Moments of the Riemann Zeta Function and Eisenstein Series I

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Abstract

It is shown that if the parameters of an Eisenstein series on $\mathrm{GL}(2k)$ are chosen so that its (integrated) L-function is the $2k$-th moment of the Riemann zeta function, then the $\binom{2k}{k}$ terms in its constant term agree with $\binom{2k}{k}$ factors appearing in a conjectural formula for the $2k$-th moment of zeta by Conrey, Farmer, Keating, Rubinstein and Snaith. When $k = 1$, an explanation for this phenomenon is found by deducing Oppenheim's generalization of the Voronoï summation formula from the Eisenstein series and representation theoretic considerations. The possibility of eliminating the problematical "arithmetic factor" is discussed.

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There is reason to expect that the $2k$-th moment of the Riemann zeta function can be related to the spectral theory of $\mathrm{GL}(k)$ or $\mathrm{GL}(2k)$. The work of Motohashi [27] supports the idea of seeking such an approach, by finding an explicit formula for the fourth moment of ζ involving special values of L-functions of Maass cusp forms for $\mathrm{SL}(2, \mathbb{Z})$. Still an automorphic attack on the higher moments of the zeta function has proved an elusive goal.

Recently Conrey, Farmer, Keating, Rubinstein and Snaith [9] gave conjectural asymptotics for the higher moments. These conjectures are supported by heuristics from Random Matrix Theory and Analytic Number Theory and by numerical computation. They are also implied by an independent conjecture of Diaconu, Goldfeld and Hoffstein [11]. We will argue that these recent conjectures provide clues as to how such an automorphic attack might be formulated. In fact, we will argue for a close connection between the $2k$-th moment of zeta and an Eisenstein series on $\mathrm{GL}(2k)$.

Once it is understood that such a connection may exist, even for the second moment, it is not immediately clear how the classical results can be related to the Eisenstein series on GL(2). The purpose of this paper is to present the evidence for a link between the $2k$-th moment and the Eisenstein series on $\mathrm{GL}(2k)$, and to establish a solid basis for this connection when $k = 1$.

The second and fourth moments of ζ are well understood. Beyond the fourth moment, there are recent conjectures, beginning with that of Conrey and Ghosh [10]. Although the moment of greatest interest is

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt, \tag{1}$$

recent authors, including Motohashi [27] and Conrey, Farmer, Keating, Rubinstein and Snaith [9] have emphasized that it is better to consider an integral such as

$$\int_0^T \zeta\left(\sigma_1+it\right)\cdots\zeta(\sigma_k+it)\zeta\left(\sigma_{k+1}-it\right)\cdots\zeta(\sigma_{2k}-it)\,dt, \tag{2}$$

since the asymptotics of such a moment reveal a structure not apparent in (1). If the asymptotics of (2) are known, then the asymptotics of (1) can be deduced as a limiting case.

The authors of [9] found that the dominant terms in (2) are $\binom{2k}{k}$ in number, and each involves a product of k^2 zeta functions. We will show that this identical structure is exhibited in the constant term of a certain Eisenstein series on $\mathrm{GL}(2k)$.

Beginning with the second moment, Ingham [16] proved that if $0 < \sigma < 1$ and $\sigma \neq \frac{1}{2}$ then

$$\int_0^T |\zeta(\sigma+it)|^2\,dt = \zeta(2\sigma)T + \frac{(2\pi)^{2\sigma-1}}{2-2\sigma}\zeta(2-2\sigma)T^{2-2\sigma} + O(T^{1-\sigma}\log(T)). \tag{3}$$

We may compare this with the constant term of the classical Eisenstein series on $\mathrm{SL}(2,\mathbb{Z})$,

$$E_\sigma(z) = \tfrac{1}{2}\,\zeta(2\sigma)\sum_{(c,d)=1}\left(\frac{y}{|cz+d|^2}\right)^\sigma, \qquad z = x+iy, \quad y > 0.$$

The series is convergent if $\mathrm{re}(\sigma) > 1$ but has meromorphic continuation to all σ. This Eisenstein series is relevant to (3) because its L-function is

$$L(s, E_\sigma) = \zeta\left(s + \sigma - \tfrac{1}{2}\right)\zeta\left(s - \sigma + \tfrac{1}{2}\right),$$

so

$$L\left(\tfrac{1}{2} + it, E_\sigma\right) = \zeta(\sigma + it)\zeta(1 - \sigma + it) = \chi(1 - \sigma + it)|\zeta(\sigma + it)|^2,$$

where $\chi(s) = \pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)^{-1}$. On the other hand, the constant term

$$\int_0^1 E_\sigma(x + iy)\,dx = \zeta(2\sigma)y^\sigma + \pi^{2\sigma-1}\frac{\Gamma(1-\sigma)}{\Gamma(\sigma)}\zeta(2 - 2\sigma)y^{1-\sigma}. \qquad (4)$$

We find that **if the Eisenstein series is selected so that its L-function matches the integrand on the left side in (3), then the zeta functions in the two components of its constant term match the two terms on the right side of (3).**

Assuming the conjectural asymptotics in [9], we will show in Section 1 that this phenomenon extends to the $2k$-th moment. For example in the fourth moment of ζ the largest terms are six in number, each a product of four zeta functions. These may be seen in the analysis in Section 1.7 of [9] of the results of Motohashi [27]. We will show that there exists an Eisenstein series on GL(4) whose L-function matches the fourth moment, and whose constant term

$$\int_0^1\int_0^1\int_0^1\int_0^1 E\left(\begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, s\right) dx\,dy\,dz\,dw$$

consists of six terms, each involving a product of four zeta functions, which match the six terms on the right-hand side of (1.7.6) in [9]. And we will check that this same precise correspondence works for all k by exhibiting an Eisenstein series on GL($2k$) whose L-function and constant term, a sum of $\binom{2k}{k}$ products of k^2 zeta functions, both match perfectly the $2k$-th moment and its conjectured asymptotics.

There is one aspect to this correspondence which remains problematical. This is the *arithmetic factor* which occurs in the conjectural asymptotics of [9]. We will discuss the arithmetic factor below in Section 2.

So far the connection that we have described between moments and Eisenstein series appears as a simple coincidence between data assocated with the Eisenstein series and data associated with the moments. The complexity of this data is sufficient that we do not believe it possible that it is coincidental. However our case will be strengthened by exhibiting a direct connection between the second moment and the Eisenstein series E_σ.

This connection comes about through a generalization, due to Oppenheim [28], of the famous Voronoï [31] summation formula. Let us state Oppenheim's formula in a smoothed version. If $a \in \mathbb{C}$ let $\sigma_a(n)$ be the classical divisor function, and let

$$\tau_a(n) = \sum_{d|n} \left(\frac{d}{n/d}\right)^a = \sigma_{2a}(n) n^{-a}$$

be the symmetrical divisor function, so $\tau_a = \tau_{-a}$. Let ϕ be a continuous function with compact support in $(0, \infty)$. In terms of standard Bessel functions (Watson [32]) let

$$\psi_s(y) = \int_0^\infty \phi(x)[-2\pi\cos(s\pi)J_{1-2s}(4\pi\sqrt{yx}) - 2\pi\sin(s\pi)Y_{1-2s}(4\pi\sqrt{yx})+ \\ 4\sin(s\pi)K_{1-2s}(4\pi\sqrt{yx})]dx. \tag{5}$$

We will show in Proposition 7 that $\psi_s(y) \longrightarrow 0$ rapidly as $y \longrightarrow \infty$, and we will prove the following theorem.

Theorem 1 *If ϕ has compact support in $(0, \infty)$ and ψ_s is given by (5) we have*

$$\sum_{n=1}^\infty \tau_{s-1/2}(n)\phi(n) = \\ \zeta(2s)\int_0^\infty \phi(x)x^{s-1/2}dx + \zeta(2-2s)\int_0^\infty \phi(x)x^{1/2-s}dx \\ + \sum_{n=1}^\infty \tau_{s-1/2}(n)\psi_s(n). \tag{6}$$

We will prove Theorem 1 by associating with ϕ a smooth vector in a principal series representation of $\mathrm{SL}(2, \mathbb{R})$. We will then consider an Eisenstein series on $\mathrm{SL}(2, \mathbb{Z})$. Since the Eisenstein series is automorphic, its value at the identity equals its value at a Weyl group element, and this relationship implies (6).

Although Oppenheim's generalization of the Voronoï summation formula is most relevant for our investigation, another generalization of Voronoï's formula, due to Wilton [34] deserves mention in this context. Wilton's formula involves the Fourier coefficients of Ramanujan's τ function, and as such is essentially a summation formula for the Fourier coefficients of an automorphic form. This, of course is how we view the coefficients $\tau_{s-1/2}$: they are Fourier coefficients of Eisenstein series. Wilton's summation formula for Ramanujan's $\tau(n)$ was not unprecedented, since Voronoï himself stated (and Hardy and Landau proved) a summation formula for $r(n)$, the number of representations of n as a sum of two squares. These coefficients are, like $\tau(n)$, the Fourier coefficients of a modular form. See Wilton [33], Berndt [4] and Miller and Schmid [26] for references to the literature of this problem.

A clear statement of the nature of the connection between Voronoï summation with the "Bessel distribution" in the representation theory of $\mathrm{GL}(2,\mathbb{R})$ may be found in Cogdell [7]. This essential insight explains exactly the reason for the appearance of (6). Another representation-theoretic approach, including a Voronoï summation formula for GL(3) is taken by Miller and Schmid [25] and [26].

It is our hope that the (thus far accidental) coincidence between Eisenstein series on $\mathrm{GL}(2k)$ and the $2k$-th moment of zeta can be explained along these lines for general k. Such a goal would obviously be highly desirable, and it seems to us that the evidence in Section 1 suggests a particular construction. In view of that evidence, we seek a representation of the standard L-function of an automorphic form on $\mathrm{GL}(2k)$ in which the parabolic subgroup with Levi factor $\mathrm{GL}(k) \times \mathrm{GL}(k)$ plays a distinguished role. Such a construction was given by Friedberg and Jacquet [13]. Their representation of the standard L-function unfolds to the so-called *Shalika model*, a unique model which only exists for self-dual automorphic forms. Fortuitously the Eisenstein series of Section 1 is self-dual for

$$|\zeta(\sigma_1+it)|^2 \cdots |\zeta(\sigma_k+it)|^2 = \zeta(\sigma_1+it)\cdots\zeta(\sigma_k+it)\zeta(\sigma_1-it)\cdots\zeta(\sigma_k-it).$$

We hope therefore that a generalization of the summation formula (6) involving "divisor sums" associated with Shalika models can be found, and that such a hypothetical summation formula will play a role in the theory of the higher moments of ζ.

We have not yet described how the Oppenheim summation formula explains Ingham's estimate (3). For $\frac{1}{2} < \sigma < 1$ we have

$$\int_0^T |\zeta(\sigma+it)|^2 dt \sim 2\pi \sum_{n<T/2\pi} \tau_{\sigma-\frac{1}{2}}(n) n^{\frac{1}{2}-\sigma}. \tag{7}$$

Application of the Oppenheim summation formula to the right hand side immediately gives the two main terms on the right hand side of (3).

When $\sigma = \frac{1}{2}$, the relationship (7) appeared in Atkinson [1] and is discussed, for example, in Jutila [18], Ivić [17] and Matsumoto [22]. When $\frac{1}{2} < \sigma < \frac{3}{4}$, this same connection was used by Atkinson [2] and Matsumoto [21] to improve the error term in (3). As these references show, this parallel runs deeper than this simple asymptotic relation, but for our purposes, (7) is sufficient to explain (3).

At first sight (7) seems very mysterious. By the functional equation

$$|\zeta(\sigma+it)|^2 = \chi(\sigma-it)\zeta(\sigma+it)\zeta(1-\sigma+it).$$

And, with re(s) sufficiently large,

$$\chi(\sigma-s)\zeta(\sigma+s)\zeta(1-\sigma+s) = \chi(\sigma-s)\sum_{n=1}^{\infty} \tau_{\sigma-1/2}(n) n^{-\frac{1}{2}-s}. \tag{8}$$

So taking $s = it$ (even though (8) is then divergent) we may regard $|\zeta(\sigma+it)|^2$ as a sort of generating function for the terms on the right-hand side of (7). But why the cut-off after $n = T/2\pi$? Very roughly, the reason is as follows. By Stirling's formula, for fixed $c \in \mathbb{R}$ we have

$$n^{-\frac{1}{2}-c-it}\chi(\sigma-c-it) \cong n^{-\frac{1}{2}-c}\left|\frac{t}{2\pi}\right|^{\frac{1}{2}-\sigma+c} \exp\left(i\left(t\log\left|\frac{t}{2\pi n}\right| - \frac{\pi}{4} - t\right)\right). \tag{9}$$

Taking $c = 0$, and substituting the series (8), ignoring the fact that it is divergent, we obtain a series of oscillatory terms. According to the principle of stationary phase, the biggest contribution to an oscillatory integral will be where the oscillations cease. We have

$$\frac{d}{dt}\left(t\log\left(\frac{t}{2\pi n}\right) - \frac{\pi}{4} - t\right) = \log\left(\frac{t}{2\pi n}\right).$$

This means that the point $t = 2\pi n$ where the oscillations cease is outside the range of integration if $n > T/2\pi$, so these terms are negligibly small and can be discarded. This outline as we have explained it is of course not rigorous but it is the essential idea of Atkinson [1]. In Section 7 we will translate this intuitive explanation into a rigorous proof following Atkinson.

Although we are optimistic that a generalization of the Oppenheim summation formula to "divisor sums" based on Shalika models may be possible and will reflect the common structure between the $2k$-th moment of ζ and the constant term of the Eisenstein series on $\mathrm{GL}(2k)$, the method by which such a formula will be applied is less clear. We have explained this when $k = 1$ by means of (7). However we do not expect to find a straightforward generalization of Atkinson [1] or of (7) to higher moments. It is worth noting that the method of Atkinson [2] and Matsumoto [21] is very different from that of Atkinson [1], and it uses the Oppenheim summation formula. Our view is that the Oppenheim summation formula is central to the second moment of ζ but there is not a unique way of applying it.

In Section 1 we will discuss the similarity between the conjectural asymptotics of the $2k$-th moment of zeta and the constant term of an Eisenstein series on $\mathrm{GL}(2k)$. In Section 2 we consider the so-called "arithmetic factor" which seems missing in this parallel, and which is also problematical because it is not a global meromorphic function when $k > 2$. We will propose a possible method of avoiding it when $k = 3$. In Section 3 we confirm that Theorem 1 is a smoothed version of Oppenheim's generalization of the Voronoï summation formula. Section 4 contains generalities on principal series representations of $\mathrm{GL}(2, \mathbb{R})$ in the particular form in which we need them, including the Bessel distribution giving a formula for the Whittaker function at a Weyl group element. Section 5 discusses the Eisenstein series associated with a certain smooth vector attached to ϕ in Theorem 1, and Section 6 deduces Theorem 1. Finally Section 7 discusses (7) by extending Atkinson [1].

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1 Eisenstein Series on $\mathrm{GL}(2k)$ and moments of ζ

Let $G = \mathrm{GL}(2k)$, let P be the standard parabolic with Levi factor $M = \mathrm{GL}(k) \times \mathrm{GL}(k)$ and let U be its unipotent radical. Let B be the standard Borel subgroup of upper triangular matrices, and let V be the unipotent radical of $B \cap M$, so that UV is the unipotent radical of B. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$. Let

$$s = \begin{pmatrix} s_1 \\ \vdots \\ s_{2k} \end{pmatrix}$$

be complex parameters. Let χ_s and δ be the quasicharacters of $B_{\mathbb{A}}$ defined by

$$\chi_s(b) = \textstyle\prod_{i=1}^{2k} |y_i|^{s_i}, \qquad \delta(b) = \textstyle\prod_{i=1}^{2k} |y_i|^{k+1-2n},$$

when

$$b = \begin{pmatrix} y_1 & * & \cdots & * \\ & y_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & y_n \end{pmatrix}.$$

Let $I(s)$ be the space of smooth functions f on $G_{\mathbb{A}}$ such that

$$f(bg) = (\delta^{1/2}\chi_s)(b)f(g).$$

The group $G_{\mathbb{A}}$ acts by right translation on $I(s)$ affording a spherical principal series representation.

Let $W = S_{2k}$ be the Weyl group of G and let $W_P = S_k \times S_k$ be the Weyl group of M. Let Φ be the root system of G. If $1 \leqslant i, j \leqslant 2k$, $i \neq j$ let $\alpha(i,j)$ denote the root corresponding to the one-parameter subgroup $X_{i,j}$ of G consisting of unipotent matrices whose only off-diagonal entries are in the i, j position. Let

$$\Phi^+ = \{\alpha(i,j) \mid 1 \leqslant i < j \leqslant 2k\},$$

$$\Phi_U = \{\alpha(i,j) \mid 1 \leqslant i \leqslant k, k+1 \leqslant j \leqslant 2k\},$$

$$\Phi_M = \{\alpha(i,j) \in \Phi \mid 1 \leqslant i,j \leqslant k \,\mathrm{or}\, k+i \leqslant i,j \leqslant 2k\}, \qquad \Phi_M^+ = \Phi_M \cap \Phi^+.$$

Thus $\Phi^+ = \Phi_U \cap \Phi_M^+$. Every coset in W/W_P has a unique representative w such that $w\Phi_M^+ \subset \Phi^+$. Let Ξ be the set of these coset representatives. Thus $|\Xi| = \binom{2k}{k}$. This is the same set Ξ defined in [9] Section 3.1.

Let $K = \prod_v K_v$ be the standard maximal compact subgroup of G where v runs through the places of $\mathbb{Q}$, and $K_v = GL(2k, \mathbb{Z}_p)$ if $v = p$ is a finite prime, $K_v = O(2k)$ if $v = \infty$. Let f_s° be the standard spherical vector in $I(s)$, defined by $f_s^\circ(bk) = \delta^{1/2}\chi_s(b)$ for $b \in B_\mathbb{A}$, $k \in K$.

If $\gamma \in G$ let $U^\gamma = U \cap \gamma B \gamma^{-1}$. If $f \in I(s)$ then for $w \in W$ the intertwining integral

$$(M(w)f)(g) = \int_{U_\mathbb{A}^w \backslash U_\mathbb{A}} f_s(w^{-1}ug)\ du$$

is in $I(ws)$. The integral is convergent if $s \in \Omega$, the domain defined by the inequalities $\mathrm{re}(s_i - s_{i+1}) > \frac{1}{2}$. As in Casselman [6] it has meromorphic continuation to all s. We have

$$M(s)\ f_s^\circ = c(w)\ f_{wu}^\circ, \tag{10}$$

where

$$c(w) = \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} c_\alpha(s).$$

Here the product is over positive roots α such that $w\alpha$ is negative, and $c_\alpha(s)$ is as follows. We represent the root α by a vector (i, j) with $1 \leq i < j \leq k$, and then

$$c_\alpha(s) = \frac{\zeta^*(s_i - s_j)}{\zeta^*(s_i - s_j + 1)},$$

where $\zeta^*(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$. The action of W on the parameter s is by matrix multiplication. Indeed, the integral in (10) decomposes as a product of local integrals, each of which is evaluated by the formula of Gindikin and Karpelevich [14], or its nonarchimedean analog (see Langlands [19] and Casselman [6]).

We consider the Eisenstein series

$$E^*(g, s) = \left\{ \prod_{1 \leqslant i < j \leqslant 2k} \zeta^*(s_i - s_j + 1) \right\} E(g, s), \qquad E(g, s) = \sum_{B_\mathbb{Q} \backslash G_\mathbb{Q}} f_s^\circ(\gamma g).$$

Also, let

$$E_M^*(g, s) = \left\{ \prod_{\alpha_{i,j} \in \Phi_M^+} \zeta^*(s_i - s_j + 1) \right\} E_M(g, s),$$

$$E_M(g,s) = \sum_{\gamma\in(B\cap M)_{\mathbb{Q}}\backslash M_{\mathbb{Q}}} f_s^\circ(\gamma g).$$

This function is essentially a pair of GL(k) Eisenstein series on the Levi subgroup $M = \mathrm{GL}(k)\times\mathrm{GL}(k)$. Let

$$Z(s) = \prod_{\substack{1\leqslant i\leqslant k\\ k+1\leqslant j\leqslant 2k}} \zeta^*(s_i - s_j + 1).$$

Proposition 1 *(i) The constant term*

$$\int_{U_{\mathbb{Q}}\backslash U_{\mathbb{A}}} E(ug,s)\,du = \sum_{w^{-1}\in\Xi} c(w)E_M(g,ws).$$

(ii) The constant term

$$\int_{U_{\mathbb{Q}}\backslash U_{\mathbb{A}}} E^*(ug,s)\,du = \sum_{w^{-1}\in\Xi} Z(ws)E_M(g,ws). \qquad (11)$$

Proof. The integral in (i) equals

$$\sum_{\gamma\in U_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/B_{\mathbb{Q}}} \int_{U_{\mathbb{Q}}\backslash U_{\mathbb{A}}} \sum_{\delta\in U_{\mathbb{Q}}^\gamma\backslash U_{\mathbb{Q}}} f(\gamma^{-1}\delta ug)\ du =$$

$$\sum_{\gamma\in U_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/B_{\mathbb{Q}}} \int_{U_{\mathbb{Q}}^\gamma\backslash U_{\mathbb{A}}} f(\gamma^{-1}ug)\ du.$$

The integrand is invariant when u is changed on the left by an element of $U_{\mathbb{A}}^\gamma$. Hence this equals

$$\sum_{\gamma\in U_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/B_{\mathbb{Q}}} \int_{U_{\mathbb{A}}^\gamma\backslash U_{\mathbb{A}}} f(\gamma^{-1}ug)\ du.$$

We factor $\gamma = \gamma_1^{-1}\,w$, where $w\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/B_{\mathbb{Q}}$ and $\gamma_1\in(P_{\mathbb{Q}}\cap wB_{\mathbb{Q}}w^{-1})\backslash P_{\mathbb{Q}}/U_{\mathbb{Q}}$. According to the Bruhat decomposition, w can be chosen so that $w^{-1}\in\Xi$. For these values of w, we have $M\cap B\subset wBw^{-1}$. Hence γ_1 can be chosen from

$(B\cap M)_{\mathbb{Q}}\backslash M_{\mathbb{Q}}$. Now replacing u by $\gamma_1^{-1}u\gamma_1$ and noting that γ_1 normalizes U, while $\gamma_1 U^{\gamma_1^{-1}w}\gamma_1^{-1} = U^w$, we get

$$\sum_w \int_{U_A^w\backslash U_A} \sum_{\gamma_1\in(B\cap M)_{\mathbb{Q}}\backslash B_{\mathbb{Q}}} f(w^{-1}u\gamma_1 g)\ du.$$

This proves (i).

To prove (ii), note that by (i) and (10) the left side of (11) is

$$\sum_{w^{-1}\in\Xi}\left\{\prod_{\substack{\alpha=\alpha(i,j)>0\\ w\alpha>0}}\zeta^*(s_i-s_j+1)\right\}\left\{\prod_{\substack{\alpha=\alpha(i,j)>0\\ w\alpha<0}}\zeta^*(s_i-s_j)\right\}E_M(g,ws).$$

In the second product, replace α by $-\alpha$, thus switching i and j, and use the functional equation. The two products may thus be combined giving

$$\sum_{w^{-1}\in\Xi}\left\{\prod_{\alpha\in w^{-1}\Phi^+}\zeta^*(s_i-s_j+1)\right\}E_M(g,ws).$$

Absorbing the $k(k-1)$ zeta functions in the normalizing factor of $E_M^*(g,ws)$, what remains is $Z(ws)$. $\square$

To make the comparison with Conrey, Farmer, Keating, Rubinstein and Snaith [9], first let $k=2$ and choose $s_1=u_1$, $s_2=u_2$, $s_3=-v_1$ and $s_4=-v_2$. The L-function $L(E,\frac{1}{2}+it)$ of $E(g,s)$ is

$$\zeta\left(\tfrac{1}{2}+it+u_1\right)\zeta\left(\tfrac{1}{2}+it+u_2\right)\zeta\left(\tfrac{1}{2}+it-v_1\right)\zeta\left(\tfrac{1}{2}+it-v_2\right),$$

and applying the functional equation of ζ in the last two factors we get the argument of [9] (1.7.1). On the other hand, the factors Z in Proposition 1 (ii) are the same as their six factors Z, except for the missing zeta function in the denominator, which is the "arithmetic factor" that we will discuss in the next section. If k is general, taking s_i to be α_i in the notation of [9], our $Z(ws)$ are exactly the $\binom{2k}{k}$ products of k^2 zeta functions that occur in the right side of their (3.1.14). Our w is their σ^{-1}. (Applying w to s is the same as applying its inverse to the indices.)

2 The Arithmetic Factor

In addition to the k^2 zeta functions, each of the $\binom{2k}{k}$ terms in the asymptotics conjectured by Conrey, Farmer, Keating, Rubinstein and Snaith [9] also involves a certain Euler product, which is only an L-function when $2k \leqslant 4$. This is the "arithmetic factor" A_k defined in their (3.1.8). This factor does not appear with the $\binom{2k}{k}$ terms in our (11), which otherwise perfectly match their terms.

Moreover the arithmetic factor is problematical for another reason. As a function of their parameters α, the arithmetic factor does not have analytic continuation everywhere but has a natural boundary if $k \geqslant 3$. This may already be seen in a classical result. If $\mathrm{re}(\sigma) > 1 - 1/k$ then Carlson proved (see Titchmarsh [30], Section 7.9)

$$\int_0^T |\zeta(\sigma + it)|^{2k} \; dt \sim \left[\sum_{n=1}^{\infty} d_k(n)^2 \, n^{-2\sigma} \right] T,$$

where $d_k(n)$ is the number of ways of expressing n as a product of k factors. Moreover

$$\sum_{n=1}^{\infty} d_k(n)^2 \, n^{-s} = \zeta(s)^{k^2} \prod_p P_k(p^{-s}),$$

where P_k is a Dirichlet polynomial. We have

$$P_1(x) = 1, \qquad P_2(x) = 1 - x^2, \qquad P_3(x) = 1 - 9x^2 + 16x^3 - 9x^4 + x^6,$$

and in general

$$P_k(x) = (1-x)^{2k-1} \sum_{n=0}^{k-1} \left(\begin{array}{c} k-1 \\ n \end{array} \right)^2 x^k.$$

Estermann [12] considered a class of Dirichlet series including the Euler product $\prod_p P_k(p^{-s})$. He proved that it is absolutely convergent for $\mathrm{re}(s) > \frac{1}{2}$ and has meromorphic continuation to $\mathrm{re}(s) > 0$ but if $k \geqslant 3$ it has a natural boundary on the line $\mathrm{re}(s) = 0$. In the notation of [9],

$$\prod_p P_k(p^{-s}) = A_k(0; s, \cdots, s, -s, \cdots, -s).$$

This natural boundary is somewhat disturbing, so let us consider how, at least for the sixth moment, it may be possible to eliminate the arithmetic factor. Estermann's discovery that the arithmetic factor does not have analytic continuation may not be an insurmountable difficulty, but it raises the question whether (1) is really the most natural object to consider. We suggest a possible alternative.

Proposition 2 *Suppose* $a = (a_1, a_2, a_3)$ *and* $b = (b_1, b_2, b_3)$ *are given. Assume that* $\sigma + a_i$ *and* $\sigma + b_i$ *have real parts* $\geqslant 1$*. Then*

$$\lim_{T_1,T_2\to\infty} \frac{1}{T}_1 \frac{1}{T_2} \int_0^{T_1} \int_0^{T_2} \zeta(3\sigma + a_1 + a_2 + a_3 + it_1 + it_2)^{-1} \times$$

$$\zeta(3\sigma + b_1 + b_2 + b_3 - it_1 - it_2)^{-1} \zeta(\sigma + a_1 + it_1)\zeta(\sigma + a_2 + it_1)\zeta(\sigma + a_3 + it_1)\times$$

$$\zeta(\sigma + b_1 - it_1)\zeta(\sigma + b_2 - it_1)\zeta(\sigma + b_3 - it_1)\times$$

$$\zeta(2\sigma + a_1 + a_2 + it_2)\zeta(2\sigma + a_1 + a_3 + it_2)\zeta(2\sigma + a_2 + a_3 + it_2)\times$$

$$\zeta(2\sigma + b_1 + b_2 - it_2)\zeta(2\sigma + b_1 + b_3 - it_2)\zeta(2\sigma + b_2 + b_3 - it_2)\ dt_2\ dt_1 =$$

$$\frac{\prod_{i=1}^{3} \prod_{j=1}^{3} \zeta(2\sigma + a_i + b_j)}{\zeta(6\sigma + a_1 + a_2 + a_3 + b_1 + b_2 + b_3)}.$$

We conjecture that if we take σ around $\frac{1}{2}$ and the a_i, b_i small, the term on the right hand side will be one of $\binom{2k}{k}$ in the asymptotics of the left hand side, paralleling those found for the sixth moment by [9]. Note that since 2σ and 3σ are farther to the right this is something like the sixth moment. If this conjecture is true, this variant may be a more natural object to consider than the sixth moment itself, since it eliminates the arithmetic factor, replacing it with a factor having meromorphic continuation.

Proof. For p a fixed prime let

$$\sigma_a(p^{k_1}, p^{k_2}) = \frac{\begin{vmatrix} \alpha_1^{k_1+k_2+2} & \alpha_2^{k_1+k_2+2} & \alpha_3^{k_1+k_2+2} \\ \alpha_1^{k_2+1} & \alpha_2^{k_2+1} & \alpha_3^{k_2+1} \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 1 & 1 \end{vmatrix}}, \qquad \alpha_i = p^{-a_i}.$$

This is a Schur polynomial in α_1, α_2 and α_3. Extend $\sigma_a(n_1, n_2)$ to all n_1 and n_2 by multiplicativity. We have

$$\sum \sigma_a(n_1, n_2)\, n_1^{-s_1}\, n_2^{-s_2} =$$

$$\frac{\zeta(s_1+a_1)\,\zeta(s_1+a_2)\,\zeta(s_1+a_3)\zeta(s_2+a_2+a_3)\,\zeta(s_2+a_3+a_1)\,\zeta(s_2+a_1+a_2)}{\zeta(s_1+s_2+a_1+a_2+a_3)}. \tag{12}$$

Indeed, both sides are Eulerian, and locally this is (9.17) on p. 155 of Bump [5]. See Stanley [29] Exercise 7.28, pages 458 and 503 and Macdonald [20], Example 7 on p. 78 for a more general statement. Also

$$\zeta(3s+a_1+a_2+a_3+b_1+b_2+b_3) \sum_{n_1,n_2} \sigma_a(n_1,n_2)\,\sigma_b(n_1,n_2)\,(n_1 n_2^2)^{-s} =$$

$$\prod_{i=1}^{3} \prod_{j=1}^{3} \zeta(s+a_i+b_j). \tag{13}$$

This follows from the Cauchy identity. See Stanley [29] p. 322.

By (12) the integrand on the left hand side of the formula in the Proposition is

$$\left[\sum_{n_1,n_2} \sigma_a(n_1,n_2)\, n_1^{-\sigma-it_1}\, n_2^{-2\sigma-it_2}\right] \left[\sum_{n_1,n_2} \sigma_b(m_1,m_2)\, m_1^{-\sigma+it_1}\, m_2^{-2\sigma+it_2}\right].$$

Proceeding as in Titchmarsh [30], Theorem 7.1, it is elementary that the diagonal terms $n_i = m_i$ give the asymptotic value, which is evaluated by (13). □

3 The Oppenheim Summation Formula

In Oppenheim [28] is proved a generalization of the Voronoï summation formula, which we now review. We will confirm that it is consistent with our Theorem 1.

Let

$$D_{1-2s}(x) = \sum_{n \leqslant x} \sigma_{1-2s}(n).$$

In its original form the formula asserted, provided that s is real and $s \in (\frac{1}{2}, \frac{3}{4})$

$$D_{1-2s}(\xi) = \zeta(2s)\xi + \frac{\zeta(2-2s)}{2-2s}\xi^{2-2s} - \tfrac{1}{2}\,\zeta(2s-1) + \Delta_{1-2s}(\xi), \tag{14}$$

where

$$\Delta_{1-2s}(\xi) = -\xi^{1-s}\sum_{n=1}^{\infty}\sigma_{1-2s}(n)n^{s-1}\times$$

$$\left\{\cos(s\pi)J_{2-2s}(4\pi\sqrt{n\xi}) + \sin(s\pi)\left[Y_{2-2s}(4\pi\sqrt{n\xi}) + (2/\pi)K_{2-2s}(4\pi\sqrt{n\xi})\right]\right\}.$$

We will show in this section that Oppenheim's summation formula implies the smoothed version in our Theorem 1, when $\frac{1}{4} < s < \frac{3}{4}$. However in Oppenheim's formula the assumption that $\mathrm{re}(s) < \frac{3}{4}$ is essential, because the series Δ_{1-2s} will not be convergent otherwise. The smoothed version does not have this limitation and we will give a proof of it for all s in Section 6.

Note that $\tau_{s-1/2}(n) = \sigma_{1-2s}(n)|n|^{s-1/2}$. Write

$$\phi(x) = |x|^{\frac{1}{2}-s}\int_x^{\infty}\Phi(\xi)d\xi, \Phi(x) = -|x|^{s-1/2}\phi'(x) - (s-\tfrac{1}{2})\,|x|^{s-3/2}\phi(x),$$

Both ϕ and Φ have compact support in $(0,\infty)$. Integrating by parts,

$$\int_0^{\infty}\Phi(x)\left[\zeta(2s)x + \frac{\zeta(2-2s)}{2-2s}x^{2-2s} - \tfrac{1}{2}\,\zeta(2s-1)\right]\,dx =$$

$$\zeta(2s)\int_0^{\infty}\phi(x)x^{s-1/2}\,dx + \zeta(2-2s)\int_0^{\infty}\phi(x)x^{1/2-s}\,dx.$$

We consider now

$$\int_0^{\infty}\Phi(x)\Delta_{1-2s}(x)\,dx.$$

First the K-Bessel contribution is $-2\pi^{-1}\sin(s\pi)\tau_{s-1/2}(n)n^{-1/2}$ times

$$-\int_0^{\infty}K_{2-2s}(4\pi\sqrt{nx})\left[x^{1/2}\phi'(x) + (s-\tfrac{1}{2})\,x^{-1/2}\phi(x)\right]\,dx.$$

Integrating by parts, this is

$$\tfrac{1}{2}\int_0^{\infty}\left[4\pi\sqrt{nx}K'_{2-2s}(4\pi\sqrt{nx}) + (2-2s)K_{2-2s}(4\pi\sqrt{nx})\right]x^{-1/2}\phi(x)\,dx. \tag{15}$$

Using the formulas (Watson [32] p. 79)

$$zK'_\nu(z) - \nu K_\nu(z) = -zK_{\nu+1}(z), \qquad zK'_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z),$$

the contribution (15) equals

$$4\sin(s\pi)\tau_{s-1/2}(n)\int_0^\infty K_{1-2s}(4\pi\sqrt{nx})\phi(x)\,dx.$$

The J and Y contributions are handled similarly using (Watson [32] pp. 45 and 66):

$$zJ'_\nu(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z), \qquad zJ'_\nu(z) + \nu J_\nu(z) = zJ_{\nu-1}(z),$$

$$zY'_\nu(z) + \nu Y_\nu(z) = zY_{\nu-1}(z),$$

and the summation formula follows.

4 Principal Series Representations

Let $s \in \mathbb{C}$, and let V_s be the space of functions $f : \mathrm{GL}(2,\mathbb{R}) \longrightarrow \mathbb{C}$ which satisfy

$$f\left(\begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} g\right) = \left|\frac{y_1}{y_2}\right|^s f(g), \tag{16}$$

and such that that the restriction of f to $K = \mathrm{SO}(2)$ is a smooth function. We do not assume that f is K-finite. The group $\mathrm{GL}(2,\mathbb{R})$ acts on V_s by right translation. Thus if $g \in \mathrm{GL}(2,\mathbb{R})$ let $(\pi_s(g)f)(x) = f(xg)$. The space V_s is the space of smooth vectors in an even principal series representation of $\pi_s : \mathrm{GL}(2,\mathbb{R}) \longrightarrow \mathrm{End}(V_s)$.

Let

$$w_0 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

It follows from the theorem of Shalika [29] that there exists a unique linear functional $\Lambda : V_s \longrightarrow \mathbb{C}$ such that for $f \in V_s$

$$\Lambda\left(\pi_s\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} f\right) = e^{2\pi i x}\Lambda(f).$$

For $\mathrm{re}(s) > \frac{1}{2}$ we may define such a functional by

$$\Lambda(f) = \int_{-\infty}^{\infty} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) e^{-2\pi i x}\, dx.$$

The Whittaker function associated with a fixed $f \in V_s$ is

$$W_f(g) = \Lambda(\pi_s(g)f) = \int_{-\infty}^{\infty} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e^{-2\pi i x}\, dx. \tag{17}$$

We will denote by $\mathcal{W}_s$ the space of all functions W_f. It is the space of smooth vectors in the Whittaker model of π_s. We have

$$W_f \begin{pmatrix} y & \\ & 1 \end{pmatrix} = |y|^{1-s} \int_{-\infty}^{\infty} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) e^{-2\pi i x y}\, dx, \tag{18}$$

which follows by a change of variables from

$$W_f \begin{pmatrix} y & \\ & 1 \end{pmatrix} = |y|^{-s} \int_{-\infty}^{\infty} f\left(w_0 \begin{pmatrix} 1 & y^{-1}x \\ & 1 \end{pmatrix}\right) e^{-2\pi i x}\, dx.$$

If $f_s \in V_s$ let

$$\tilde{f}_{1-s}(g) = \int_{-\infty}^{\infty} f_s\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx. \tag{19}$$

Again this integral is convergent if $\mathrm{re}(s) > \frac{1}{2}$, and we have $\tilde{f}_{1-s} \in V_{1-s}$. Thus $M_s : V_s \longrightarrow V_{1-s}$ is an intertwining operator. It is known that both the Whittaker integral and the intertwining integral (19) have meromorphic continuation to the entire s plane. That is, one may fix a smooth function on $\mathrm{SO}(2)/\{\pm I\}$, which then extends uniquely to a function satisfying (16) for any s. Then both (17) and (19) are meromorphic functions of s. However we will not need this fact. For our purposes we only need the integrals (17) and (19) in the case $\mathrm{re}(s) > \frac{1}{2}$, where both are absolutely convergent.

Now we wish to choose the function f so that $y^{-1/2} W_f \begin{pmatrix} y & \\ & 1 \end{pmatrix} = \phi(y)$ where ϕ is a prescribed function. We assume that ϕ is smooth and compactly supported in $(0, \infty)$. For this particular f the values of $W_f \begin{pmatrix} y & \\ & 1 \end{pmatrix}$ will be zero when $y < 0$. We define a smooth function $f_{s,\phi} \in V_s$ by

$$f_{s,\phi}(g) =$$

$$\begin{cases} \left|\frac{y_1}{y_2}\right|^s \int_0^\infty \phi(u)\, u^{s-1/2} e^{2\pi i x u}\, du & \text{if } g = \begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}; \\ 0 & \text{if } g = \begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix}. \end{cases} \tag{20}$$

According to the Bruhat decomposition, every element of $\mathrm{GL}(2,\mathbb{R})$ is expressible uniquely in one of these forms.

Proposition 3 *The function $f_{s,\phi}$ is smooth and satisfies (16).*

Proof. It is clear that $f_{s,\phi}$ satisfies (16). It follows from this that smoothness is equivalent to the smoothness of its restriction to $\mathrm{SO}(2)$. We have (assuming $\sin(\theta) \neq 0$)

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$$

with $y_1 = \sin(\theta)^{-1}$, $y_2 = \sin(\theta)$, $* = \cos(\theta)$ and $x = \cot(\theta)$. So

$$f_{s,\phi} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \sin(\theta)^{-2s} \int_0^\infty \phi(u) u^{s-1/2} e^{2\pi i u \cot(\theta)}\, du.$$

The issue is smoothness where $\sin(\theta) = 0$, but the Fourier transform here is a rapidly decreasing function of $\cot(\theta)$, so $f_{s,\phi}$ is smooth at these points. □

Let $W_{s,\phi} = W_f$ with $f = f_{s,\phi}$.

Proposition 4 *We have*

$$W_{s,\phi} \begin{pmatrix} y & \\ & 1 \end{pmatrix} = \begin{cases} \sqrt{y}\, \phi(y) & \text{if } y > 0; \\ 0 & \text{if } y < 0. \end{cases}$$

Proof. Since ϕ is compactly supported in $(0,\infty)$ both cases are contained in

$$W_{s,\phi} \begin{pmatrix} y & \\ & 1 \end{pmatrix} = \sqrt{|y|}\, \phi(y)$$

which we prove. Using (18) and the definition of $f_{s,\phi}$ the left side equals

$$|y|^{1-s} \int_{-\infty}^\infty \int_0^\infty \phi(u)\, u^{s-1/2} e^{2\pi i x u} du\, e^{-2\pi i x}\, dx = \sqrt{|y|}\, \phi(y)$$

by Fourier inversion. □

If $f_s = f_{s,\phi}$ we will denote the function $\tilde{f}_{1-s}$ defined by (19) as $\tilde{f}_{1-s,\phi}$.

Proposition 5 *We have* $f_{s,\phi}(I) = \tilde{f}_{1-s,\phi}(I) = 0$. *Moreover*

$$f_{s,\phi}(w_0) = \int_0^\infty \phi(x) x^{s-1/2}\, dx \tag{21}$$

and

$$\zeta(2s-1)\tilde{f}_{1-s,\phi}(w_0) = \zeta(2-2s) \int_0^\infty \phi(x) x^{-s+1/2}\, dx. \tag{22}$$

Proof. We have $f_{s,\phi}(I) = 0$ by definition. On the other hand

$$\tilde{f}_{1-s,\phi}(I) = \int_{-\infty}^\infty \int_{-\infty}^\infty \phi(u)\, u^{s-1/2} e^{2\pi i x u}\, du\, dx.$$

By the Fourier inversion formula, this is the value of $\phi(u)\, u^{s-1/2}$ at $u = 0$. However this smooth function has compact support strictly to the right of 0 so $\tilde{f}_{1-s,\phi}(I) = 0$.

The first formula (21) follows immediately from (20). As for (22),

$$\tilde{f}_{1-s,\phi}(w_0) = \int_{-\infty}^\infty f_{s,\phi}\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} w_0 \right) dx =$$

$$\int_{-\infty}^\infty f_{s,\phi}\left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} w_0 \begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix} \right) dx =$$

$$\int_{-\infty}^\infty |x|^{-2s} f_{s,\phi}\left(w_0 \begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix} \right) dx$$

Replace x by $-x^{-1}$ in this identity to obtain

$$\tilde{f}_{1-s,\phi}(w_0) = \int_{-\infty}^\infty |x|^{2(s-1)} f_{s,\phi}\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right) dx =$$

$$\int_{-\infty}^\infty |x|^{2(s-1)} \int_0^\infty \phi(u) u^{s-1/2} e^{2\pi i x u}\, du\, dx.$$

Although we began by assuming that $\mathrm{re}(s) > 1$, this integral is convergent for all values of s. Indeed, $\phi(u)\, u^{s-1/2}$ is compactly supported and smooth, so its Fourier transform is Schwartz class. To evaluate this integral, we may therefore assume that $\frac{1}{2} < s < 1$. In this case,

$$\int_{-\infty}^\infty |x|^{2(s-1)}\, e^{2\pi i x u}\, dx = 2^{2-2s} (\pi u)^{1-2s}\, \sin(\pi s)\, \Gamma(2s-1).$$

Proceeding formally at first if we interchange the order of integration and obtain

$$\tilde{f}_{1-s,\phi}(w_0) = 2^{2-2s}\pi^{1-2s}\sin(\pi s)\Gamma(2s-1)\int_0^\infty \phi(u)\, u^{1/2-s}\, du.$$

Now using the functional equation of ζ in the form

$$2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\,\zeta(s) = \zeta(1-s),$$

we obtain (22). The interchange of the order of integration may be justified by tilting the line of integration with respect to x off the real axis into the upper half plane at a positive angle in both the positive and negative directions. $\square$

We now recall a formula of Cogdell and Piatetski-Shapiro [8] for the action of the Weyl group element w_0 on the Whittaker model. If $W \in \mathcal{W}_s$ then this formula asserts that for $\pi = \pi_s$ we have

$$W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} w_0\right) = \int_{\mathbb{R}^\times} J_\pi(uy) W\begin{pmatrix} u & \\ & 1 \end{pmatrix} d^\times u, \tag{23}$$

where the "Bessel Function" (introduced by Gelfand and Kazhdan)

$$J_\pi(u) = \begin{cases} -\frac{\pi\sqrt{|u|}}{\sin(\pi(s-\frac{1}{2}))}\left[J_{2s-1}(4\pi\sqrt{u}) - J_{1-2s}(4\pi\sqrt{u})\right] & \text{if } u>0; \\ -\frac{\pi\sqrt{|u|}}{\sin(\pi(s-\frac{1}{2}))}\left[I_{2s-1}(4\pi\sqrt{|u|}) - I_{1-2s}(4\pi\sqrt{|u|})\right] & \text{if } u<0. \end{cases}$$

This formula was stated in [8] p. 57 when π is unitary, so that either $s = \frac{1}{2}+it$ with t real, or s is real in $(0,1)$. In their notation, $ir = s-1/2$. A proof may be found in Baruch and Mao [3], Theorem 4.4 and Appendix 1.

We will prove (23) without assuming unitaricity for the particular Whittaker function $W_{s,\phi}$. The proof is close to that of Baruch and Mao (which was explained to us by Moshe Baruch) except that the compact support of ϕ allows us to dispense with their convergence factor.

We have (Watson [32], p. 64 and p. 78):

$$Y_\nu(z) = \frac{\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z)}{\sin(\pi\nu)},$$

$$K_\nu(z) = -\frac{\pi}{2}\left(\frac{I_\nu(z) - I_{-\nu}(z)}{\sin(\pi\nu)}\right).$$

Using these relations and standard trigonometric identities,

$$J_\pi(u) = \begin{cases} -2\pi\sqrt{u}\left[\cos(\pi s)J_{1-2s}(4\pi\sqrt{u}) + \sin(\pi s)Y_{1-2s}(4\pi\sqrt{u})\right] & \text{if } u > 0; \\ 4\sqrt{|u|}\sin(\pi s)K_{1-2s}(4\pi\sqrt{|u|}) & \text{if } u < 0. \end{cases} \tag{24}$$

Theorem 2 *Assume that* $\mathrm{re}(s) > \frac{1}{2}$. *If* $W = W_{s,\phi}$ *then (23) is valid.*

Proof. We apply (18) to the function $\pi_s(w_0)f_{s,\phi}$, then use the definition of $f_{s,\phi}$ obtain

$$W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} w_0\right) = |y|^{1-s}\int_{-\infty}^{\infty} f_{s,\phi}\left(w_0\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} w_0\right) e^{-2\pi ixy}\, dx =$$

$$|y|^{1-s}\int_{-\infty}^{\infty} f_{s,\phi}\left(\begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} x^{-1} & \\ & x \end{pmatrix} w_0 \begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix}\right) e^{-2\pi ixy}\, dx =$$

$$|y|^{1-s}\int_{-\infty}^{\infty}\int_0^{\infty} |x|^{-2s}\phi(u)\, u^{s-1/2}e^{-2\pi i(xy+x^{-1}u)}\, du\, dx.$$

Since ϕ is compactly supported, it is not hard to justify interchanging the order of integration. We then combine the positive and negative contributions of x to write the inner integral as

$$2\int_0^{\infty} x^{-2s}\cos(2\pi(xy + x^{-1}u))\, dx.$$

Then using (24) and Gradshteyn and Ryzhik [15] 3.871 on p. 470, which is applicable when $\mathrm{re}(s) \in (\frac{1}{2}, 1)$, we obtain (23). For $\mathrm{re}(s) > \frac{1}{2}$, the result follows by analytic continuation. Note that Y_ν is denoted as N_ν in [15]. $\square$

5 Eisenstein series

Let ϕ be as in Theorem 1 and let $f_{s,\phi}$ be as in Section 4. Assuming $\mathrm{re}(s) > 1$, we consider the Fourier expansion of the Eisenstein series

$$E_\phi(g, s) = \tfrac{1}{2}\zeta(2s)\sum_{\Gamma_\infty\backslash\Gamma} f_{s,\phi}(\gamma g).$$

Proposition 6 *If* $\mathrm{re}(s) > 1$ *we have*

$$\int_0^1 E_\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, s\right) e^{-2\pi i n x}\, dx = \tau_{s-1/2}(n) n^{-1/2} W_{s,\phi}\left(\begin{pmatrix} n & \\ & 1 \end{pmatrix} g\right) \tag{25}$$

if $n \neq 0$, *while*

$$\int_0^1 E_\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, s\right) dx = \zeta(2s) f_{s,\phi}(g) + \zeta(2s-1)\tilde{f}_{1-s,\phi}(g). \tag{26}$$

Proof. Assuming $n \neq 0$ we have

$$\int_0^1 E_\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, s\right) e^{-2\pi i n x}\, dx =$$

$$\tfrac{1}{2}\,\zeta(2s) \int_0^1 \sum_{\Gamma_\infty \backslash \Gamma} f_{s,\phi}\left(\gamma \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e^{-2\pi i n x}\, dx.$$

Let $c \in \mathbb{Z}$ and let $\Gamma_c = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \right\}$ be the subset of Γ with prescribed c. If $n \neq 0$ then the contribution of Γ_c vanishes when $c = 0$ so we may write this as

$$\tfrac{1}{2}\,\zeta(2s) \sum_{c \neq 0} \int_{-\infty}^{\infty} \sum_{\Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} f_{s,\phi}\left(\gamma \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e^{-2\pi i n x}\, dx =$$

$$\zeta(2s) \sum_{c>0} \sum_{\substack{d \bmod c \\ a \bmod c \\ ad \equiv 1 (\bmod c) \\ (d,c)=1}}$$

$$\int_{-\infty}^{\infty} f_{s,\phi}\left(\begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} \begin{pmatrix} & -c^{-1} \\ c & \end{pmatrix} \begin{pmatrix} 1 & d/c \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e^{-2\pi i n x}\, dx.$$

To evaluate this we may use (16), and also make a variable change $x \longrightarrow x - d/c$ to obtain

$$\zeta(2s) \sum_{c>0} \sum_{\substack{d \bmod c \\ (d,c)=1}} e^{2\pi i n d/c} c^{-2s} \int_{-\infty}^{\infty} f_{s,\phi}\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e^{-2\pi i n x}\, dx.$$

According to Ramanujan's identity

$$\zeta(2s)\sum_{c>0}\sum_{\substack{d \bmod c\\(d,c)=1}} e^{2\pi i n d/c}c^{-2s}=\sigma_{1-2s}(n).$$

Also the variable change $x\longrightarrow n^{-1}x$ shows that

$$\int_{-\infty}^{\infty} f_{s,\phi}\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x\\ & 1\end{pmatrix}g\right)e^{-2\pi i n x}\,dx = n^{s-1}W_{s,\phi}\left(\begin{pmatrix} n & \\ & 1\end{pmatrix}g\right).$$

Since $\sigma_{1-2s}(n)n^{s-1}=\tau_{s-1/2}(n)n^{-1/2}$, we have proved (25).

Next we prove (26). The contribution of $c=0$ is $\zeta(2s)f_{s,\phi}(g)$. The contribution of $c\neq 0$ is evaluated like (25) and we obtain

$$\zeta(2s)\sum_{c>0}\sum_{\substack{d \bmod c\\(d,c)=1}} c^{-2s}\int_{-\infty}^{\infty} f_{s,\phi}\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x\\ & 1\end{pmatrix}g\right)dx =$$

$$\zeta(2s)\sum_{c>0}\varphi(c)c^{-2s}\tilde{f}_{1-s,\phi}(g),$$

where φ is Euler's phi function. We have

$$\zeta(2s)\sum_{c>0}\varphi(c)c^{-2s}=\zeta(2s-1).$$

This proves (26). $\square$

6 Proof of Theorem 1

Let ϕ have compact support in $(0,\infty)$ and define $\psi_s:\mathbb{R}\longrightarrow\mathbb{C}$ as in (5).

Proposition 7 *For any s then $\psi_s(y)=O(y^{-N})$ as $y\longrightarrow\infty$, for all N.*

Proof. We make use of the asymptotic expansions (Watson, p. 199)

$$J_\nu(z)=\sqrt{\tfrac{2}{\pi}}\cos\left(z-\tfrac12\nu\pi-\tfrac14\pi\right)z^{-1/2}$$

$$+\sqrt{\tfrac{2}{\pi}}\,\frac{(\nu+\frac12)(\nu+\frac32)}{2}\sin\left(z-\tfrac12\nu\pi-\tfrac14\pi\right)z^{-3/2}+O(z^{-5/2}),$$

$$Y_\nu(z) = \sqrt{\tfrac{2}{\pi}} \sin\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) z^{-1/2}$$

$$+\sqrt{\tfrac{2}{\pi}}\, \frac{(\nu+\frac{1}{2})(\nu+\frac{3}{2})}{2} \cos\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) z^{-3/2} + O(z^{-5/2}),$$

and $K_\nu(z) \sim \sqrt{2/(\pi z)}e^{-z}$. The term in brackets in (5) is therefore

$$\sqrt{\tfrac{2}{\pi}} \Big[-\cos(\sqrt{xy} + 2s\pi - 3\pi/4)x^{-1/4}y^{-1/4}$$

$$+ \frac{(\frac{3}{2}-2s)(\frac{5}{2}-2s)}{2} \sin(\sqrt{xy} - 3\pi/4)\, x^{-3/4}y^{-3/4} + O((xy)^{-5/4})\Big].$$

Now let us consider

$$\int_0^\infty \phi(x) \cos(\sqrt{xy} + 2s\pi - 3\pi/4)\, x^{-1/4}\, dx.$$

The variable change $x = u^2$ makes this

$$2\int_0^\infty \phi(u^2) \cos(u\sqrt{y} + 2s\pi - 3\pi/4)\, u^{1/2}\, du.$$

This is the Fourier transform of a smooth function, evaluated at $\sqrt{y}$, hence is of rapid decay as $y \longrightarrow \infty$. The second term is similarly of rapid decay. This leaves us with the error term which is dominated by the convergent integral $\int \phi(x)\, x^{-5/4} dx$ times $y^{-5/4}$. This proves that the integral is $O(y^{-5/4})$. Taking more terms of the asymptotic expansion gives better error terms. □

We may now prove Theorem 1. We have the Fourier expansion of the Eisenstein series

$$E_\phi(g,s) = \sum_{n\in\mathbb{Z}} \int_0^1 E_\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, s\right) e^{-2\pi i n x} dx.$$

This is clear if $\mathrm{re}(s) > 1$ and we at first assume that s is in this region. By Propositions 6, 5 and 4, it follows that $E_\phi(I,s)$ is the left side of (6). On the other hand by Propositions 6 and 5 and Theorem 2, $E_\phi(w_0,s)$ equals the right side of (6). Of course they are equal since $E_\phi(w_0,s)$ is automorphic, and the Theorem is proved when $\mathrm{re}(s) > 1$. The general case follows by analytic continuation, since using Proposition 7 it is not hard to see that both sides of (6) are entire functions of s.

7 The second moment and the divisor problem

We will use the method of Atkinson [1] to explain the relationship (7).

Matsumoto [21] obtains a better error term than Ingham [16], at least when $\sigma < \frac{3}{4}$, and Ingham's error term is better than we obtain by the method that we will use. Matsumoto's result can be written

$$\int_0^T |\zeta(\sigma+it)|^2\,dt = \zeta(2\sigma)T + \frac{(2\pi)^{2\sigma-1}}{2-2\sigma}\zeta(2-2\sigma)T^{2-2\sigma} + O(T^{1/(1+4\sigma)}\log(T)^2).$$

His method, adapted from Atkinson [2] is based on a more subtle application of the Oppenheim summation formula. This work is continued in Matsumoto and Meurman [24] and [23].

We think the original approach of Atkinson [2] is still useful in giving the crucial relationship (7) quite simply. Atkinson's proof is repeated in Titchmarsh [30].

Theorem 3 *If* $\frac{1}{2} < \sigma < 1$ *we have*

$$\left|\int_0^T |\zeta(\sigma+it)|^2\,dt - 2\pi\sum_{n<T/2\pi} \tau_{\sigma-\frac{1}{2}}(n)n^{\frac{1}{2}-\sigma}\right| = O(T^{2\sigma-\frac{1}{2}}).$$

Proof. First we note that

$$\left|\int_0^T |\zeta(\sigma+it)|^2\,dt - \frac{1}{2i}\int_{\sigma+\varepsilon-iT}^{\sigma+\varepsilon+iT} \zeta(\sigma+s)\zeta(\sigma-s)\,ds\right| = O(T^{\frac{1}{2}+\varepsilon}).$$

Indeed, rewriting $\int_0^T |\zeta(\sigma+it)|^2\,dt = \frac{1}{2}\int_{-T}^T |\zeta(\sigma+it)|^2\,dt$, this difference consists of the constant residue $\pi\zeta(2\sigma-1)$ plus the contributions of two short integrals

$$\frac{1}{2i}\int_{\pm iT}^{\sigma+\varepsilon\pm iT} \zeta(\sigma+s)\zeta(\sigma-s)\,ds = O(T^{\frac{1}{2}+\varepsilon}).$$

We have

$$\frac{1}{2i}\int_{\sigma+\varepsilon-iT}^{\sigma+\varepsilon+iT} \zeta(\sigma+s)\zeta(\sigma-s)\,ds = \frac{1}{2i}\sum_{n=1}^{\infty}\tau_{\sigma-\frac{1}{2}}(n)\int_{\sigma+\varepsilon-iT}^{\sigma+\varepsilon+iT} n^{-\frac{1}{2}-s}\chi(\sigma-s)\,ds.$$

If $n > T/2\pi$, using (9) and Lemma 4.3 on p. 61 of Titchmarsh [30], the n-th term here is

$$O\left(\frac{n^{-\frac{1}{2}-\sigma-\varepsilon}T^{\frac{1}{2}+\varepsilon}}{\log|\frac{T}{2\pi n}|}\right).$$

Since $\sum \tau_{\sigma-\frac{1}{2}}(n)n^{-\frac{1}{2}-\sigma-\varepsilon} < \infty$ we may discard these terms. We are left with the problem of estimating

$$\frac{1}{2i}\sum_{n<T/2\pi}\tau_{\sigma-\frac{1}{2}}(n)\int_{\sigma+\varepsilon-iT}^{\sigma+\varepsilon+iT} n^{-\frac{1}{2}-s}\chi(\sigma-s)\,ds.$$

Using (9) we have

$$\int_{\pm iT}^{\sigma+\varepsilon\pm iT} n^{-\frac{1}{2}-s}\chi(\sigma-s)\,ds = O\left(n^{-1/2}T^{\frac{1}{2}+\varepsilon}\right).$$

Now by Theorem 1 or a Tauberian theorem

$$\sum_{n\leqslant x}\tau_{\sigma-\frac{1}{2}}(n)n^{\frac{1}{2}-\sigma} \sim \zeta(2\sigma)\,x,$$

and by partial summation it follows that

$$\sum_{n\leqslant x}\tau_{\sigma-\frac{1}{2}}(n)n^{-\frac{1}{2}} = O(x^{\sigma}). \tag{27}$$

Therefore

$$\frac{1}{2i}\sum_{n<T/2\pi}\tau_{\sigma-\frac{1}{2}}(n)\int_{\pm iT}^{\sigma+\varepsilon\pm iT} n^{-\frac{1}{2}-s}\chi(\sigma-s)\,ds = O(T^{\sigma}).$$

Thus we may move the path of integration back into the critical strip and approximate $\int_0^T |\zeta(\sigma+it)|^2\,dt$ by

$$\frac{1}{2i}\sum_{n<T/2\pi}\tau_{\sigma-\frac{1}{2}}(n)\int_{-iT}^{iT} n^{-\frac{1}{2}-s}\chi(\sigma-s)\,ds.$$

We would like to change the limits to $-i\infty$ and $i\infty$. By Titchmarsh's Lemma 4.3,

$$\int_{iT}^{i\infty} n^{-\frac{1}{2}-s}\chi(\sigma-s)\,ds = O\left(\frac{T^{\sigma-1/2}}{\sqrt{n}\log(T/2\pi n)}\right).$$

applying (27) again, the error in changing the limits is $O(T^{2\sigma-1/2})$. We consider therefore

$$\frac{1}{2i}\sum_{n<T/2\pi}\tau_{\sigma-\frac{1}{2}}(n)\int_{-i\infty}^{i\infty}n^{-\frac{1}{2}-s}\chi(\sigma-s)ds.$$

Since

$$\frac{1}{2}\int_{-\infty}^{\infty}x^{-\frac{1}{2}-it}\chi(\sigma-it)\,dt=2\pi x^{\frac{1}{2}-\sigma}\cos(2\pi x),$$

and since $\cos(2\pi n)=1$ when n is an integer, we get

$$2\pi\sum_{n<T/2\pi}\tau_{\sigma-\frac{1}{2}}(n)n^{\frac{1}{2}-\sigma}.$$

□

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