Factorial Schur Functions and the Yang-Baxter Equation

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Abstract. Factorial Schur functions are generalizations of Schur functions that have, in addition to the usual variables, a second family of "shift" parameters. We show that a factorial Schur function times a deformation of the Weyl denominator may be expressed as the partition function of a particular statistical-mechanical system (six-vertex model). The proof is based on the Yang-Baxter equation. There is a deformation parameter t which may be specialized in different ways. If $t = -1$, then we recover the expression of the factorial Schur function as a ratio of alternating polynomials. If $t = 0$, we recover the description as a sum over tableaux. If $t = \infty$ we recover a description of Lascoux that was previously considered by the second author. We also are able to prove using the Yang-Baxter equation the asymptotic symmetry of the factorial Schur functions in the shift parameters. Finally, we give a proof using our methods of the dual Cauchy identity for factorial Schur functions. Thus using our methods we are able to give thematic proofs of many of the properties of factorial Schur functions.

Dedicated to Professor Fumihiro Sato

1 Introduction

Factorial Schur functions are generalizations of ordinary Schur functions $s_\lambda(z) = s_\lambda(z_1, \cdots, z_n)$ for which a surprising amount of the classical theory remains valid. In addition to the usual spectral parameters $z = (z_1, \cdots, z_n)$ and the partition λ they involve a set $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots)$ of shifts that can be arbitrary complex numbers (or formal variables), and are denoted $s_\lambda(z|\alpha)$. In the original paper of Biedenharn and Louck [BL], only the special case where $\alpha_n = 1 - n$ was considered. Their motivation, inspired by questions from mathematical physics, was to the decomposition of tensor products of representations with using particular bases. It turns out that factorial Schur functions are the same as double Schubert polynomials for Grassmannian permutations, and in this form they appeared even earlier in Lascoux and Schützenberger [LS1], whose motivation (from algebraic geometry) was completely different.

Other early foundational papers are Chen and Louck [CL], who gave new foundations based on divided difference operators, and Goulden and Hamel [GH] where the analogy between Schur functions and factorial Schur functions was further developed. In particular they gave a Jacobi-Trudi identity. See Louck [Lou] for further historical remarks.

Biedenharn and Louck (in the special case $\alpha_n = 1 - n$) defined $s_\lambda(z|\alpha)$ to be a sum over Gelfand-Tsetlin patterns, and this definition extends to the general case. Translated into the equivalent language of tableaux, their definition is equivalent to (14) below. It was noticed independently by Macdonald [Mcd1] and by Goulden and Greene [GG] that one could generalize the factorial Schur functions of Biedenharn and Louck by making use of an arbitrary set α of shifts. Macdonald observed an alternative definition of the factorial Schur functions as a ratio of two alternating polynomials, generalizing the Weyl character formula. This definition is (7) below.

Both Macdonald and Goulden and Greene also noticed a relationship with what are called *supersymmetric Schur functions*. These are symmetric functions in two sets of variables, $z = (z_1, z_2, \cdots)$ and $w = (w_1, w_2, \cdots)$. They are defined in terms of the ordinary Schur functions by

$$s_\lambda(z\|w) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(z) s_{\nu'}(w),$$

where $c^\lambda_{\mu\nu}$ is the Littlewood-Richardson coefficient, μ and ν run through partitions and ν' is the conjugate partition. The relationship between the factorial Schur functions and the supersymmetric Schur functions is this: although the $s_\lambda(z|\alpha)$ are symmetric in the z_i they are not symmetric in the α_i. Nevertheless, as the number n of the parameters z_i tends to infinity, they become symmetric in the α_i in a certain precise sense, and in the limit, they stabilize. Thus in a suitable sense

$$\lim_{n \longrightarrow \infty} s_\lambda(z|\alpha) = s_\lambda(z\|\alpha). \tag{1}$$

Another important variant of the factorial Schur functions are the *shifted Schur functions* that were proposed by Olshanskii, and developed by Okounkov and Olshanskii [OO2], [OO1]. Denoted $s^*_\lambda(x_1, \cdots, x_n)$, they are essentially the same as the factorial Schur functions of Biedenharn and Louck, but incorporate shifts in the parameters so that they are no longer symmetric in the usual sense, but at least satisfy the stability property $s^*_\lambda(x_1, \cdots, x_n) = s^*_\lambda(x_1, \cdots, x_n, 0)$. These were applied to the representation theory of the infinite symmetric group.

Molev and Sagan [MS] give various useful results for factorial Schur functions, including a Littlewood-Richardson rule. A further Littlewood-Richardson rule was found by Kreiman [Kr]. Knutson and Tao [KT] show that factorial Schur functions correspond to Schubert classes in the equivariant cohomology of Grassmanians. See also Mihalcea [Mi] and Ikeda and Naruse [IN].

Tokuyama [To] gave a formula for Schur functions that depends on a parameter t. This formula may be regarded as a deformation of the Weyl character formula. It was shown by Hamel and King [HK] that Tokuyama's formula could be generalized and reformulated as the evaluation of the partition function for a statistical system based on the six-vertex model in the free-fermionic regime. Brubaker, Bump and Friedberg [BBF] gave further generalizations of the results of Hamel and King, with new proofs based on the Yang-Baxter equation. More specifically, they used the fact that the six-vertex model in the free-fermionic regime satisfies a parametrized Yang-Baxter equation with nonabelian parameter group $\mathrm{GL}(2) \times \mathrm{GL}(1)$ to give statistical-mechanical systems whose partition functions were Schur functions times a deformation of the Weyl denominator. This result generalizes the results of Tokuyama and of Hamel and King.

Our main new result (Theorem 1) is a Tokuyama-like formula for factorial Schur functions. This is a simultaneous generalization of [BBF] and of the representations of Lascoux [La2] and of McNamara [McN]. As in [BBF] we will consider partition functions of statistical-mechanical systems in the free-fermionic regime. A significant difference between this paper and that was that in [BBF] the Boltzmann weights were constant along the rows, depending mainly on the choice of a parameter z_i. Now we will consider systems in which we assign a parameter z_i to each row, but also a shift parameter α_j to each column. Furthermore, we will make use of a deformation parameter t that applies to the entire system. We will show that the partition function may be expressed as the product of a factor (depending on t) that may be recognized as a deformation of the Weyl denominator, times the factorial Schur function $s_\lambda(z|\alpha)$.

The proof of Theorem 1 depends on the Yang-Baxter equation. We feel that it is significant that the Yang-Baxter equation can be made a central tool in the theory of factorial Schur functions. The results in the paper after Theorem 1 are mainly already known, but we will reprove them using our methods—either deducing them from Theorem 1 or giving proofs using the same tool (free-fermionic Yang-Baxter equation).

By specializing t in the formula of Theorem 1, we will obtain different formulas for the factorial Schur functions. Taking $t = -1$, we obtain the representation as a ratio of alternating polynomials, which was Macdonald's generalization of the Weyl character formula. This is the formula we take as the definition of the factorial Schur functions, though other definitions are possible.

There are two specializations t in which the Weyl denominator in Theorem 1 reduces to a monomial. Taking $t = 0$, we obtain the tableau definition of the factorial Schur functions. When $t = \infty$ we recover another representation of the factorial Schur functions. Indeed Lascoux [La2] found six-vertex model representations of Grassmannian Schubert polynomials. A proof of this representation based on the Yang-Baxter equation was subsequently found by McNamara [McN]. It is this representation that we obtain when $t = \infty$.

Although we do not prove the supersymmetric limit (1), we will at least prove the key fact that the $s_\lambda(z|\alpha)$ are asymptotically symmetric in the α_j as the number n of parameters z_i tends to infinity. We will obtain this by another application of the Yang-Baxter equation. We also give a proof of the dual Cauchy identity for factorial Schur functions using our methods.

In addition to [La2] and [McN], Zinn-Justin [ZJ1], [ZJ2] gave another interpretation of factorial Schur functions as transition matrices for a lattice model that may be translated into a free-fermionic five-vertex model. It is unclear whether Zinn-Justin's representation may also be obtained from Theorem 1 ours by specialization, but it is certainly very similar.

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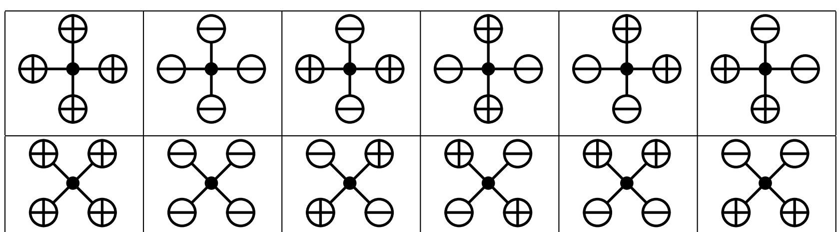
2 Yang-Baxter equation

We review the six-vertex model and a case of the Yang-Baxter equation from [BBF]. We will consider a planar graph. Each vertex v is assumed to have exactly four edges adjacent to it. *Interior edges* adjoin two vertices, and around the boundary of the graph we allow *boundary edges* that adjoin only a single vertex. Every vertex has six numbers $a_1(v), a_2(v), b_1(v), b_2(v), c_1(v), c_2(v)$ assigned to it. These are called the *Boltzmann weights* at the vertex. By a *spin* we mean an element of the two-element set $\{+, -\}$. In addition to the graph, the Boltzmann weights at each vertex, we will also assign a spin to each boundary edge. Once we have specified the graph, the Boltzmann weights at the vertices, and the boundary spins, we have specified a statistical system $\mathfrak{S}$.

A *state* $\mathfrak{s}$ of the system will be an assignment of spins to the interior edges. Given a state of the system, every edge, boundary or interior, has a spin assigned to it. Then every vertex will have a definite configuration of spins on its four adjacent edges, and we assume these to be in one of the two orientations listed in (2). Then let $\beta_{\mathfrak{s}}(v)$ equal $a_1(v), a_2(v), b_1(v), b_2(v), c_1(v)$ or $c_2(v)$ depending on the configuration of spins on the adjacent edges. If v does not appear in the table, the weight is zero.



$a_1(v)$	$a_2(v)$	$b_1(v)$	$b_2(v)$	$c_1(v)$	$c_2(v)$

(2)

The *Boltzmann weight of the state* $\beta(\mathfrak{s})$ is the product $\prod_v \beta_{\mathfrak{s}}(v)$ of the Boltzmann weights at every vertex. We only need to consider configurations in which the spins adjacent to each vertex are in one of the configurations from the above table; if this is true, the state is called *admissible*. A state that is not admissible has Boltzmann weight zero.

The *partition function* $Z(\mathfrak{S})$ is $\sum_{\mathfrak{s}} \beta(\mathfrak{s})$, the sum of the Boltzmann weights of the states. We may either include or exclude the inadmissible states from this sum, since they have Boltzmann weight zero.

If at the vertex v we have

$$a_1(v)a_2(v) + b_1(v)b_2(v) - c_1(v)c_2(v) = 0,$$

the vertex is called *free-fermionic*. We will only consider systems that are free-fermionic at every vertex.

Korepin, Boguliubov and Izergin [KBI] describe a nonabelian parametrized Yang-Baxter equation for the free-fermionic six-vertex model with parameter group $\Gamma = SL(2,\mathbb{C})$. Concretely this means that there is a map $R : \Gamma \to \mathrm{End}(V \otimes V)$, where V is a two-dimensional vector space, such that if $\gamma, \delta \in \Gamma$ then

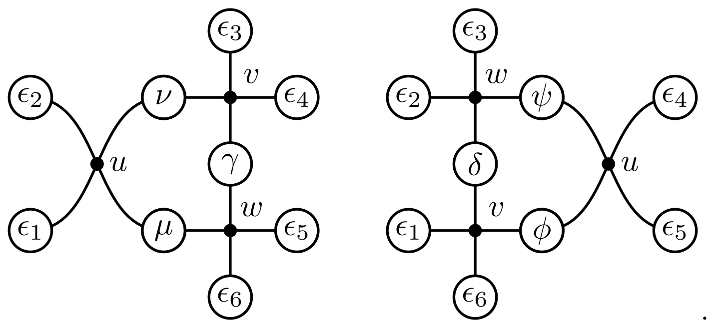
$$R(\gamma)_{12}R(\gamma\delta)_{13}R(\delta)_{23} = R(\delta)_{23}R(\gamma\delta)_{13}R(\gamma)_{12},$$

where if $R \in \mathrm{End}(V \otimes V)$ then R_{ij} means $R \times I_V$ acting on $V \otimes V \otimes V$ with R acting on the i, j tensor components, and the identity on the remaining component. Scalar matrices can obviously be added to Γ so their actual group is $SL(2,\mathbb{C}) \times \mathbb{C}^\times$. A statement with a slightly larger parameter group $GL(2,\mathbb{C}) \times \mathbb{C}^\times$ is in Brubaker, Bump and Friedberg [BBF]. The nonzero components of $R(\gamma)$ if written with respect to a standard basis of $V \otimes V$ will be the Boltzmann weights of a free-fermionic vertex. This has the following explicit reformulation:

Proposition 1. ***[BBF, Theorem 3]*** *Let v, w be vertices with free-fermionic Boltzmann weights. Define another type of vertex u with*

$$\begin{aligned}
a_1(u) &= a_1(v)a_2(w) + b_2(v)b_1(w), \\
a_2(u) &= b_1(v)b_2(w) + a_2(v)a_1(w), \\
b_1(u) &= b_1(v)a_2(w) - a_2(v)b_1(w), \\
b_2(u) &= -a_1(v)b_2(w) + b_2(v)a_1(w), \\
c_1(u) &= c_1(v)c_2(w), \\
c_2(u) &= c_2(v)c_1(w).
\end{aligned}$$

Then for any assignment of edge spins $\varepsilon_i \in \{\pm\}$ $(i = 1,2,3,4,5,6)$ the *following two configurations have the same partition function:*



. (3)

Note that by the definition of the partition function, the interior edge spins (labeled ν, γ, μ and δ, ψ, ϕ) are summed over, while the boundary edge spins, labeled ε_i are invariant. In order to obtain this from Theorem 3 of [BBF] one replaces the R-matrix $\pi(R)$ in the notation of that paper by a constant multiple.

3 Bijections

In this section we will define some combinatorial bijections that we will need later. One of the sets is the set of states of a statistical-mechanical system, as in the last section, and we start by defining that.

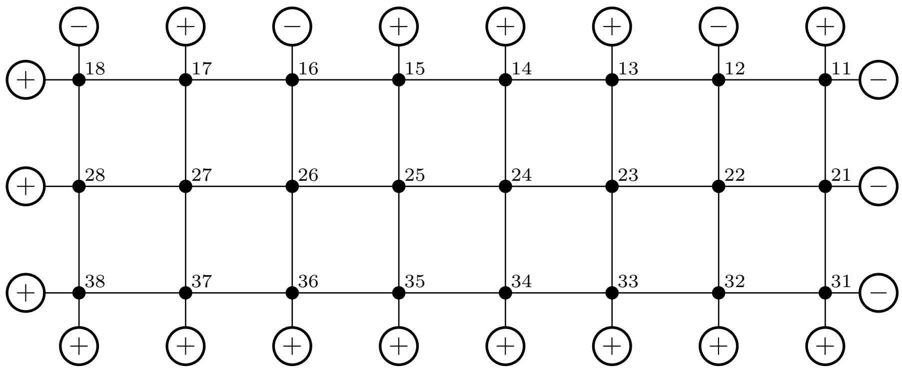
We will make use of two special n-tuples of integers, namely

$$\rho = (n, \cdots, 3, 2, 1), \qquad \delta = (n-1, n-2, \cdots, 2, 1, 0).$$

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a partition, so $\lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0$. Let us consider a lattice with n rows and $n + \lambda_1$ columns. We will index the rows from 1 to n. We will index the columns from 1 to $n + \lambda_1$, *in reverse order*. We put the vertex $v_\Gamma(i, j, t)$ at the vertex in the i row and j column.

We impose the following boundary edge spins. On the left and bottom boundaries, every edge is labeled $+$. On the right boundary, every edge is labeled $-$. On the top, we label the edges indexed by elements of form $\lambda_j + n - j + 1$ for $j = 1, 2, \cdots, n$ with a $-$ spin. These are the entries in $\lambda + \rho$. The remaining columns we label with a $+$ spin.

For example, suppose that $n = 3$ and that $\lambda = (5, 4, 1)$, so $\lambda + \rho = (8, 6, 2)$. Since $\lambda_j + n - j + 1$ has the values $8, 6$ and 2, we put $-$ in these columns. We label the vertex $v_\Gamma(i,j,t)$ in the i row and j column by ij as in the following diagram:



(4)

Let this system be called $\mathfrak{S}^{\Gamma}_{\lambda,t}$.

We recall that a *Gelfand-Tsetlin pattern* is an array

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} p_{11} & & p_{12} & & \cdots & & p_{1n} \\ & p_{22} & & & & p_{2n} & \\ & & \ddots & & \iddots & & \\ & & & p_{nn} & & & \end{array} \right\} \tag{5}$$

in which the rows are interleaving partitions. The pattern is *strict* if each row is strongly dominant, meaning that $p_{ii} > p_{i,i+1} > \cdots > p_{in}$.

A *staircase* is a semistandard Young tableau of shape $(\lambda_1 + n, \lambda_1 + n - 1, \ldots, \lambda_1)'$ filled with numbers from $\{1, 2, \ldots, \lambda_1 + n\}$ with the additional condition that the diagonals are weakly decreasing in the south-east direction when written in the French notation.

An example of a staircase for $\lambda = (5,4,1)$ is

8			
7	8		
6	7	8	
5	6	6	7
4	5	5	5
3	3	3	4
2	2	2	3
1	1	1	1

(6)

Proposition 2. *Let λ be a partition of length $\leqslant n$. There are natural bijections between the following three sets of combinatorial objects*

1. *States of the six-vertex model $\mathfrak{S}^{\Gamma}_{\lambda,t}$,*

2. *Strict Gelfand-Tsetlin patterns with top row $\lambda+\rho$, and*

3. *Staircases whose rightmost column consists of the integers between* 1 *and λ_1+n that are not in $\lambda+\rho$.*

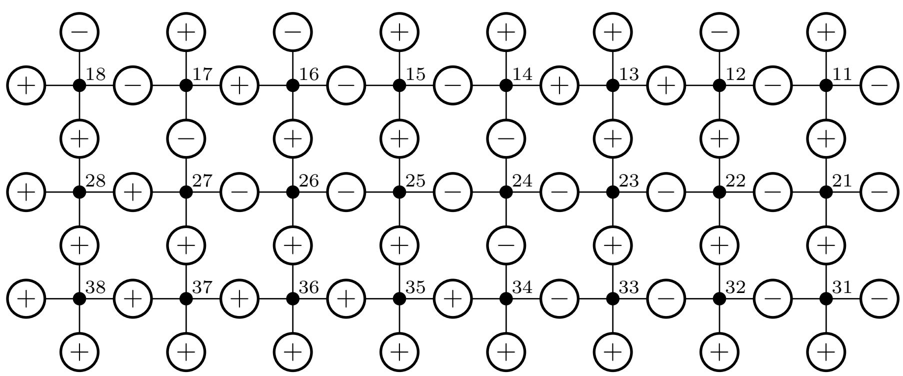
Proof. Suppose we start with a state of the six-vertex model $\mathfrak{S}^{\Gamma}_{\lambda,t}$. Let us record the locations of all minus spins that live on vertical edges. Between rows k and $k+1$, there are exactly $n-k$ such minus spins for each k. Placing the column numbers of the locations of these minus spins into a triangular array gives a strict Gelfand-Tsetlin pattern with top row $\lambda+\rho$.

Given a strict Gelfand-Tsetlin pattern $\mathfrak{T}=(t_{ij})$ with top row $\lambda+\rho$, we construct a staircase whose rightmost column is missing $\lambda+\rho$ in the following manner: We fill column $j+1$ with integers $u_1,\ldots,u_{n-j+\lambda_1}$ such that

$$\{1,2,\ldots,n+\lambda_1\}=\{u_1,\ldots,u_{n-j+\lambda_1}\}\sqcup\{t_{n+1-j,1},\ldots t_{n+1-j,j}\}.$$

It is easily checked that these maps give the desired bijections. $\square$

We give an example. As before let $n=3$ and $\lambda=(5,4,1)$. Here is an admissible state:



Then the entries in the i-th row of $\mathfrak{T}$ are to be the columns j in which a $-$ appears above the (i,j) vertex. Therefore

$$\mathfrak{T}=\left\{\begin{array}{ccccc}8&&6&&2\\&7&&4&\\&&4&&\end{array}\right\}.$$

Taking the complements of the rows (including a fourth empty row) gives the following sets of numbers:

$$\begin{array}{cccccccc} 7 & 5 & 4 & 3 & 1 & & & \\ 8 & 6 & 5 & 3 & 2 & 1 & & \\ 8 & 7 & 6 & 5 & 3 & 2 & 1 & \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

so the corresponding staircase is (6).

4 A Tokuyama-like formula for Factorial Schur Functions

A good primary reference for factorial Schur functions is Macdonald [Mcd1]. They are also in Macdonald [Mcd2], Ex. 20 in Section I.3 on p.54.

Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be a sequence of complex numbers or formal variables. If $z \in \mathbb{C}$ let

$$(z|\alpha)^r = (z + \alpha_1) \cdots (z + \alpha_r).$$

Macdonald [Mcd1] gives two formulas for factorial Schur functions that we will also prove to be equivalent by our methods. Let $\mu = (\mu_1, \cdots, \mu_n)$ where the μ_i are nonnegative integers. Let $z_1, \cdots, z_n$ be given. Define

$$A_\mu(z|\alpha) = \det((z_i|\alpha)^{\mu_j})_{i,j}$$

where $1 \leqslant i, j \leqslant n$ in the determinant. We will also use the notation

$$\boldsymbol{z}^\mu = \prod_i z_i^{\mu_i}.$$

Let $\delta = (n-1, n-2, \cdots, 0)$ and let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a partition of length at most n. Define

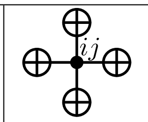
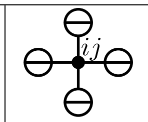
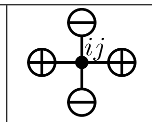
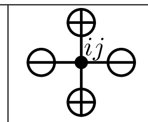
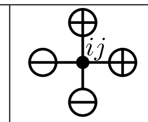
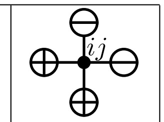
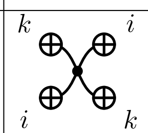
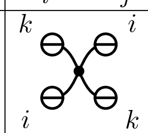
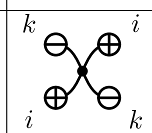
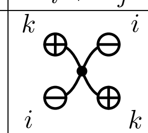
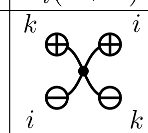
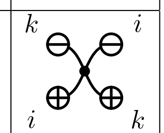
$$s_\lambda(z|\alpha) = \frac{A_{\lambda+\delta}(z|\alpha)}{A_\delta(z|\alpha)}. \tag{7}$$

The denominator here is actually independent of α and is given by the Weyl denominator formula:

$$A_\delta(z|\alpha) = \prod_{i<j}(z_i - z_j) \tag{8}$$

Indeed, it is an alternating polynomial in the z_i of the same degree as the right-hand side, and so the ratio is a polynomial in α that is independent of z_i. To see that the ratio is independent of α, one may compare the coefficients of z^δ on both sides of (8). Since both the numerator and the denominator is an alternating function of the z_i, the ratio $s_\lambda(z|\alpha)$ is a symmetric polynomial in $z_1, \cdots, z_n$.

Now let us consider two types of Boltzmann weights. Let $z_1, \cdots, z_n$ be given, and let $\alpha_1, \alpha_2, \alpha_3, \cdots$ another sequence of complex numbers, and t another parameter. If $1 \leqslant i, k \leqslant n$ and if $j \geqslant 0$ are integers, we will use the following weights.

$v_\Gamma(i,j,t)$						
	1	$z_i - t\alpha_j$	t	$z_i + \alpha_j$	$z_i(t+1)$	1
$v_{\Gamma\Gamma}(i,k,t)$						
	$tz_i + z_k$	$tz_k + z_i$	$t(z_k - z_i)$	$z_i - z_k$	$(t+1)z_i$	$(t+1)z_k$

Lemma 1. *We may take $u = v_{\Gamma\Gamma}(i,k,t)$, $v = v_\Gamma(i,j,t)$ and $w = v_\Gamma(k,j,t)$ in Proposition 1.*

Proof. The relation $a_1(u) = a_1(v)a_2(w) + b_2(v)b_1(w)$ becomes

$$tz_i + z_k = 1 \cdot (z_k - t\alpha_j) + (z_i + \alpha_j)t,$$

and all the other relations are checked the same way. $\square$

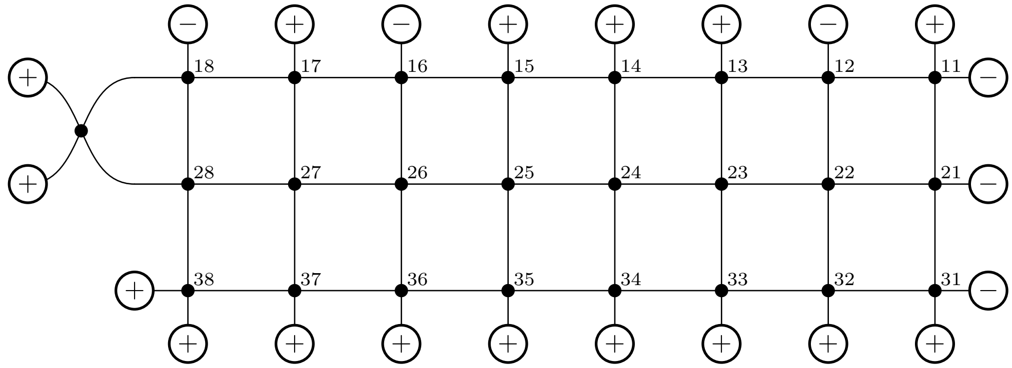
Proposition 3. *The function*

$$\left[\prod_{i>j}(tz_j + z_i)\right] Z(\mathfrak{S}^\Gamma_{\lambda,t}), \tag{9}$$

is symmetric under permutations of the z_i.

Proof. Let $\mathfrak{S} = \mathfrak{S}^\Gamma_{\lambda,t}$. It is sufficient to show that (9) is invariant under the interchange of z_i and z_{i+1}. The factors in front are permuted by this interchange with one exception, which is that $tz_i + z_{i+1}$ is turned into $tz_{i+1} + z_i$.

Therefore we want to show that $(tz_i+z_{i+1})Z(\mathfrak{S}) = (tz_{i+1}+z_i)Z(\mathfrak{S}')$ where $\mathfrak{S}'$ is the system obtained from $\mathfrak{S}$ by interchanging the weights in the $i, i+1$ rows. Now consider the modified system obtained from $\mathfrak{S}$ by attaching $v_{\Gamma\Gamma}(i, i+1, t)$ to the left of the $i, i+1$ rows. For example, if $i = 1$, this results in the following system:



Referring to (2), there is only one admissible configuration for the two interior edges adjoining the new vertex, namely both must be $+$, and so the Boltzmann weight of this vertex will be tz_i+z_{i+1}. Thus the partition function of this new system equals $(tz_i + z_{i+1})Z(\mathfrak{S})$. Applying the Yang-Baxter equation repeatedly, this equals the partition function of the system obtained from $\mathfrak{S}'$ by adding $v_{\Gamma\Gamma}(i, i+1, t)$ to the right of the $i, i+1$ rows, that is, $(tz_{i+1} + z_i)Z(\mathfrak{S}')$, as required. This proves that (9) is symmetric. □

Theorem 1. *We have*

$$Z(\mathfrak{S}^{\Gamma}_{\lambda,t}) = \left[\prod_{i<j}(tz_j + z_i)\right] s_\lambda(z|\alpha).$$

Proof. We show that the ratio

$$\frac{Z(\mathfrak{S}^{\Gamma}_{\lambda,t})}{\prod_{i<j}(tz_j + z_i)} \tag{10}$$

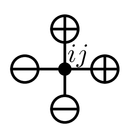
is a polynomial in the z_i, and that it is independent of t. Observe that (9) is an element of the polynomial ring $\mathbb{C}[z_1, \cdots, z_n, t]$, which is a unique factorization domain. It is clearly divisible by $tz_j + z_i$ when $i > j$, and since it is symmetric, it is therefore divisible by all $tz_j + z_i$ with $i \neq j$. These are coprime, and therefore it is divisible by their product, in other words (10) is a polynomial. We note that the numerator and the denominator have the same degree in t, namely $\frac{1}{2}n(n-1)$. For the denominator this is clear and for the numerator, we note that each term is a monomial whose degree is the number of vertices with a $-$ spin on the vertical edge below. This is the number of vertical edges labeled $-$ excluding those at the top, that is, the number of entries in the Gelfand-Tsetlin pattern in Lemma 2 *excluding* the first row. Therefore each term in the sum $Z(\mathfrak{S}^{\Gamma}_{\lambda,t})$ is a monomial of degree $\frac{1}{2}n(n-1)$ and so the sum has at most this degree; therefore the ratio (10) is a polynomial of degree 0 in t, that is, independent of t.

To evaluate it, we may choose t at will. We take $t=-1$.

Let us show that if the pattern occurs with nonzero Boltzmann weight, then every row of the pattern (except the top row) is obtained from the row above it by discarding one element. Let μ and ν be two partitions that occur as consecutive rows in this Gelfand-Tsetlin pattern:

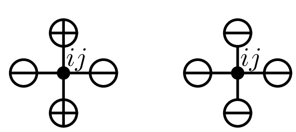
$$\left\{\begin{array}{ccccccccc} \mu_1 & & \mu_2 & & \cdots & & \cdots & & \mu_{n-i+1} \\ & \nu_1 & & \nu_2 & & \cdots & & \nu_{n-i} & \end{array}\right\},$$

where μ_k are the column numbers of the vertices (i,μ_k) that have a $-$ spin on the edge above the vertex, and ν_k are the column numbers of the vertices (i,ν_k) that have a $-$ spin on the edge below it. Because $t=-1$, the pattern



does not occur, or else the Boltzmann weight is zero, and the term may be discarded. Therefore every $-$ spin below the vertex in the i-th row must be matched with a $-$ above the vertex. It follows that every ν_i equals either μ_i or μ_{i+1}. Thus the partition ν is obtained from μ by discarding one element.

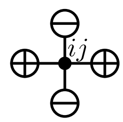
Let μ_k be the element of μ that is not in ν. It is easy to see that in the horizontal edges in the i-th row, we have a $-$ spin to the right of the μ_k-th column and $+$ to the left. Since $t=-1$, the Boltzmann weights of the two patterns:



both have the same Boltzmann weight $z_i+\alpha_j$. We have one such contribution from every column to the right of μ_k and these contribute

$$\prod_{j=1}^{\mu_k-1}(z_i+\alpha_j)=(z_i|\alpha)^{\mu_k-1}.$$

For each column $j=\mu_l$ with $l<k$ we have a pattern



and these contribute -1. Therefore the product of the Boltzmann weights for this row is

$$(-1)^{k-1}(z_i|\alpha)^{\mu_k-1}.$$

Since between the i-th row μ and the $(i+1)$-st row ν of the Gelfand-Tsetlin pattern one element is discarded, there is some permutation σ of $\{1,2,3,\cdots,n\}$ such that ν is obtained by dropping the $\sigma(i)$-th element of μ. In other words,

$$\mu_k-1=(\lambda+\rho)_{\sigma(i)}-1=(\lambda+\delta)_{\sigma(i)}$$

and we conclude that

$$Z(\mathfrak{S}^{\Gamma}_{\lambda,-1})=\sum_{\sigma\in S_n}\pm\prod_i(z_i|\alpha)^{(\lambda+\delta)_{\sigma(i)}}.$$

The signs may be determined as follows. First, take all $\alpha_i=0$ and $t=-1$. The term corresponding to σ is then $\pm\prod_i z_i^{(\lambda+\delta)_{\sigma(i)}}$. The ratio (10) is symmetric, and with $t=-1$ the denominator is antisymmetric. This shows

$$Z(\mathfrak{S}^{\Gamma}_{\lambda,-1})=\pm\sum_{\sigma\in S_n}(-1)^{l(\sigma)}\prod_i(z_i|\alpha)^{(\lambda+\delta)_{\sigma(i)}}=\pm A_{\lambda+\delta}(z|\alpha).$$

We still need to determine the leading $\pm$. Using (8) we see that (10) equals $\pm s_\lambda(z|\alpha)$. To evaluate the sign, we may take $t=0$ and all $z_i=1$. Then the partition function is a sum of positive terms, and $s_\lambda(1,\cdots,1)$ is positive, proving that (10) equals $s_\lambda(z|\alpha)$. $\square$

5 The Combinatorial Definition of Factorial Schur Functions

Macdonald also gives a combinatorial formula for the factorial Schur function as a sum over semi-standard Young tableaux of shape λ in $\{1, 2, \cdots, n\}$. Let T be such a tableau. This generalizes the well-known combinatorial formula for Schur functions. If i, j are given such that $j \leqslant \lambda_i$, let $T(i, j)$ be the entry in the i-th row and j-th column of T. Let

$$T^*(i, j) = T(i, j) + j - i, \tag{11}$$

and define

$$(z|\alpha)^T = \prod_{(i,j)} (z_{T(i,j)} + \alpha_{T^*(i,j)}). \tag{12}$$

For example, if $\lambda = (4, 2, 0)$ and $n = 3$, then we might have

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}\,, \qquad T^* = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} \tag{13}$$

and

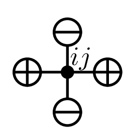
$$(z|\alpha)^T = (z_1 + \alpha_1)(z_1 + \alpha_2)(z_1 + \alpha_3)(z_3 + \alpha_6)(z_2 + \alpha_1)(z_2 + \alpha_2).$$

Theorem 2. *Let λ be a partition. Then*

$$s_\lambda(z|\alpha) = \sum_T (z|\alpha)^T, \tag{14}$$

where the sum is over semistandard Young tableaux with shape λ in $1, 2, 3, \cdots, n$.

This formula, expressing the factorial Schur function as a sum over semi-standard Young tableaux, is equivalent (in a special case) to a formula of Biedenharn and Louck [BL], who made it the definition of the factorial Schur function. For them, the sum was over the set of Gelfand-Tsetlin patterns with prescribed top row, but this is in bijection with tableaux. In this generality, the formula is due to Macdonald [Mcd1].

When $t = 0$, the Boltzmann weight for  is zero, so we are limited to states omitting this configuration. If $\mathfrak{T} = \mathfrak{T}_{\lambda+\rho}$ is the Gelfand-Tsetlin pattern corresponding to this state, and if the entries of $\mathfrak{T}$ in are denoted p_{ik} as in (5), it is easy to see that the equality $p_{i-1,k-1} = p_{i,k}$ would cause this configuration to appear at the i, j position, where $j = p_{i,k}$. Therefore $p_{i-1,k-1} > p_{i,k}$. This inequality implies that we may obtain another Gelfand-Tsetlin pattern $\mathfrak{T}_\lambda$ with top row λ by subtracting $\rho_{n-i+1} = (n-i+1, n-i, \cdots, 1)$ from the i-th row of $\mathfrak{T}_{\lambda+\rho}$.

Consider the example of $\lambda = (4, 2, 0)$ with $n = 3$. Then

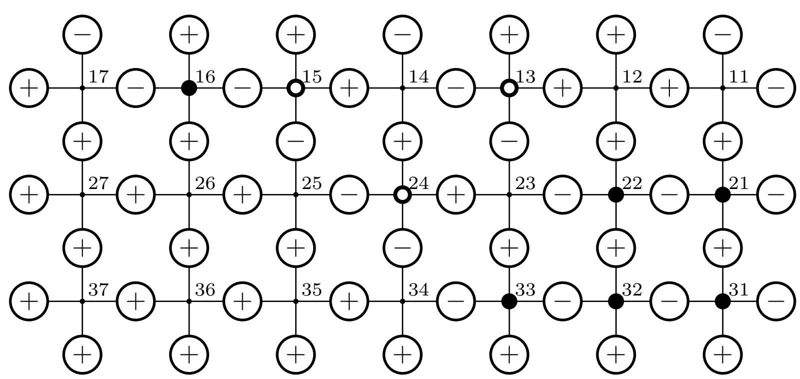
$$\text{if} \quad \mathfrak{T}_{\lambda+\rho} = \left\{ \begin{array}{ccccc} 7 & & 4 & & 1 \\ & 5 & & 3 & \\ & & 4 & & \end{array} \right\} \quad \text{then} \quad \mathfrak{T}_\lambda = \left\{ \begin{array}{ccccc} 4 & & 2 & & 0 \\ & 3 & & 2 & \\ & & 3 & & \end{array} \right\}.$$

We associate with $\mathfrak{T}_\lambda$ a tableau $T(\mathfrak{T}_\lambda)$ of shape λ. In this tableau, removing all boxes labeled $\boxed{n}$ from the diagram produces a tableau whose shape is the second row of $\mathfrak{T}_\lambda$. Then removing boxes labeled $\boxed{n-1}$ produces a tableau whose shape is the third row of $\mathfrak{T}_\lambda$, and so forth. Thus in the example, $T(\mathfrak{T}_\lambda)$ is the tableau T in (13). Let w_0 denote the long element of the Weyl group S_n, which is the permutation $i \mapsto n+1-i$.

Proposition 4. *Let $t = 0$, and let $\mathfrak{s} = \mathfrak{s}(\mathfrak{T}_{\lambda+\rho})$ be the state corresponding to a special Gelfand-Tsetlin pattern $\mathfrak{T}_{\lambda+\rho}$. With $\mathfrak{T}_\lambda$ as above and $T = T(\mathfrak{T}_\lambda)$ we have*

$$w_0 \left(\prod_{v \in \mathfrak{s}} \beta_{\mathfrak{s}}(v) \right) = \boldsymbol{z}^{w_0(\delta)} (z|a)^T. \tag{15}$$

Before describing the proof, let us give an example. With $\mathfrak{T}_{\lambda+\rho}$ as above, the state $\mathfrak{s}(\mathfrak{T}_{\lambda+\rho})$ is



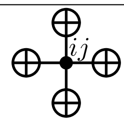
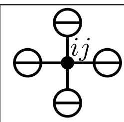
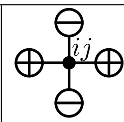
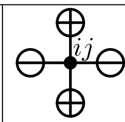
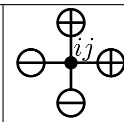
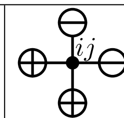
The locations labeled $\circ$ produce powers of z_i, and the locations labeled $\bullet$ produce shifts of the form $z_i + \alpha_j$. The weight of this state is

$$\prod_{v \in \mathfrak{s}} \beta_{\mathfrak{s}}(v) = z_1^2 z_2 (z_1 + \alpha_6)(z_2 + \alpha_2)(z_2 + \alpha_1)(z_3 + \alpha_3)(z_3 + \alpha_2)(z_3 + \alpha_1).$$

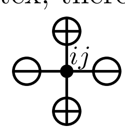
Applying w_0 interchanges $z_i \longleftrightarrow z_{4-i}$. The factor $z_1^2 z_2$ becomes $z_3^2 z_2 = \boldsymbol{z}^{w_0(\delta)}$, and the terms that remain agree with

$$\prod_{(i,j)} (z_{T(i,j)} + \alpha_{T^*(i,j)}).$$

Proof. We are using the following Boltzmann weights.

Gamma Ice						
Boltzmann weight	1	z_i	0	$z_i + \alpha_j$	z_i	1

We have contributions of z_i from vertices that have $-$ on the vertical edge below, and there is one of these for each entry in the Gelfand-Tsetlin pattern. These contribute a factor of $\boldsymbol{z}^{\delta}$. Applying w_0 to $\boldsymbol{z}$ leads to $\boldsymbol{z}^{w_0(\delta)}$.

Considering the contribution from the (i, j) vertex, there will be a factor of $z_i + \alpha_j$ when the vertex has the configuration  . In the above example, the locations are labeled by $\bullet$.

Let $\mathfrak{T}_{\lambda+\rho}$ be the Gelfand-Tsetlin pattern (5). Let $\mathfrak{T}_{\lambda}$ and $T = T(\mathfrak{T}_{\lambda})$ be as described above, and let T^* be as in (11). Let the entries in $\mathfrak{T}_{\lambda+\rho}$ and $\mathfrak{T}_{\lambda}$ be denoted $p_{i,j}$ and $q_{i,j}$, with the indexing as in (5). Thus $q_{i,j} = p_{i,j} - n + j - 1$. We will make the convention that $q_{i,n+1} = p_{i,n+1} = 0$. The condition for $\bullet$ in the i, j position of the state is that for some k with $i \leqslant k \leqslant n$ we have $p_{i+1,k+1} < j < p_{i,k}$. Translating this in terms of the $q_{i,j} = p_{i,j} - n + j - 1$ the condition becomes $q_{i+1,k+1} + n - k + 1 \leqslant j < q_{i,k} + n - k$. The effect of w_0 is to interchange $z_i \leftrightarrow z_{n-i+1}$, and therefore

$$w_0 \left(\prod_{v \in \mathfrak{s}} \beta_{\mathfrak{s}}(v) \right) = \boldsymbol{z}^{w_0(\delta)} \prod_{i=1}^{n} \prod_{k=i}^{n} \prod_{j=q_{i+1,k+1}+n-k+1}^{q_{i,k}+n-k} (z_{n+1-i} + \alpha_j). \tag{16}$$

On the other hand, in the tableau T, the location of the entries equal to $\boxed{n+1-i}$ in the $(k+1-i)$-th row is between columns $q_{i+1,k+1}+1$ through $q_{i,k}$, and if j is one of these columns then

$$T(k+1-i,j) = n+1-i, \qquad T^*(k+1-i,j) = n+j-k.$$

Therefore in the notation (12) we have

$$(z|\alpha)^T = \prod_{i=1}^{n} \prod_{k=i}^{n} \prod_{j=q_{i+1,k+1}+1}^{q_{i,k}} (z_{n+1-i} + \alpha_{n+j-k}).$$

This equals (16) and the proof is complete. $\square$

We now give the proof of Theorem 2. Summing over states, the last Proposition implies that

$$Z_\lambda(w_0(\boldsymbol{z}), \alpha, 0) = \boldsymbol{z}^{w_0(\delta)} \sum_T (z|a)^T.$$

Since $s_\lambda^\Gamma(\boldsymbol{z}, \alpha, 0) = s_\lambda^\Gamma(w_0(\boldsymbol{z}), \alpha, 0)$ we have

$$s_\lambda^\Gamma(z, \alpha, 0) = \frac{Z_\lambda(w_0(\boldsymbol{z}), \alpha, 0)}{\prod_{i>j} w_0(\boldsymbol{z})_j} = \frac{\boldsymbol{z}^{w_0(\delta)} \sum_T (z|a)^T}{\boldsymbol{z}^{w_0(\delta)}} = \sum_T (z|a)^T,$$

and the statement follows.

6 The limit as t tends to infinity

Let μ be a partition, and let α_μ denote the sequence

$$\alpha_\mu = (\alpha_{\mu_1+n}, \alpha_{\mu_2+n-1}, \cdots, \alpha_{\mu_n+1}).$$

If λ is a partition then λ' will denote the conjugate partition whose Young diagram is the transpose of that of λ.

Theorem 3. (Vanishing Theorem) *We have*

$$s_\lambda(-\alpha_\mu|\alpha) = \begin{cases} 0 & \text{if } \lambda \not\subset \mu, \\ \prod_{(i,j)\in\lambda} (\alpha_{n-i+\lambda_i+1} - \alpha_{n-\lambda'_j+j}) & \text{if } \lambda = \mu. \end{cases}$$

See Okounkov [Ok] Section 2.4 and Molev and Sagan [MS]. In view of the relationship between Schubert polynomials for Grassmannian permutations and factorial Schur functions, this is equivalent to an older vanishing statement for Schubert polynomials. Vanishing properties for Schubert polynomials are implied by Theorem 9.6.1 and Proposition 9.6.2 of Lascoux [La1], which are related to the results of Lascoux and Schützenberger [LS2, LS3]. We will prove it in this section by our methods.

We examine the behavior of our six-vertex model as we send the parameter t to infinity. The first result of this section may be construed as a rederivation of a theorem of Lascoux [La2, Theorem 1]. However the approach we take is to interpret it as giving us a proof of the equivalence of factorial Schur functions and double Schubert polynomials for Grassmannian permutations. We also obtain the vanishing theorem for factorial Schur functions (Theorem 3).

We start with two simple lemmas. To state these, we describe the six admissible arrangements of edge spins around a vertex as types a_1, a_2, b_1, b_2, c_1 and c_2 respectively, when reading from left to right in the diagram (2).

Lemma 2. *In each state, the total number of sites of type a_2, b_1 and c_1 is equal to $n(n-1)/2$.*

Proof. This number is equal to the number of minus spins located in the interior of a vertical string. $\square$

Let μ be the partition $(\lambda+\delta)'$.

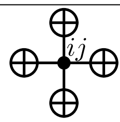
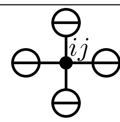
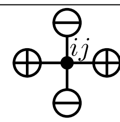
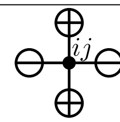
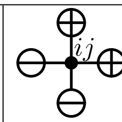
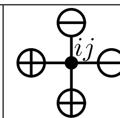
Lemma 3. *In each state, the number of occurrences of a_2, b_2 and c_1 patterns in the i-th column is equal to μ_i.*

Proof. This number is equal to the number of minus spins located on a horizontal string between the strings labeled i and $i+1$. This count is known since for any rectangle, knowing the boundary conditions on the top, bottom and rightmost sides determines the number of such spins along the leftmost edge. $\square$

As a consequence of Lemma 2, the result of taking the limit as $t \to \infty$ can be interpreted as taking the leading degree term in t for each of the Boltzmann weights $v_\Gamma(i,j,t)$ as in Section 4. Thus with the set of Boltzmann weights $(1, -\alpha_j, 1, z_i + \alpha_j, z_i, 1)$ and corresponding partition function $Z(\mathfrak{S}^\Gamma_{\lambda,\infty})$, we have

$$Z(\mathfrak{S}^\Gamma_{\lambda,\infty}) = z^\delta s_\lambda(z|\alpha).$$

By Lemma 3, if we consider our ice model with the series of Boltzmann weights in the following diagram, then the corresponding partition function is given by dividing $Z(\mathfrak{S}^{\Gamma}_{\lambda,\infty})$ by $(-\alpha)^{\mu}$.

Gamma Ice						
Boltzmann weight	1	1	1	$-z_i/\alpha_j - 1$	$-z_i/\alpha_j$	1

Let us denote the partition function for this set of Boltzmann weights by $Z(\mathfrak{S}^{\Gamma'}_{\lambda,\infty}(z|\alpha))$. Thus we obtain the result of [McN, Theorem 1.1],

$$Z(\mathfrak{S}^{\Gamma'}_{\lambda,\infty}(z|\alpha)) = \frac{\boldsymbol{z}^{\delta}}{(-\boldsymbol{\alpha})^{(\lambda+\delta)'}} s_{\lambda}(z|\alpha). \tag{17}$$

We now pause to introduce the notions of double Schubert polynomials and Grassmannian permutations so that we can make the connection to [La2, Theorem 1] precise.

A permutation is *Grassmannian* if it has a unique (right) descent.

Let n and m be given and let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a partition such that $\lambda_1 \leqslant m$. Then there is associated with λ a Grassmannian permutation $w_\lambda \in S_{n+m}$. This is the permutation such that

$$w_\lambda(i) = \begin{cases} \lambda_{n+1-i} + i & \text{if } i \leqslant n, \\ i - \lambda'_{i-n} & \text{if } i > n, \end{cases}$$

where λ' is the conjugate partition. This has $w_\lambda(n+1) < w_\lambda(n)$ and no other descent.

Let $x_1, \cdots, x_{n+m}$ and $y_1, \cdots, y_{n+m}$ be parameters. We define the divided difference operators as follows. If $1 \leqslant i < n+m$ and f is a function of the x_i let

$$\partial_i f(x_1, \cdots, x_{n+m}) = \frac{f - s_i f}{x_i - x_{i+1}}$$

where $s_i f$ is the function obtained by interchanging x_i and x_{i+1}. Then if $w \in S_{n+m}$, let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression of w as a product of simple reflections. Then let $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$. This is well-defined since the divided difference operators ∂_i satisfy the braid relations. Let w_0 be the long element of S_{n+m}. Then the *double Schubert polynomials*, which were defined by Lascoux and Schützenberger [LS1] are given by

$$\mathfrak{S}_w(x,y) = \partial_{w^{-1}w_0}\left(\prod_{i+j\leqslant n+m}(x_i - y_j)\right).$$

The theory of factorial Schur functions is a special case of the theory of double Schubert polynomials developed by Lascoux and Schützenberger [LS1, LS2]. Although the comparison is well-known, there does not appear to be a truly satisfactory reference in the literature. We give a new proof in the thematic spirit of this paper.

Theorem 4. *The factorial Schur functions are equal to double Schubert polynomials for Grassmannian permutations. More precisely,*

$$\mathfrak{S}_{w_\lambda}(x,y) = s_\lambda(x|-y)$$

Proof. The proof is a comparison of (17) with [La2, Theorem 1]. To translate the left-hand side in (17) into the staircase language of Lascoux we use the bijection in Proposition 2. In Theorem 1 of [La2] if Lascoux' x is our z and his y is our $-\alpha$, then the left-hand side of his identity is exactly the partition function on the left-hand side of our (17). The monomial $x^{\rho_r}y^{-\langle\tilde{u}\rangle}$ on the right-hand side of his identity equals the monomial $z^\rho(-\alpha)^{-(\lambda+\delta)'}$ on the right-hand side of (17). Lascoux' $\mathbb{X}_{\tilde{u},u^\omega}$ is the double Schubert polynomial $\mathfrak{S}_{w_\lambda}$. The statement follows. $\square$

To conclude this section, we shall use this description to give a proof of the characteristic vanishing property of Schur functions. In lieu of (17) above, the following result is clearly equivalent to Theorem 3.

Theorem 5. *For two partitions λ and μ of at most n parts, we have*

$$Z(\mathfrak{S}^{\Gamma'}_{\lambda,\infty}(-\alpha_\mu|\alpha)) = 0 \textit{ unless } \lambda \subset \mu,$$

$$Z(\mathfrak{S}^{\Gamma'}_{\lambda,\infty}(-\alpha_\lambda|\alpha)) = \prod_{(i,j)\in\lambda}\left(\frac{\alpha_{n+1-i+\lambda_i}}{\alpha_{n-\lambda'_j+j}} - 1\right).$$

Proof. Fix a state, and assume that this state gives a non-zero contribution to the partition function $Z(\mathfrak{S}^{\Gamma'}_{\lambda,\infty}(-\alpha_\mu|\alpha))$. Under the bijection between states of square ice and strict Gelfand-Tsetlin patterns, let k_i be the leftmost entry in the i-th row of the corresponding Gelfand-Tsetlin pattern. We shall prove by descending induction on i the inequality

$$n+1-i+\mu_i \geq k_i.$$

For any j such that $k_{i+1} < j < k_i$, there is a factor $(x_i/\alpha_j - 1)$ in the Boltzmann weight of this state. We have the inequality $n+1-i+\mu_i > n+1-(i+1)+\mu_{i+1} \geq k_{i+1}$ by our inductive hypothesis. Since $(\alpha_\mu)_i = \alpha_{n+1-i+\mu_i}$, in order for this state to give a non-zero contribution to $Z(\mathfrak{S}^{\Gamma'}_{\lambda,\infty}(-\alpha_\mu|\alpha))$, we must have that $n+1-i+\mu_i \geq k_i$, as required.

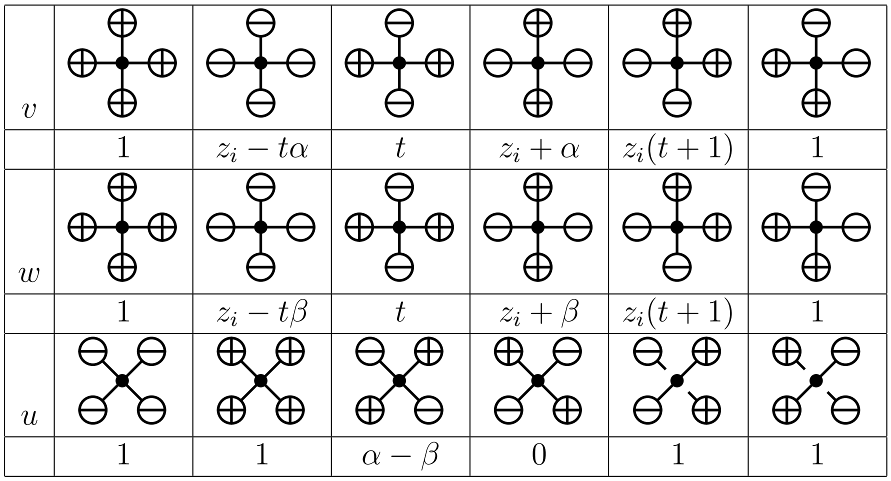
Note that for all i, we have $k_i \geq n+1-i+\lambda_i$. Hence $\mu_i \geq \lambda_i$ for all i, showing that $\mu \supset \lambda$ as required, proving the first part of the theorem.

To compute $Z(\mathfrak{S}^{\Gamma'}_{\lambda,\infty}(-\alpha_\lambda|\alpha))$, notice that the above argument shows that there is only one state which gives a non-zero contribution to the sum. Under the bijection with Gelfand-Tsetlin patterns, this is the state with $p_{i,j} = p_{1,j}$ for all i, j. The formula for $Z(\mathfrak{S}^{\Gamma'}_{\lambda,\infty}(-\alpha_\lambda|\alpha))$ is now immediate. $\square$

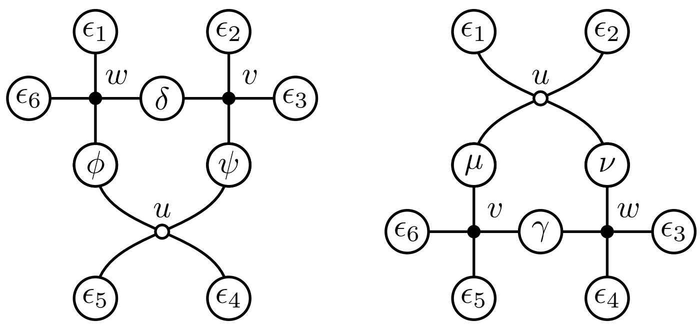
7 Asymptotic Symmetry

Macdonald [Mcd1] shows that the factorial Schur functions are *asymptotically symmetric* in the α_i as the number of parameters z_i increases. To formulate this property, let σ be a permutation of the parameters $\alpha = (\alpha_1, \alpha_2, \cdots)$ such that $\sigma(\alpha_j) = \alpha_j$ for all but finitely many j. Then we will show that if the number of parameters z_i is sufficiently large, then $s_\lambda(z|\sigma\alpha) = s_\lambda(z|\alpha)$. How large n must be depend on both λ and on the permutation σ.

We will give a proof of this symmetry property of factorial Schur functions using the Yang-Baxter equation. In the following theorem we will use the following Boltzmann weights:



Theorem 6. *With the above Boltzmann weights, and with* $\epsilon_1, \cdots, \epsilon_6$ *fixed spins* $\pm$*, the following two systems have the same partition function.*



Proof. This may be deduced from Theorem 3 of [BBF] with a little work. By rotating the diagram 90° and changing the signs of the horizontal edge spins, this becomes a case of that result. The R-matrix is not the matrix for u given above, but a constant multiple. We leave the details to the reader. $\square$

Let σ_i be the map on sequences $\alpha = (\alpha_1, \alpha_2, \cdots)$ that interchanges α_i and α_{i+1}. Let λ be a partition. We will show that sometimes:

$$s_\lambda(z|\alpha) = s_\lambda(z|\sigma_i \alpha) \tag{18}$$

and sometimes

$$s_\lambda(z|\alpha) = s_\lambda(z|\sigma_i \alpha) + s_\mu(z|\sigma_i \alpha)(\alpha_i - \alpha_{i+1}), \tag{19}$$

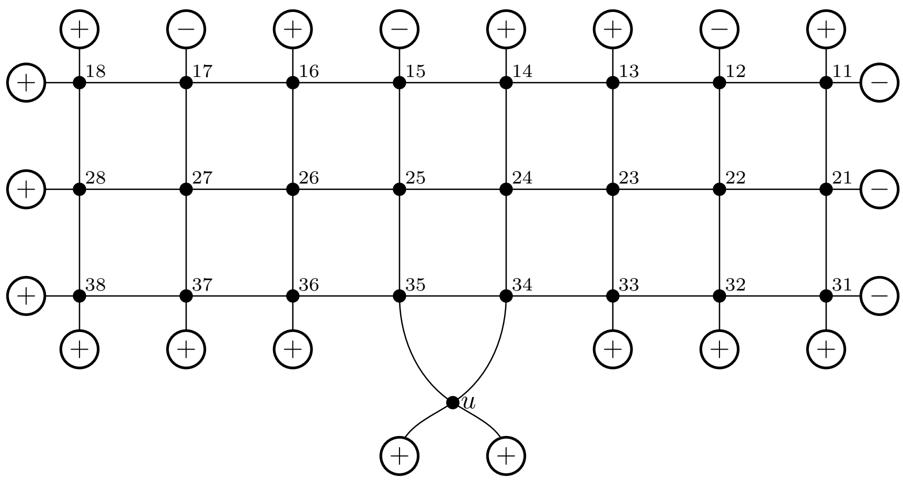
where μ is another partition. The next Proposition gives a precise statement distinguishing between the two cases (18) and (19).

Proposition 5. *(i) Suppose that $i+1 \in \lambda+\rho$ but that $i \notin \lambda+\rho$. Let μ be the partition characterized by the condition that $\mu+\rho$ is obtained from $\lambda+\rho$ by replacing the unique entry equal to $i+1$ by i. Then (19) holds with this μ.*

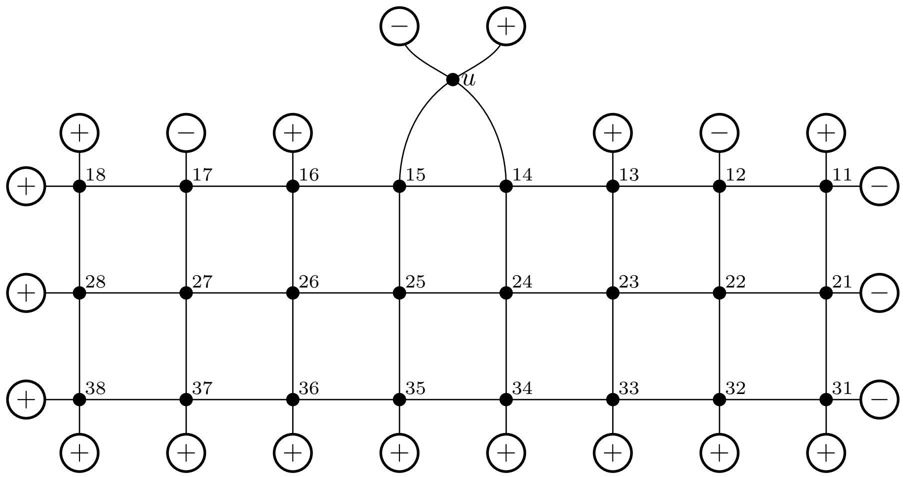
(ii) If either $i+1 \notin \lambda+\rho$ or $i \in \lambda+\rho$ then (18) holds.

For example, suppose that $\lambda = (3,1)$ and $n = 5$. Then $\lambda+\rho = (8,5,3,2,1)$. If $i = 4$, then $i+1 = 5 \in \lambda+\rho$ but $i \notin \lambda+\rho$, so (19) holds with $\mu+\rho = (8,4,3,2,1)$, and so $\mu = (3)$.

Proof. We will use Theorem 6 with $\alpha = \alpha_i$ and $\beta = \alpha_{i+1}$. The parameter t may be arbitrary for the following argument. We first take $i = n$ and attach the vertex u below the i and $i+1$ columns, arriving at a configuration like this one in the case $n = 3$, $\lambda = (4,3,1)$.



There is only one legal configuration for the spins of the edges between u and the two edges above it, which in the example connect with $(3,5)$ and $(3,4)$: these must both be $+$. The Boltzmann weight at u in this configuration is unchanged, and the partition function of this system equals that of $\mathfrak{S}^{\Gamma}_{\lambda,t}$. After applying the Yang-Baxter equation, we arrive at a configuration with the u vertex above the top row, as follows:



Now if we are in case (i), the spins of the two edges are $-$, $+$ then there are two legal configurations for the vertex, and separating the contribution these we obtain

$$Z(\mathfrak{S}_{\lambda,t}) = (\sigma_i Z(\mathfrak{S}_{\lambda,t})) + (\alpha - \beta)\,(\sigma_i Z(\mathfrak{S}_{\mu,t}))\,.$$

If we are in case (ii), there is only one legal configuration, so $Z(\mathfrak{S}_{\lambda,t}) = (\sigma_i Z(\mathfrak{S}_{\lambda,t}))\,.$ □

Corollary 1. *Let λ be a partition, and let l be the length of λ. If $n \geqslant l + i$ then $s_\lambda(z|a) = s_\lambda(z|\sigma_i a)$.*

Proof. For if l is the length of the partition λ, then the top edge spin in column j is $-$ when $j \leqslant n - l$, and therefore we are in case (i). □

This Corollary implies Macdonald's observation that the factorial Schur functions are asymptotically symmetric in the α_i.

8 The Dual Cauchy Identity

Let m and n be positive integers. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \leq m$, we define a new partition $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_m)$ by

$$\hat{\lambda}_i = |\{j \mid \lambda_j \leq m - i\}|.$$

After a reflection, it is possible to fit the Young diagrams of λ and $\hat{\lambda}$ into a rectangle.

We shall prove the following identity, known as the dual Cauchy identity. Another proof may be found in Macdonald [Mcd1] (6.17). In view of the relationship between factorial Schur functions and Schubert polynomials, this is equivalent to a statement on page 161 of Lascoux [La1]. See also Corollary 2.4.8 of Manivel [Ma] for another version of the Cauchy identity for Schubert polynomials.

Theorem 7 (Dual Cauchy Identity)**.** *For two finite alphabets of variables* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_m)$*, we have*

$$\prod_{i=1}^{n} \prod_{j=1}^{m} (x_i + y_j) = \sum_{\lambda} s_\lambda(x|\alpha) s_{\hat{\lambda}}(y| - \alpha).$$

The sum is over all partitions λ *with at most* n *parts and with* $\lambda_1 \leq m$*.*

Proof. The proof will consist of computing the partition function of a particular six-vertex model in two different ways. We will use the weights $v_\Gamma(i, j, t)$ introduced in Section 4 with $t = 1$ and for our parameters $(z_1, \ldots, z_{m+n})$, we will take the sequence $(y_m, \ldots, y_1, x_1, \ldots, x_n)$.

As for the size of the six-vertex model we shall use and the boundary conditions, we take a $(m + n) \times (m + n)$ square array, with positive spins on the left and lower edges, and negative spins on the upper and right edges. By Theorem 1 (with $\lambda = 0$), the partition function of this array is

$$\prod_{i<j} (y_i + y_j) \prod_{i,j} (y_i + x_j) \prod_{i<j} (x_i + x_j).$$

We shall partition the set of all states according to the set of spins that occur between the rows with parameters y_1 and x_1. Such an arrangement of spins corresponds to a partition λ in the usual way, i.e. the negative spins are in the columns labeled $\lambda_i + n - i + 1$. In this manner we can write our partition function as a sum

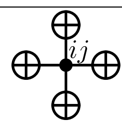
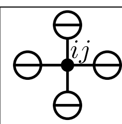
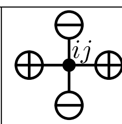
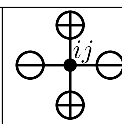
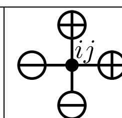
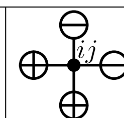
$$\sum_{\lambda} Z_\lambda^{\text{top}} Z_\lambda^{\text{bottom}}.$$

Here Z_λ^{top} is the partition function of the system with m rows, $m + n$ columns, parameters $y_m, \ldots, y_1$ and boundary conditions of positive spins on the left, negative spins on the top and right, and the λ boundary condition spins on the bottom. And $Z_\lambda^{\text{bottom}}$ is the partition function of the system with n rows, $m+n$ columns, spectral parameters $x_1, \ldots, x_n$ and the usual boundary conditions for the partition λ.

By Theorem 1, we have

$$Z_\lambda^{\text{bottom}} = \prod_{i<j}(x_i + x_j) s_\lambda(x|\alpha).$$

It remains to identify Z_λ^{top}. To do this, we perform the following operation to the top part of our system. We flip all spins that lie on a vertical strand, and then reflect the system about a horizontal axis. As a consequence, we have changed the six-vertex system that produces the partition function Z_λ^{top} into a system with more familiar boundary conditions, namely with positive spins along the left and bottom, negative spins along the right hand side, and along the top row we have negative spins in columns $\hat{\lambda}_i + m - i + 1$. The Boltzmann weights for this transformed system are (since $t = 1$)

Gamma Ice						
Boltzmann weight	1	$y_i + \alpha_j$	1	$y_i - \alpha_j$	$2y_i$	1

Now again we use Theorem 1 to conclude that

$$Z_\lambda^{\text{top}} = \prod_{i<j}(y_i + y_j) s_{\hat{\lambda}}(y| - \alpha).$$

Comparing the two expressions for the partition function completes the proof. $\square$

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