that η is now not a character of G, but of its τ-fixed points.

Let $G = \mathrm{GL}(n, \mathbb{F}_{q^r})$, and let $\tau : G \longrightarrow G$ be the Frobenius automorphism, corresponding to the Galois automorphism $x \longmapsto x^q$ of $\mathbb{F}_q$. The fixed subgroup τ is $H = \mathrm{GL}(n, \mathbb{F}_q)$. Let $N : G \longrightarrow G$ be the "norm map" defined by (1). We let G act on itself by twisted conjugacy: $g : x \longrightarrow g \cdot x \cdot {}^\tau g^{-1}$. The orbits are naturally called *twisted conjugacy classes.* Since

$$N(gx^\tau g^{-1}) = gN(x)g^{-1},$$

N takes twisted conjugacy classes to ordinary conjugacy classes. It is not necessarily true that $N(x) \in H$. However $N(x)$ is conjugate in G to elements of a unique conjugacy class in H, so N induces a bijection between the twisted conjugacy classes of G and the conjugacy classes of H.

If η is a class function on H, we extend it to a class function on G by:

$$\eta(g) = \begin{cases} \eta(h) & \text{if } g \text{ is conjugate in } G \text{ to } h \in H; \\ 0 & \text{if } g \text{ is not conjugate to an element of } H. \end{cases} \tag{15}$$

The second case is only for definiteness, since η will appear principally in composition with $N : G \longrightarrow G$ and $N(g)$ is *always* conjugate to an element of H.

Suppose that χ is an irreducible character of G which is τ-invariant, that is, such that ${}^\tau\chi = \chi$. If (π, V) is an irreducible G-module affording χ, then ${}^\tau\pi \cong \pi$, so there exists an intertwining map $I : V \longrightarrow V$ such that

$$I \circ \pi(g) = \pi({}^\tau g) \circ I. \tag{16}$$

This identity only determines I up to a scalar, but Shintani's correspondence will specify I uniquely.

Theorem 11 (Shintani) *There is a bijection between the τ-invariant irreducible characters χ of G and the irreducible characters η of H. In this correspondence, we have*

$$\eta(N(g)) = \mathrm{tr}(I \circ \pi(g)), \tag{17}$$

where η is extended to a class function on G by (15), and $I : V \longrightarrow V$ is a linear transformation satisfying (16), where (π, V) is a representation affording the character χ. The map I satisfies $I^r = \pm 1$, and $I^r = 1$ if r is even.

If χ and η are related as in Shintani's theorem, then we say that χ is the *base change* of η.

Theorem 12 *Let ξ and η be irreducible characters of G and H, respectively. Then*

$$\frac{1}{|G|}\sum_{g \in G} \xi(g)\overline{\eta(N(g))} = \begin{cases} \eta(1)/\xi(1) & \text{if } \xi \text{ is the base change of } \eta; \\ 0 & \text{otherwise}. \end{cases}$$

We have extended η to a class function on G by (15).

Proof Let χ be the irreducible character of G which is the base change of η. By Shintani's theorem this means that $\eta(N(g)) = \mathrm{tr}(I \circ \pi)$ where (π, V) is a representation with character χ and $I : V \longrightarrow V$ is as in (17).

Suppose first that $\xi = \chi$. Let $\langle \cdot, \cdot \rangle$ be a G-invariant inner product on V. Then it is also $\hat{G}$-invariant. Let $v_1, \cdots, v_d$ be an orthonormal basis of V. The trace of any linear transformation $T : V \longrightarrow V$ equals $\sum_i \langle Tv_i, v_i \rangle$. Since $\eta(N(g)) = \mathrm{tr}(I \circ \pi(g)) = \mathrm{tr}(\pi(g) \circ I)$, we have

$$\frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\eta(N(g))} = \frac{1}{|G|} \sum_{g \in G} \sum_{i,j} \langle \pi(g) v_i, v_i \rangle \, \overline{\langle \pi(g) I v_j, v_j \rangle}.$$

Interchanging the order of summation and using Schur orthogonality for matrix coefficients this equals

$$\frac{1}{\chi(1)} \sum_{i,j} \langle v_i, I v_j \rangle \, \overline{\langle v_i, v_j \rangle}.$$

We have $\langle v_i, v_j \rangle = \delta_{ij}$ (Kronecker δ), so we obtain

$$\frac{1}{\chi(1)} \sum_{i} \langle v_i, I v_i \rangle = \frac{1}{\chi(1)} \overline{\mathrm{tr}(I)}.$$

But using (17) with $g = 1$, $\mathrm{tr}(I) = \eta(1)$, so this is just $\eta(1)/\chi(1)$.

If ξ and χ are different characters, we proceed similarly, but we have

$$\frac{1}{|G|} \sum_{g \in G} \sum_{i,j} \langle \pi'(g) v_i, v_i \rangle \, \overline{\langle \pi(g) I v_j, v_j \rangle}$$

where (π', V') is an irreducible representation affording ξ, and this is zero by Schur orthogonality.

□

There is considerable literature generalizing Shintani's results to other algebraic groups over finite fields, and our Theorem 12 should extend to these. Another example is as follows. Let H be a finite group (nonabelian if the example is not to be too trivial). Let $G = H \times \cdots \times H$ (r copies); let τ be the automorphism cyclicly permuting the components. An analog of Theorem 12 is true for this example, also.

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