Multiple Dirichlet Series and Automorphic Forms, I

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Abstract

This is the first of a series of three lectures concerning multiple Dirichlet series arising from sums of twisted automorphic L-functions. This lecture begins with an historical overview, explaining how such series arise from Rankin-Selberg constructions. Then more recent work, using Hartogs's continuation principle in place of such constructions, is described. Applications to the nonvanishing of L-functions and to other problems are also discussed.

I wish to thank Steven J. Miller, who took TEX notes on my conference lecture as I was delivering it. His file was very valuable in preparing these notes.—Sol Friedberg.

1 Multiple Dirichlet Series and Automorphic Forms, I (Lecture by Sol Friedberg)

This is the first of a three part mini-course, with the remaining two lectures to be presented by Gautam Chinta. I will start with an overview and historically-based introduction.

1.1 The Family of Twisted L-Functions

Fix an $n \geq 2$ and let F be a global field containing n n^{th} roots of unity. (Though the reader may focus on number fields, later, in one of Gautam Chinta's talks, we'll see examples in the function field case.)

A basic problem is the following: let π be an automorphic representation of $\mathrm{GL}(r)$ over the field F, with standard L-function

$$L(s, \pi) = \sum c(n) ||n||^{-s}.$$

(In this lecture I will write the finite part of the L-function as $L(s, \pi)$ for convenience.) We want to study the family of twisted L-functions

$$L(s, \pi \times \chi) = \sum c(n) \chi(n) ||n||^{-s}$$

where we fix π and vary the twist by a character χ; χ will range over the idèle class characters of order *exactly* n. We may also wish to modify the problem a little bit, and suppose instead that χ ranges over the subset of idèle class characters of order exactly n with χ_v specified at a finite number of places. (We will also use the notation L-function $L(s, \pi, \chi)$ for the twisted L-function.)

1.2 Questions

What kind of questions can we ask? The first is non-vanishing.

1.2.1 Non-vanishing

1. Non-vanishing: given a point in the critical strip s_0 (with $0 < \mathfrak{Re}(s_0) < 1$), can one show there exist infinitely many χ as above with $L(s_0, \pi \times \chi) \neq 0$? Goes back to Shimura, Rohrlich, and Waldspurger. A natural question: if we had perfect knowledge about RH then the interesting choice is $s_0 = \frac{1}{2}$,

where sometimes the L-function is zero and sometimes not (for example, elliptic curves of rank greater than zero where when you twist you get rank 0).

2. If $n = 2$ (quadratic twists) and $\pi = \widetilde{\pi}$ (self-dual) and if $\epsilon(\frac{1}{2}, \pi \times \chi) = -1$ for all twists χ under consideration, can one show there exist infinitely many χ such that $L'(\frac{1}{2}, \pi \times \chi) \neq 0$? Note that under these hypotheses, the functional equation guarantees a zero of odd order for each twisted L-function at the center of the critical strip.

Note: In these questions, we are not assuming π is cuspidal – $L(s, \pi)$ could be a product. Then the first question becomes that of establishing a simultaneous non-vanishing theorem. For example, take two independent $\mathrm{GL}(2)$ holomorphic modular forms. It isn't know if there is a twist such that both do not vanish at the center of the critical strip (our L-functions are normalized so that $s \to 1 - s$). Using Multiple Dirichlet Series, one can establish simultaneous non-vanishing for points s_0 in the critical strip but sufficiently far from the center of the strip (Chinta-Friedberg-Hoffstein).

1.2.2 Distribution

1. Study the distribution of $L(s, \pi \times \chi)$ as we vary χ as above. For example, we study

$$\sum_{\mathrm{cond}(\chi) < \mathrm{X}} L(s, \pi \times \chi)^k a(s, \pi, d) \ \sim \ \text{what?} \tag{1}$$

   Here $a(s, \pi, d)$ is some weight factor.

One approach that has been fruitful is the Multiple Dirichlet Series approach, and becomes a nice way to introduce the whole field. If these are the objects we want to study as we vary χ, why not construct a function of two variables that adds them up. For example,

$$Z(s, w) \ = \ \sum_d \frac{L(s, \pi \times \chi_d) a(s, \pi, d)}{|d|^w}. \tag{2}$$

Above we have $\mathfrak{Re} s, \mathfrak{Re} w > 1$ and χ_d corresponds to $F(\sqrt[n]{d})/F$, and is given by an n^{th} power residue symbol. We have put in a weight function $a(s, \pi, d)$. We will say quite a bit more about this weight factor later.

The goal, the approach, is to construct this function. We understand its behavior in s as it is a sum of $\mathrm{GL}(r)$ L-functions; we want to obtain analytic information in the new variable w. The s and w information gets smeared together (more than you might think). You might think we just sum over extensions (as d corresponds to certain cyclical extensions), but will see this isn't the whole story.

Similarly we could look at a sum over more variables:

$$\sum \frac{L(s_1, \pi_1, \chi_d) L(s_2, \pi_2, \chi_d) \cdots}{|d|^w}. \tag{3}$$

1.3 A First Example

Why is this series a reasonable thing to construct? Goes back to Siegel and half integral weight Eisenstein series. Let $j(\gamma, z)$ be the theta multiplier: it is (in standard notation) $\epsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2}$. Note that $\left(\frac{c}{d}\right)$ is a (quadratic) Kronecker symbol. We can build

$$\widetilde{E}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} j(\gamma, z)^{-1} \mathfrak{Im}(\gamma z)^s. \tag{4}$$

Maass showed (1937) that the m^{th} Fourier coefficient of $\widetilde{E}(z, s)$ is essentially equal to $L(2s, \chi_m)$ where χ_m is a quadratic character given by a Legendre symbol.

What does *essentially equal* mean? It means it isn't exactly the right formula, but correct up to 2-factors (something happens at $p = 2$), archimedean factors (suppressing from the notation) and most importantly correction factors that take into account that m might not be square-free. This factor will be a product of polynomials in $||v||^{-s}$ at the places v such that $\mathrm{ord}_v(m) \geq 2$.

Siegel took Maass' observation and said

$$\int_0^\infty \left(\widetilde{E}(y, s) - \text{const term}\right) y^w d^\times y \approx \sum_m \frac{L(2s, \chi_m)}{m^w}. \tag{5}$$

Goldfeld-Hoffstein in 1984 used this to get asymptotics for

$$\sum_{0 < m < X} L(2s, \chi_m). \tag{6}$$

Similarly can do for m negative. Later Goldfeld-Hoffstein-Patterson used these Eisenstein series over an imaginary quadratic field together with the Asai integral to get similar results for L-functions attached to CM elliptic curves, and then Hoffstein and Rosen used the method over the rational function field ($\mathbb{F}_q(T)$).

Goldfeld and Hoffstein anticipated the difficulty of settling a similar question for automorphic forms in the higher rank case. They write:

> At present, however, we cannot obtain mean value theorems for quadratic twists of an arbitrary L-function associated to an automorphic form... These appear to be difficult problems and their solution may ultimately involve the analytic number theory of $\mathrm{GL}(n)$.

1.4 Other Examples of MDS's Arising From Rankin-Selberg Integrals

There are other examples of Rankin-Selberg integrals that give rise to multiple Dirichlet series (a more modern point of view to look at Siegel's work this way). A number of interesting examples can be understood as follows: in the previous Section, we saw that the Mellin transform, which gives a standard L-function if applied to something of integral weight, gives a Multiple Dirichlet Series of the desired type when applied to an Eisenstein series of half-integral weight. Note that the integral is no longer an Euler product in that case. In a similar way we can look at other integrals that give Euler products (Rankin-Selberg type integrals) when applied to an automorphic form. Replacing the automorphic form by a metaplectic Eisenstein series (like the half-integral weight Eisenstein series $\widetilde{E}$), one can hope that the resulting object is an interesting multiple Dirichlet series.

1.4.1 Examples

1. Let π be a $\mathrm{GL}(2)$ automorphic form. Then Bump-Friedberg-Hoffstein construct a half-integral weight Eisenstein series on GSp_4. Take an integral (represents a spin L-function, called Novodvorsky integral, when applied in the non-metaplectic case) and get a similar type construction for a general $\mathrm{GL}(2)$ automorphic form, that is, a sum of quadratic twists of a $\mathrm{GL}(2)$ standard L-function, i.e. a function $Z(s,w)$ as above.

   (There is also another construction of Friedberg-Hoffstein that obtains such a sum of twisted $\mathrm{GL}(2)$ without using GSp_4.)

2. Let π be a $\mathrm{GL}(3)$ automorphic form. Work of Bump-Friedberg-Hoffstein-Ginzburg (never published) obtains the double Dirichlet series as an integral of an Eisenstein series on the double cover of GSp_6.

3. Suzuki, Banks-Bump-Lieman, generalizing earlier work of Bump-Hoffstein: there is a metaplectic Eisenstein series on the n-fold cover of $\mathrm{GL}(n)$ (induced from the theta function on the n-fold cover of $\mathrm{GL}(n-1)$) whose Whittaker coefficients are n-th order twists of a given $\mathrm{GL}(1)$ L-series. One can then take an integral and get a sum of twists of $\mathrm{GL}(1)$:

$$\sum_d \frac{L(s, \xi\chi_d^{(n)})}{|d|^w}, \tag{7}$$

where ξ is on $\mathrm{GL}(1)$ and is fixed. One is then able to control such sums, at least modulo technical difficulties, discussed below. (This series has been studied by Friedberg-Hoffstein-Lieman, using a different method to be explained below.)

4. Similarly, working with n-th order twists,

$$\sum \frac{|L(s, \chi_m)|^2}{m^w}. \tag{8}$$

can be obtained as a Rankin-Selberg integral of metaplectic Eisenstein series. (This series has been studied by A. Diaconu, again using the new method.)

The point is these multiple series come out in natural ways from constructions like this.

1.4.2 Obstructions

Why isn't the rest of the talk just doing these integrals? These integrals involve some difficult things:

1. truncation (can be handled, general theory of Arthur, though complicated to do in practice; needed as these Eisenstein integrals are not convergent);

2. bad primes (bad primes are difficult to handle in Rankin-Selberg type integrals, unlike Langlands-Shahidi method, and this is particularly true in the case of integrals involving metaplectic automorphic forms, where the primes dividing n present additional complications);

3. integrals of archimedean Whittaker functions (the general theory here is not as well developed).

Since many properties of L-functions are already known, one might hope that one can write down and study multiple Dirichlet series *without* Rankin-Selberg integrals. Remarkably, this is possible in many cases, and it is one main goal of this conference to explain how. However, we note that such integrals do play a role in the study of higher twists, as we shall explain at the end of this lecture.

1.5 Conceptual Overview

Question: why should these be well-behaved in w? There are lots of cases where we think they should be better behaved then we can prove.

1.5.1 Heuristic: BFH 1996

Let $F = \mathbb{Q}$, $n = 2$ (quadratic twists only!). We start with an L-function

$$L(s, \pi) \;=\; \sum_n c(n) n^{-s}. \tag{9}$$

The family of objects of interest is $L(s, \pi \times \chi_m)$:

$$L(s, \pi, \chi_m) \;=\; \sum_n c(n) \left(\frac{m}{n}\right) n^{-s}, \tag{10}$$

though this is not quite right (problem if m, n not both square-free). We set

$$Z(s, w) \;=\; \sum_m \frac{L(s, \pi, \chi_m)}{m^w}. \tag{11}$$

We have, using the definition of the L-function, that

$$Z(s, w) \;=\; \sum_m \sum_n c(n) \left(\frac{m}{n}\right) n^{-s} m^{-w}. \tag{12}$$

We reverse the order of summation, to obtain

$$Z(s, w) \;\approx\; \sum_n c(n) L(w, \chi_n) n^{-s}. \tag{13}$$

Note that we started with a sum of $L(s, \pi, \chi_m)$, that is, a sum of $\mathrm{GL}(r)$ L-functions, and we obtain a sum of $L(w, \chi_n)$, that is, a sum of $\mathrm{GL}(1)$ L-functions!!

For the moment, this is only a heuristic, as it assumes $\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right)$ and all numbers square-free and prime to the conductor.

We need to keep track of how functional equations work while we twist. We have a functional equation sending

$$L(s, \pi \times \chi_m) \;\to\; |m|^{r(\frac{1}{2}-s)} L(1-s, \widetilde{\pi} \times \chi_m); \tag{14}$$

this assumes m is square-free. Thus $Z(s, w)$ satisfies two types of functional equations:

1. First we have the functional equations from $s \to 1-s$, but because of the power of m we have $w \to w + r(s - \frac{1}{2})$. Thus there is a link between the two:
$$(s, w) \;\to\; (1-s, w + r(s - 1/2)). \tag{15}$$

2. The second, coming from $w \to 1-w$ in the equality (13), is
$$(s, w) \;\to\; (s + w - 1/2, 1 - w). \tag{16}$$

The observation is that these functional equations generate a finite group of functional equations for $\mathrm{GL}(1)$, $\mathrm{GL}(2)$ and $\mathrm{GL}(3)$, but an infinite group for $\mathrm{GL}(4)$ (affine Weyl group) and higher. This suggests that a good $Z(s, w)$ (here *good* deals with all numbers including those that are not square-free) should continue to $\mathbb{C}^2$ for $\mathrm{GL}(1)$, $\mathrm{GL}(2)$ and $\mathrm{GL}(3)$ but to a proper subregion for $\mathrm{GL}(4)$ and higher.

What about poles? There should be a pole at $w = 1$ (since $\zeta(w)$ arises in equation (13) when n is 1). If this does have a pole at $w = 1$, then by (11) this implies the non-vanishing of $L(s_0, \pi \times \chi_m)$ for infinitely many χ_m! Similarly if all epsilon factors are -1 then one gets a non-vanishing theorem for $L'(1/2, \pi \times \chi_m)$ from the pole of $\frac{\partial}{\partial s} Z(s, w)$ at $w = 1$. (In fact there are several polar divisors meeting at $(1/2, 1)$ and one must check that they do not cancel.) Then standard methods involving contour integrals give mean value theorems.

Consider the case of $\mathrm{GL}(4)$ and higher. There the group of functional equations is infinite. If we take this infinite group and move this line $w = 1$ around, the poles accumulate in what looks like a barrier to continuation. See BFH's 1996 article in the Bulletin AMS. So we shouldn't expect continuation to all of $\mathbb{C}^2$. However, if we could get continuation up to the conjectured barrier, that would be very significant; we would get a tremendous amount of information (Lindelöf in twisted aspect, simultaneous non-vanishing at the center of the critical strip). At the moment this problem seems difficult.

The situation for $\mathrm{GL}(1)$, $\mathrm{GL}(2)$ and $\mathrm{GL}(3)$ is different. There we can make the heuristic rigorous and thereby prove continuation to $\mathbb{C}^2$ without using Rankin-Selberg integrals. Applications (non-vanishing, mean-value theorems) then follow. The key point is to take advantage of the finite group of functional equations, and Hartogs's Continuation Principle.

1.6 Hartogs's Continuation Principle

Definition 1.1 (Tube Domain)**.** *An open set* $\Omega \subset \mathbb{C}^m$ *is called a* tube domain *if there is an open set* $\omega \in \mathbb{R}^m$ *such that* $\Omega = \{s \in \mathbb{C}^m : \mathfrak{Re}(s) \in w\}$*. We write* $\Omega = T(\omega)$ *to denote this relation.*

Definition 1.2 (Convex Hull)**.** *If* $R \subset \mathbb{R}^m$ *or* $\mathbb{C}^m$ *and* $m \geq 2$*, let* $\hat{R}$ *be the* convex hull *of* R*.*

It is easy to see that if $\Omega = T(\omega)$ then $\hat{\Omega} = T(\hat{\omega})$.

With this as background, the relevant result is

Theorem 1.3. *If* Ω *is a connected tube domain, then any holomorphic function in* Ω *can be extended to a holomorphic function on* $\hat{\Omega}$*.*

When we continue complex functions, we get to take convex hulls *for free*!

1.7 Continuation of $Z(s,w)$ to $\mathbb{C}^2$ for $\mathrm{GL}(r)$ if $r \leq 3$

We can now sketch the continuation of $Z(s,w)$. First we introduce some weight functions so that the interchange of summations is actually valid. What we mean by this is that interchange of summation implicitly assumed everything was square-free, which is not the case. In the work on Rankin-Selberg integral representations for $Z(s,w)$, these weight factors arise from Fourier coefficients; remarkably, they are exactly the ones needed to make the interchange of summation work.

Thus we look at

$$\sum L(s, \pi \times \chi_m) a(s, \pi, m) \xi(m) m^{-w}, \tag{17}$$

where ξ is on $\mathrm{GL}(1)$), and we write this as a sum of series

$$\sum L(w, \xi\chi_n) b(w, \xi, \pi, n) n^{-s}. \tag{18}$$

This comes (for the correct choice of weight factors $a(\cdot)$, $b(\cdot)$) from interchanging sums and using the Law of Quadratic Reciprocity.

The sum of L-functions in s is also a sum of L-functions in w. We can use the convexity bounds on each of these L-functions to extend the regions of convergence, and, on these enlarged regions, to prove the corresponding functional equations. Thus the heuristic arguments can be made rigorous, though we need to study how the ϵ-factors change under twisting. As Fisher and Friedberg show, by a congruence sieving one can arrange it so that the sign of the ϵ-factor does not vary. So the interchange and functional equation gives another multiplie Dirichlet series that is basically of the same form. (As we will explain later, for higher order twists something more complicated happens!)

We iterate this procedure until we get a region whose convex hull is $\mathbb{C}^2$, and then use Hartogs. There are finitely many poles, but we can remove these by multiplying by a finite product of linear terms.

1.7.1 Example: Quadratic Twists of $\mathrm{GL}(3)$

Everything hasn't been done in the full generality it could be, but this will show many key features. Recall if π' is a cuspidal automorphic representation of $\mathrm{GL}(2)$ then there is the Gelbart-Jacquet lift $\mathrm{Ad}^2(\pi')$ which is an automorphic representation of $\mathrm{GL}(3)$. (This was referred to in other conference talks as the symmetric square lift.)

Theorem 1.4 (Bump-Friedberg-Hoffstein: Shalika Volume)**.** *Let π' be on $GL_2(\mathbb{A}_{\mathbb{Q}})$. Let M be a finite set of places including 2, ∞, primes dividing $\mathrm{cond}(\pi')$. Then there exist infinitely many quadratic characters χ_d such that d falls in a given quadratic residue class mod v for all $v \in M$ (mod 8 if $v = 2$) and such that* $L(\frac{1}{2}, \mathrm{Ad}^2(\pi') \times \chi_d) \neq 0$.

Remarks

1. $\mathbb{Q}$ is not essential. Moreover, with a little more work one could specify χ_v for all places $v \in M$. And one should be able to do non-lifts with extra work.

2. If π is on $\mathrm{GL}(3)$ this method gives a new proof that the symmetric square $L(s, \pi, \mathrm{sym}^2)$ is holomorphic (more precisely, one sees that the product $\zeta(3s-1)L(s, \pi, \mathrm{sym}^2)$ is holomorphic excpet at $s = 1, 2/3$). We shall see why the symmetric square arises presently.

Sketch of the proof.

1. In the paper it is shown that not only do weight factors exist, but they are unique (complicated set of recursion relations). These weight factors allow us to interchange summations.

2. Establish the functional equations

$$\begin{aligned} \alpha(s,w) &= (1-s, w+3s-3/2) \\ \beta(s,w) &= (s+w-1/2, 1-w). \end{aligned} \tag{19}$$

(Note that α and β generate a dihedral group of order 12.)

3. Use α, β, α and get a continuation to a region whose convex hull is $\mathbb{C}^2$. In fact, one shows that

$$w(w-1)(3s+w-5/2)(3s+2w-3)(3s=w-3/2)\times \text{bad prime factor} \times Z(s,w)$$

has analytic continuation to $\mathbb{C}^2$.

□

Theorem 1.5. *Suppose* π *is automorphic on* $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$ *with trivial central character. Then for* $\sigma = \pm 1$ *we have*

$$\sum_{d>0} L_M(\frac{1}{2}, \pi, \chi_{\sigma d}) a(\frac{1}{2}, \pi, \sigma d) e^{-d/X} = CX \log X + C'X + C'' + O(X^{-3/4}), \tag{20}$$

where C *is a non-zero multiple of*

$$\lim_{s \to 1} (s-1/2) L_M(2s, \pi, \mathrm{sym}^2); \tag{21}$$

The term C arises by contour integration as the residue of the pole at $w = 1$. Note that by equation (13), this residue arises from the summands indexed by n a perfect square, so it is approximately $\sum c(n^2) n^{-2s}$, which is related to $L(2s, \pi, \mathrm{sym}^2)$.

To complete the proof of this Theorem, suppose that $\pi = \mathrm{Ad}^2(\pi')$. Then

$$L(s, \pi, \mathrm{sym}^2) = \zeta(s) L(s, \mathrm{sym}^4(\pi'), \chi_{\pi'}^2). \tag{22}$$

Here $\chi_{\pi'}$ denotes the central character of π'. Using this equality, one can see that $L(s, \pi, \mathrm{sym}^2)$ has a simple pole at $s = 1$. The proof in our paper uses the Kim-Shahidi result on the automorphicity of $\mathrm{sym}^4(\pi')$ as well as the Jacquet-Shalika nonvanishing theorem to conclude that the second term does not vanish at $s = 1$, and hence that $C \neq 0$. Prof. Shahidi has kindly informed me that a simpler proof that $L(1, \mathrm{sym}^4(\pi'), \chi_{\pi'}^2) \neq 0$ is available in an older paper of his.

If we took something on $\mathrm{GL}(3)$ that isn't a lift then $C = 0$; thus this provides an analytic way to tell if something is a lift or not from the asymptotic behavior of its quadratically-twisted L-functions.

1.8 Summary of the Quadratic Twist Case

For $\mathrm{GL}(1), \mathrm{GL}(2)$ and $\mathrm{GL}(3)$: quadratic twists continue to $\mathbb{C}^2$ (resp. $\mathbb{C}^3$, $\mathbb{C}^4$ for the multi-variable sums corresponding to $\mathrm{GL}(1) \times \mathrm{GL}(1)$ and $\mathrm{GL}(1) \times \mathrm{GL}(2)$, $\mathrm{GL}(1) \times \mathrm{GL}(1) \times \mathrm{GL}(1)$). The weight factors needed to make the heuristic rigorous (i.e. to show that a sum of Euler products in s is also a sum of Euler products in w) are unique.

Though the heuristics are easiest to explain over $\mathbb{Q}$, we emphasize that the method works over a general global field (Fisher-Friedberg). For example, over a function field we get a rational function in q^{-s} and q^{-w} with a specified denominator; this comes from the functional equations. For example, to any algebraic curve over a finite field and a conductor one gets a finite dimensional vector space of rational functions of two complex variables; see Fisher-Friedberg 2004 for details and examples.

1.9 Higher Twists

The most recent work is here, where some new phenomena occur. The heuristic we wrote earlier goes into the trash bin. We cannot use it: the functional equations involve Gauss sums – we cannot ignore them. We can use higher reciprocity, but we are naturally in the world of Gauss sums. The basic idea of adopting the Hartogs's method is that we need to consider several different families of multiple Dirichlet series that are linked by functional equations. The weight functions often involve Gauss sums.

The basic fact is that on $GL(r)$ the twist of an L-function by an n-th order character χ_m has an epsilon factor involving $G(\chi_m)^r$ where G is a normalized Gauss sum. Moreover, when one carries out the operations of "take the functional equation" and "interchange summation", these two operations are not commuting involutions (even ignoring scattering matrix, bad primes, etc.) in contrast to the quadratic-twist case. Instead, these operations give rise to the linked families of MDS of the previous paragraph.

1.9.1 n-Fold Twists of $\mathrm{GL}(1)$

This is work of Friedberg-Hoffstein-Liemann 2004. We end up with two different families, the n-th order twists of the original L-function and a multiple Dirichlet series built up from infinite sums of n-th order Gauss sums. These latter sums arise as the Fourier coefficients of Eisenstein series on the n-fold cover of $\mathrm{GL}(2)$, and they can thus be controlled by using the theory of metaplectic Eisenstein series. Note that automorphic methods, which could be for the most part avoided in the quadratic twist case, seem unavoidable in many problems involving n-th order twists for $n > 2$.

In the case at hand, the continuation of the MDS to $\mathbb{C}^2$ is obtained by using the functional equation of the Fourier coefficients of Eisenstein series on the n-fold cover of $\mathrm{GL}(2)$, which is inherited from the Eisenstein series themselves. Note that earlier we mentioned such a sum could be approached by an integral of an Eisenstein series on the n-fold cover of $\mathrm{GL}(n)$. Thus the Hartogs-based method allows one to replace the use of Eisenstein series on the n-fold cover of $\mathrm{GL}(n)$ with the use of Eisenstein series on the n-fold cover of $\mathrm{GL}(2)$, which are considerably simpler. We shall see a similar reduction to $\mathrm{GL}(2)$ in the work of Brubaker, Bump, and Friedberg on Weyl group multiple Dirichlet series that is the subject of the other minicourse.

1.9.2 Cubic Twists of $\mathrm{GL}(2)$

This is work of Brubaker-Friedberg-Hoffstein in 2005.

Let $K = \mathbb{Q}(\sqrt{-3})$. For $d \in O_K$,$d \equiv 1 \bmod 3$ let $\|d\|$ denote the absolute norm of d. Let $\chi_d^{(n)}(a) = \left(\frac{d}{a}\right)_n$ denote the Hecke character associated by class-field theory to the extension $K(\sqrt[n]{d})/K$, with $n = 2, 3, 6$. Let $P(s; d)$ denote a certain Dirichlet polynomialdefined in our paper. $P(s; d)$ depends on π but we suppressthis from the notation. $P(s; d)$ is a complicated object, but has the propertiesthat if one factors$d = d_1 d_2^2 d_3^3$ with each $d_i \equiv 1 \bmod 3$, d_1 square-free,$d_1 d_2^2$ cube-free, then $P(s; d) = 1$ if $d_3 = 1$ and also for fixed d_1, d_2, the sum

$$\sum_{d_3 \equiv 1 \mod 3} \frac{P(s; d_1 d_2^2 d_3^3)}{\|d_3\|^{3w}}$$

converges absolutely for $\mathfrak{Re} w > 1/2$ and $\mathfrak{Re} s \geq 1/2$.

Then we prove

Theorem 1.6. *Let $\pi = \otimes \pi_v$ be an automorphic representation of $GL(2, \mathbb{A}_K)$ such that $L(s, \pi, \chi)$ isentire for all Hecke characters χ such that $\chi^3 = 1$. Let S be a* *finite set of primes including the archimedean prime and the primes dividing* $2, 3$ *and thelevel of* π*.Then, for any sufficiently large positive integer* k*, the asymptotic formula*

$$\sum_{\|d\|<X} L_S(s,\pi,\chi^{(3)}_{d_1 d_2^2}) P(s;d) \left(1 - \frac{\|d\|}{X}\right)^k \sim \frac{1}{k+1} c^{(3)}(s,\pi) X$$

holds for any s *with* $\mathfrak{Re}s \geq 1/2$*. The constant* $c^{(3)}(s,\pi)$ *isnon-zero, and is given by*

$$c^{(3)}(s,\pi) = c_S L_S(3s,\pi,\vee^3)\zeta_S(6s)\zeta_S(6s+1)^{-1} \prod_{p \notin S} (1-\gamma_p^3 \|p\|^{-3s-1})(1-\delta_p^3\|p\|^{-3s-1}),$$

where ζ_S *denotes the Dedekind zeta function of* K *with the Euler factors at the places in* S *removed,* γ_p, δ_p *are the Satake parameters of the representation* π_p*, and* c_S *is a non-zero constant.*

An immediate consequence of this, the convergence of the basic sum, and the usualconvexity bound for$L(1/2,\pi,\chi^{(3)}_{d_1 d_2^2})$ is

Corollary 1.7. *Let* π *be as in the Main Theorem. Then there exist infinitely many cube-free* d *such that* $L(1/2,\pi,\chi_d^{(3)}) \neq 0$*. More precisely, let* $N(X)$ *denote the number of such* d *with* $\|d\| \leq X$*. Then for any* $\epsilon > 0$*,* $N(X) \gg X^{1/2-\epsilon}$*.*

Sketch of the Proof: Define the multiple Dirichlet series

$$Z_1(s,w) = \sum_{d \equiv 1 \mod 3, (d,S)=1} \frac{L_S(s,\pi,\chi^{(3)}_{d_1 d_2^2}) P(s;d)}{\|d\|^w}.$$

(Here the sum is over all $d \in O_K$ with $d \equiv 1 \mod 3$ and $\mathrm{ord}_v(d) = 0$ for all finite $v \in S$.) This series convergesabsolutely for $\mathfrak{Re}(s), \mathfrak{Re}(w) > 1$. We establish the continuation of this function to a larger region. Let

$$Z^*(s,w) = Z_1(s,w)\,\zeta_S(6s+6w-5)\,\zeta_S(12s+6w-8) \times \prod_{p\notin S}(1-\gamma_p^3\|p\|^{2-3s-3w})^{-1}(1-\delta_p^3\|p\|^{2-3s-3w})^{-1},$$

where γ_p, δ_p are the Satake parameters of the representation π_p.We show that$Z^*(s,w)$ has a meromorphic continuation to the half plane $\mathfrak{Re}(s+w) > 1/2$ and is analytic in this region except for polarlines at $w = 1$, $w = 0$, $w = 5/3-2s$, $w = 3/2-2s$, $w = 4/3-2s$, $w = 7/6-s$, $w = 1-s$, $w = 5/6-s$. We alsoshow that theresidue at$w = 1$ satisfies

$$\mathrm{Res}_{\mathrm{w=1}}\mathrm{Z}^*(\mathrm{s,w}) = \mathrm{c_S}\,\mathrm{L_S}(3\mathrm{s},\pi,\mathrm{sym}^3)\,\zeta_{\mathrm{S}}(6\mathrm{s})\,\zeta_{\mathrm{S}}(12\mathrm{s}-2)$$

and is an analytic function of s for $\mathfrak{Res} > -1/2$, except possibly at the points $s = 1/3, 1/4, 1/6, 0$, which require a more detailed analysis. The properties of thesymmetric cube L-series have been completely described by Kim and Shahidi.

The First Two Series and the First Functional Equation

This step is based on the exact functional equation for the cubically-twisted L-series.Write $d = d_1 d_2^2 d_3^3$as above. Ignoring bad primes such asthose dividing the level of π and the infinite place,$L(s,\pi,\chi^{(3)}_{d_1d_2^2})$ has a functional equation of the form

$$L(s,\pi,\chi^{(3)}_{d_1d_2^2}) \to \epsilon_\pi G(\chi^{(3)}_{d_1d_2^2})^2 L(1-s,\tilde{\pi},\bar{\chi}^{(3)}_{d_1d_2^2})\|d_1d_2\|^{1-2s}.$$

Here $\tilde{\pi}$ denotes the contragredient of π,ϵ_π (the central value of the usual epsilon-factor for π) has absolute value 1 and$G(\chi^{(3)}_d)$ is the usual Gauss sum associated to $\chi^{(3)}_d$,normalized to have absolute value 1. The crucial factor $\|d_1d_2\|^{1-2s}$ arises as part of the epsilon-factor of the twisted L-function since $\pi \otimes \chi^{(3)}_d$ is ramified at the primes dividing d_1d_2. This functional equationgives rise to a functional equation for the double Dirichlet series Z_1,reflecting $Z_1(s,w)$ into a second double Dirichletseries

$$Z_6(s,w) = \sum \frac{L_S(s,\tilde{\pi},\bar{\chi}^{(3)}_{d_1d_2^2})\,G(\chi^{(3)}_{d_1d_2^2})^2\,P(1-s;d_1d_2^2d_3^3)\,\|d_2d_3^3\|^{1-2s}}{\|d_1d_2^2d_3^3\|^w}.$$

More precisely, the functional equation above induces a transformationrelating $Z_1(s,w)$ to $Z_6(1-s,w+2s-1)$.(The exact transformation is somewhat complicated due to bad primes.)

The Second Functional Equation Next we study the series $Z_6(s,w)$ itself.The appearance of $G(\chi^{(3)}_{d_1d_2^2})^2$, the square of a cubic Gauss sum,introduces, via the Hasse-Davenport relation, a conjugate 6$^{\text{th}}$ order Gausssum. However, the Fourier coefficients of Eisenstein series on the 6-fold cover of $GL(2)$ may be written as sums of Gauss sums

$$\sum_{d\equiv 1 \mod 3,(d,S)=1} \frac{G^{(6)}(m,d)}{\|d\|^w},$$

and accordingly series of this type possess a functional equation in w. We show, using this functional equation, that $Z_6(s,w)$ possesses a functional equation as$(s,w) \to (s+2w-1, 1-w)$, transforming into itself.

The Third Series and the Third Functional Equation

We show that the order of summation in $Z_1(s,w)$written as a doubly-indexed Dirichlet series can beinterchanged, leading to an expression of the form

$$Z_1(s,w) = \sum \frac{L_S(w,\chi^{(3)}_{m_1m_2^2})\, Q(w; m_1m_2^2m_3^3)}{\|m_1m_2^2m_3^3\|^s},$$

where Q is once again a specific Dirichlet polynomial depending on $\pi$and the L-series on the right are Hecke L-series.Applying the functional equation in w for the Hecke L-series weare led to introduce the third double Dirichlet series

$$Z_3(s,w) = \sum \frac{L_S(w,\bar\chi^{(3)}_{m_1m_2^2})\, G(\chi^{(3)}_{m_1m_2^2})\, Q(1-w; m_1m_2^2m_3^3)\, \|m_2m_3^3\|^{1/2-w}}{\|m_1m_2^2m_3^3\|^s}.$$

The functional equation for the Hecke L-series induces a transformationrelating $Z_1(s,w)$ to $Z_3(s+w-1/2, 1-w)$.

Continuing Z_3 Once again, the series Z_3may be studied using metaplectic Eisenstein series. Indeed, we showthat this series is a sum of cubictwists of Rankin-Selberg convolutions of π with the theta function on the3-fold cover of $GL(2)$. (Recall that this function is the residue of anEisenstein series on the 3-fold cover of $GL(2)$; see Patterson's Crelle paper.) From the meromorphic continuation of the twisted Rankin-Selberg convolutionswe deduce a corresponding continuation for Z_3.

Applying Hartogs's Theorem We now apply Hartogs's theorem to obtain the continuation of these 3 functions.The functions $Z_1(s,w)$ and $Z_6(s,w)$ have overlapping regions of absoluteconvergence. If the functional equation interchanging $Z_1(s,w)$ and$Z_6(s,w)$ is used several times, the convexity principle for several-complex variables applied to the union of translates of these regions impliesan analytic continuation of $Z_1(s,w)$ and $Z_6(s,w)$ to the half plane $\mathfrak{Re}(w+s) > 3/2$. The relations with $Z_3(s,w)$ then imply an analyticcontinuation to the half plane $\mathfrak{Re}(w+s) > 1/2$, which is what we require forour applications.

Remarks:

1. A furtherfunctional equation transforming $Z_3(s,w)$ into itself as$(s,w) \to (1-s, w+4s-2)$, can be proved. This then allows ananalytic continuation of all three double Dirichlet series to $\mathbb{C}^2$. This alsogives rise to a group of functional equations which is non-abelianand of order 384. These computations have not been written down in detail.

2. As mentioned above, in the quadratic twist case the double Dirichletseries for $r = 1, 2, 3$ can be identified, up to afinite number of places, with certain integral transforms ofmetaplectic Eisenstein series. In the case at hand, althoughthere is no known way to construct the double Dirichlet series as asimilar integral transform (or as a Rankin-Selberg convolution),there is a natural candidate attachedto the cubic cover of G_2, and it is possible that our complicatedformulas reflect in a certain sense combinatorial issues arising fromthat group.

3. One may also obtain a mean value result for the product of two Hecke L-functionsin different variables when they aresimultaneously twisted by cubic characters. This has been accomplishedby BEN BRUBAKER in his Brown University doctoral dissertation.

1.10 Concluding Remarks

The Multiple Dirichlet Series that continue to a product of complex planes are ready-made for establishing distribution results via contour integration. We emphasize that the MDS method applies over a general global field containing sufficiently many roots of unity; thus such mean value theorems may be established without being constrained by the proliferation of Gamma factors in higher degree extensions. The most natural theorems to prove involve sums of L-functions times weighting factors $a(s,\pi,d)$.

There are many additional recent applications of Multiple Dirichlet Series to automorphic forms and analytic number theory. These will be discussed by Chinta is his lectures. For completeness, we note some of them now.

1. A proof that one twist of a $\mathrm{GL}(2)$ L-function of order n (a prime) has nonzero central value implies that infinitely many do (Brubaker, Bukur, Chinta, Frechette, Hoffstein).

2. Generalization of the Luo-Ramakrishnan theorem (which characterizes a $\mathrm{GL}(2)$ modular form by the central values of its quadratically-twisted L-functions) to all number fields, due to my doctoral student Li Ji.

3. A generalization of the Luo-Ramakrishan theorem to $\mathrm{GL}(3)$ over $\mathbb{Q}$ by Chinta and Diaconu.

4. The use of unweighted multiple Dirichlet series to prove simultaneous non-vanishing theorems inside the critical strip and also a distribution theorem at $s = 1$, by Chinta, Friedberg, and Hoffstein. (Note: Unfortunately, Chinta did not have time to discuss this in his talk. But I am hoping that he will include it in his lecture notes.)

5. Chinta's mean value theorem for biquadratic extensions of $\mathbb{Q}$.

6. Work of Diaconu and Tian applying MDS to study twisted Fermat curves over totally real fields.

7. Work of Diaconu, Goldfeld, and Hoffstein relating MDS to predictions coming from random matrix theory.

An additional set of lectures by Bump and Brubaker describes the very new theory of MDS's attached to a reduced root system, which we have dubbed Weyl Group Multiple Dirichlet Series. As this list shows, the theory of MDS seems quite rich. In the rest of this conference these matters will be developed further.