PROBLEMS IN MULTIPLE DIRICHLET SERIES

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0. Let Φ be an irreducible simply-laced root system of rank r. In [QMDS] is defined an action of the Weyl group of Φ on the set of power series in r indeterminates. This action plays a key role in defining, establishing the functional equations of, and proving the analytic continuation of Weyl group multiple Dirichlet series constructed from quadratic twists. In particular, constructing the multiple Dirichlet series with the requisite properties is shown to be equivalent to constructing a rational function f invariant under the (somewhat complicated) Weyl group action mentioned above and satisfying certain limiting conditions. I propose here some problems related to and extending the results of [QMDS]. These notes are not meant to be complete or self-contained. I will be more precise in my talks.

1. The methods of [QMDS] give an alternative to [WMD1,WMD2,WMD3] in defining the p-parts of multiple Dirichlet series, at least in the quadratic case. What is the precise relationship between these two approaches? Presumably, Sol will talk more of this in his writeup. I just want to point out that there is a striking similarity between the Weyl Character Formula and the averaging procedure used to construct the invariant function f. Relating [QMDS] to the Gelfand-Tsetlin conjecture of [WMD3] would involve extracting information on the coefficients of a polynomial constructed as a over a Weyl group. In the case of the Weyl Character Formula, the classical multiplicity theorems—e.g. Freudenthal, Kostant, Demazure—serve this purpose. Are there analogues of any of these in our setting?

2. In the case of the rational function field $\mathbb{F}_q[t]$, there is a "duality" between the $\mathfrak{p}$-part polynomial of a multiple Dirichlet series and the multiple Dirichlet series itself. For example, let $\mathfrak{p}$ be a prime of norm p, g_i be the Gauss sum $g_i(1,\mathfrak{p})$ and τ_i the Gauss sum associated to the finite field. Then the $\mathfrak{p}$-part of the A_2 series is given by

$$1 + g_1x + g_2y + pg_1g_2xy^2 + pg_1g_2x^2y + +p^2g_1^2g_2x^2y^2$$

and the numerator of the A_2 series by

$$1 + \tau_1q\,x + \tau_1q\,y + \tau_1\tau_2q^3\,x^2\,y + \tau_1\tau_2q^3xy^2 + \tau_1^2\tau_2q^4\,x^2\,y^2.$$

I first noticed this duality for quadratic series (in which case it can be explained by a certain uniqueness property), but I find it remarkable that it

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seems to continue to be true for higher order twists—see the evidence given in [FFWMD] and in the next section.

For $n = 2$, it would be very interesting to investigate if this duality continues to hold in the case of infinite root systems. There is the obvious complication that one is confronted with nonuniqueness issues but one could still hope that some sort of duality exists. More precisely I would make the following conjecture. (For clarity of exposition and reasons of laziness, I'll restrict my attention to the affinization of D_4.) In their beautiful work, Adrian and Alina show that despite a non-unique choice of weighting polynomials $\mathcal{P}$, there does exist a choice $\mathcal{P}_0$ for which the multiple Dirichlet series $Z(\underline{s}, \mathcal{P}_0)$ has the full group of functional equations and analytic continuation to a sufficiently large domain to deduce asymptotics for the fourth moment of quadratic L-functions over the rational function field $\mathbb{F}_q[t]$.

Let $\mathcal{P}(x_1, x_2, x_3, x_4, y; \mathfrak{p})$ be a choice of weighting polynomials. By this I mean that $\mathcal{P}$ is the $\mathfrak{p}$-part polynomial divided by normalizing factors, that the $\mathfrak{p}$-dependence is polynomial in p and that the associated multiple Dirichlet series has a full group of functional equations. Then I conjecture that

- $Z(x_1, x_2, x_3, x_4, y; \mathcal{P}) = \mathcal{P}(qx_1, qx_2, qx_3, qx_4; 1/q)$.
- There exists a choice $\mathcal{P}_0$ of weighting polynonials for which $Z(\cdot; \mathcal{P}_0)$ has analytic continuation to the entire Tit's cone.
- (More optimistic) This choice $\mathcal{P}_0$ is given by averaging the constant function over the (infinite) Weyl group.

If the conjecture is false, it should be possible to quickly disprove the first item by simply computing low order terms on both sides. I don't have a feel for what goes on in the higher genus case–even for finite root sytems. In fact, not even for A_2. I think this is an attractive and approachable area for further study.

On an unrelated (to multiple Dirichlet series) but potentially interesting note, I remark that if one sets $p = 1$ in the action of [QMDS] one gets invariant functions which have a simple product structure. Does this also happen in the affine case? Can one get new infinite product identities like Macdonald's?

3. In [FFWMD] is suggested (sort of) an extension of the action of [QMDS] to higher order twists. I conjecture that functions invariant under this action give the $\mathfrak{p}$-part polynomial of the $\Phi^{(n)}$ multiple Dirichlet series for arbitrary Φ and n. In the paper, I verified this for $A_2^{(n)}$ for all n and for $A_3^{(n)}$ for $n = 3, 4$. The questions of the previous sections can therefore be formulated in this context as well. In addition, the function field multiple Dirichlet series can provide a means for learning about coefficients of the theta function on the n-fold cover of GL_2. However there is the big problem that some of my computations seem incompatible with the interpretation of the Patterson/Bump-Hoffstein conjecture given in [WMD2]. I'm not sure what

to make of this. Probably it's just that the conjecture needs to be formulated more preceisely.

References

[WMD1] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein, *Weyl group multiple Dirichlet series I.*

[WMD2] B. Brubaker, D. Bump, and S. Friedberg, *Weyl group multiple Dirichlet series II: The stable case.*

[WMD3] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein, *Weyl group multiple Dirichlet series III: Eisenstein series and twisted unstable* A_r.

[FFWMD] G. Chinta, *Multiple Dirichlet series over the rational function field.*

[QMDS] G. Chinta and P. Gunnells, *Weyl group multiple Dirichlet series constructed from quadratic characters.*