Whittaker Functions and Demazure Operators

Ben Brubaker[1], Daniel Bump[2], and Anthony Licata[3]

January 16, 2014

Dedicated to the memory of Steve Rallis

Abstract

We show that elements of a natural basis of the Iwahori fixed vectors in a principal series representation of a reductive p-adic group satisfy certain recursive relations. The precise identities involve operators that are variants of the Demazure-Lusztig operators, with correction terms, which may be calculated by a combinatorial algorithm that is identical to the computation of the fibers of the Bott-Samelson resolution of a Schubert variety. This leads to an action of the affine Hecke algebra on functions on the maximal torus of the L-group. A closely related action was previously described by Lusztig using equivariant K-theory of the flag variety, leading to the proof of the Deligne-Langlands conjecture by Kazhdan and Lusztig. In the present paper, the action is applied to give a simple formula for the basis vectors of the Iwahori Whittaker functions. We also show that these Whittaker functions can be expressed as nonsymmetric Macdonald polynomials.

1 Introduction

One of Steve Rallis' many enthusiasms was the proof of the Casselman-Shalika formula, exploiting the property of the Iwahori fixed vectors in the

[1]Department of Mathematics, The University of Minnesota, 127 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455

[2]Department of Mathematics, Stanford University, Stanford CA 94305-2125, USA

[3]Mathematical Science Institute, The Australian National University, John Dedman Building 27, Union Lane, Canberra ACT 0200, Australia

Whittaker model. This paper reconsiders these Iwahori fixed vectors for the unramified principal series of a split, reductive Chevalley group G over a non-archimedean local field F.

By the Iwasawa decomposition, $G = BK$ with B the standard Borel subgroup of G and K a certain maximal compact subgroup, and $B = NT$ with N a maximal unipotent subgroup and T a maximal F-split torus. Given an unramified quasi-character of T (i.e. a character trivial on $T(\mathfrak{o})$ where $\mathfrak{o}$ denotes the ring of integers of F), then we may form the induced principal series representation $\pi := \mathrm{Ind}_B^G(\tau)$.

Let B_- be the Borel subgroup opposite to B and let N_- denote its unipotent radical. Fix a character ψ of $N_-(F)$ lying in an open T-orbit. A Whittaker functional is a linear map

$$\Omega_\tau : \mathrm{Ind}_B^G(\tau) \longrightarrow \mathbb{C} \quad \text{with} \quad \Omega_\tau(\pi(n_-)f) = \psi(n_-)\Omega_\tau(f)$$

for all $f \in \mathrm{Ind}_B^G(\tau)$ and all $n_- \in N_-$. Rodier showed that the space of such functionals is one-dimensional [28], building on earlier work of Gelfand and Graev, Gelfand and Kazhdan, Piatetski-Shapiro, and Shalika. This is true even if $\mathrm{Ind}_B^G(\tau)$ is reducible. However we will assume that τ is in general position, and that this induced representation is irreducible.

Let J be the Iwahori subgroup which is the preimage of $B_-(\mathbb{F}_q)$ in the special maximal compact subgroup $G(\mathfrak{o})$. Let $M(\tau) := \mathrm{Ind}_B^G(\tau)^J$, the space of Iwahori fixed vectors in the principal series. Then to any $f \in M(\tau)$, we may associated an "Iwahori Whittaker function" defined by

$$\mathcal{W}_{\tau,f}(g) := \Omega_{\hat{\tau}}(\pi(g)f), \tag{1}$$

where $\hat{\tau}$ indicates that we are taking the contragredient representation of $\mathrm{Ind}_B^G(\tau)$. For most τ, the dimension of $M(\tau)$ is equal to the order of the Weyl group W of G. Here we study Iwahori Whittaker functions for the so-called "standard basis" of $M(\tau)$, denoted $\{\Phi_w\}_{w\in W}$ and made from characteristic functions on J-double cosets of K. See Section 2 for their precise definition and [16] for further information. The notation in (1) will be slightly modified in Section 3, corresponding to a convenient normalization of the Whittaker function.

Iwahori fixed vectors were employed in Casselman [7] and Casselman and Shalika [8] in their computation of the spherical and Whittaker functionals on the unique (up to constant) spherical vector in unramified principal series. Clever use of intertwining operators allowed them to avoid explicit computation of the Whittaker functional on all but one Iwahori fixed vector. Later Reeder [26] gave a closed formula for Iwahori Whittaker functions of the so-called "Casselman basis" and an answer in terms of a Lefschetz trace for a certain subset of standard basis elements – those $w \in W$ corresponding to standard parabolic subgroups of G.

In this paper, we build Iwahori Whittaker functions for any standard basis element from certain operators on the ring $\mathcal{O}(\hat{T})$ of regular functions on the dual torus of T. The precise form of the operators will be determined by formulas relating the Whittaker functional and intertwining operators found in [8].

To explain this, we may identify an unramified character τ of $T(F)$ with an element $\boldsymbol{z}$ of the dual torus $\hat{T}(\mathbb{C})$, and we may associate an element $a_{-\lambda}$ of $T(F)$ with a weight λ of $\hat{T}(\mathbb{C})$. The Whittaker function $\mathcal{W}_{\tau,f}(a_{-\lambda})$ vanishes unless λ is dominant. We may also think of λ as a rational character of $\hat{T}(\mathbb{C})$ which we may apply to $\boldsymbol{z}^{-1}$, the Langlands parameter of the contragredient, and we will denote the result as $\boldsymbol{z}^{-\lambda}$. Recall that for each simple reflection $s_i \in W$, the Demazure operator ∂_i is defined as follows:

$$\partial_i f(\boldsymbol{z}) := (1 - \boldsymbol{z}^{-\alpha_i})^{-1}(f(\boldsymbol{z}) - \boldsymbol{z}^{-\alpha_i} f(s_i \boldsymbol{z})). \tag{2}$$

The operators we use to describe Iwahori Whittaker functions are:

$$\mathfrak{D}_i = (1 - q^{-1}\boldsymbol{z}^{-\alpha_i})\partial_i, \quad \text{and} \quad \mathfrak{T}_i = \mathfrak{D}_i - 1. \tag{3}$$

Using these operators, we give three answers for the value of Iwahori Whittaker functions – in terms of Hecke (**A**)lgebras, geometry of (**B**)ott-Samelson varieties, and (**C**)ombinatorics of Macdonald polynomials.

For the first answer, let v be a parameter that may be either a complex number or an indeterminate, and let $\mathcal{H}_v$ be the Iwahori Hecke algebra with generators T_i satisfying $T_i^2 = (v-1)T_i + v$ and the braid relations. In Section 4, we show that the map $T_i \mapsto \mathfrak{T}_i$ gives a representation of this Hecke algebra on $\mathcal{O}(\hat{T}(\mathbb{C}))$.

Let $\tilde{\mathcal{H}}_v$ be the (extended) affine Iwahori Hecke algebra (Section 4), obtained from $\mathcal{H}_v$ by adjoining an abelian subalgebra isomorphic to the group algebra of the weight lattice.

Theorem A (Theorem 2, Section 4)**.** *To any $w \in W$ and any dominant weight λ,*

$$\mathcal{W}_{\tau,\Phi_w}(a_{-\lambda}) = (*)\, \mathfrak{T}_w \boldsymbol{z}^{\lambda}. \tag{4}$$

The action of $\mathcal{H}_{q^{-1}}$ on $\mathcal{O}(\hat{T}(\mathbb{C}))$ extends to $\tilde{\mathcal{H}}_{q^{-1}}$, and the resulting module is antispherical.

The constant $(*)$ in (4) is an unimportant normalization. The "antispherical" module is the $\widetilde{\mathcal{H}}_v$-module obtained from inducing the sign character of $\mathcal{H}_v$. At least the statement of this result (or closely related ones) has appeared in the literature. See for example [2] and [27].

Turning to the second answer, we demonstrate that the recursive formula is closely connected to the geometry of Bott-Samelson varieties, which provide a resolution of singularities for Schubert varieties in the flag variety G/B, and are a key ingredient in the proof that Demazure characters compute the cohomology of line bundles over Schubert varieties ([4], [12], [1]). To any $w \in W$, the construction of the Bott-Samelson variety over the Schubert variety X_w depends on a reduced decomposition for w.

Let $\mathfrak{w} = s_{i_1} \cdots s_{i_k}$ be a choice of reduced word for w. To $\mathfrak{w}$ and any dominant weight $\lambda \in \Lambda$, the group of rational characters of $\hat{T}$, define an operator on $\mathcal{O}(\hat{T})$ by

$$\boldsymbol{Z}_{\mathfrak{w}} := \mathfrak{D}_{i_1} \cdots \mathfrak{D}_{i_k}.$$

When we want to emphasize the connection to Schubert varieties, we write $\boldsymbol{Y}_w := \mathfrak{T}_w$, and in view of Theorem A, applying the operator to $\boldsymbol{z}^\lambda$ with λ dominant gives the Whittaker function.

Theorem B (Theorem 6, Section 7)**.** *Given any $w \in W$ with a corresponding reduced word $\mathfrak{w}$,*

$$\boldsymbol{Z}_{\mathfrak{w}} = \sum_{u \leqslant w} P_{\mathfrak{w},u}(q^{-1}) \boldsymbol{Y}_u,$$

where q denotes the cardinality of the residue field of F, $\leqslant$ is the Bruhat order, and $P_{\mathfrak{w},u}$ is the Poincaré polynomial of the fiber in the Bott-Samelson variety $Z_{\mathfrak{w}}$ over any point in the open Schubert cell Y_u contained the Schubert variety X_w.

We remark here that similar connections between the geometry of Bott-Samelson and Schubert varieties and representations of Hecke algebra also appear in the theory of Soergel bimodules; see for example [30] and [32]. It would be interesting to establish a direct connection between Soergel bimodules and Whittaker models, though we do not address this question in the current paper. There is a close connection between Soergel bimodules and equivariant cohomology of Schubert varieties, but the problem at hand seems even closer to equivariant K-theory, a connection which might be important to develop.

Finally, we connect Iwahori Whittaker functions for the standard basis to specializations of non-symmetric Macdonald polynomials using results of Ion [19]. Recall that (symmetric) Macdonald polynomials are a Weyl group invariant two-parameter (commonly q and t) family of polynomials associated to a finite, irreducible root system. Members are indexed by anti-dominant elements of the corresponding weight lattice Λ. They simultaneously generalize many known classes of symmetric functions and possess deep connections to zonal spherical functions for real and p-adic groups, and to the representation theory of affine Kac-Moody groups. See [25] for details.

Non-symmetric Macdonald polynomials, defined more recently by Heckman, Opdam, Macdonald, Cherednik, and Sahi, are a family $\{E_\lambda(q,t)\}$ where λ may now be any element of the full weight lattice Λ. By symmetrizing over the W-orbit of λ, we obtain the symmetric Macdonald polynomial. Non-symmetric Macdonald polynomials are an important tool in the study of double affine Hecke algebras, but their interpretation in terms of matrix coefficients for classical or p-adic representations has been investigated in [9, 10, 11]. In certain specializations of q, t, one may say more. Ion has shown that $E_\lambda(q,\infty)$ are Demazure characters of basic representations of affine Kac-Moody groups [17] and $E_\lambda(\infty,t)$ are matrix coefficients for $M(\tau)$ using the spherical functional [18]. (See also [6].) The following result gives a further specialization.

Theorem C (Theorem 7, Section 8)**.** *Given a dominant weight λ and a $w \in W$, the Iwahori Whittaker function for the standard basis element Φ_w at $a_{-\lambda}$*

$$\mathcal{W}_{\tau,\Phi_w}(a_{-\lambda}) = b(\tau,w) w_0 E_{w_0 w(\lambda+\rho)}(0, v^{-1}).$$

Here v^{-1} is the cardinality of the residue field of F, $b(\tau,w)$ is a simple, explicit constant depending only on τ and w, and w_0 acts on E by permuting the variables $\boldsymbol{z}$.

We caution the reader that notations for non-symmetric Macdonald polynomials differ among various authors, and here we are using the notation in [19]. As a further point of confusion, the parameter t in $E_\lambda(q,t)$ is associated to the cardinality of the residue field, even though the letter q is commonly used in the literature of p-adic group representations.

In addition to these three theorems on Iwahori Whittaker functions, in Section 6 we provide efficient proofs of the Casselman-Shalika formula for the spherical Whittaker function and the Demazure character formula, using specializations of the parameter appearing in the operators $\mathfrak{T}_i$.

We thank Bogdan Ion for pointing out that our recursion could be used to connect to a specialization of the nonsymmetric Macdonald polynomial. We also thank Gautam Chinta, Solomon Friedberg, Paul Gunnells, David Kazhdan, Daniel Orr, Arun Ram, Mark Reeder, and Anne Schilling for helpful conversations. Schilling, Mark Shimozono, and Nicolas Thiéry wrote Sage code for non-symmetric Macdonald polynomials was instrumental in helping us to refine our results. This work was supported by NSF grants DMS-0652817, DMS-0844185, and DMS-1001079.

2 Preliminaries

The papers of Casselman [7] and Casselman and Shalika [8] will serve as the basic references for the local theory. We will slightly change the point of view: our Whittaker functionals will be computed with respect to the opposite maximal unipotent subgroup to the Borel from which we induce.

Let G be a split reductive Chevalley group. By this we mean an affine algebraic group scheme over $\mathbb{Z}$, whose Lie algebra $\mathfrak{g}_\mathbb{Z}$ has a fixed Chevalley basis defined over $\mathbb{Z}$ corresponding to a root system $\Delta^\vee$. This is the dual of the root system Δ for the Langlands dual group $\hat{G}$. We will call elements of $\Delta^\vee$ *coroots*. They are roots of G or coroots of $\hat{G}$. If T and $\hat{T}$ are standard maximal split tori of G and $\hat{G}$, then the ambient spaces of $\Delta^\vee$ and Δ are the groups $\Lambda^\vee = X^*(T)$ and $\Lambda = X^*(\hat{T})$ of rational characters of T and $\hat{T}$. Note that Δ is the root system of $\hat{T}$, not T, since this root system plays a bigger role in describing the Whittaker function. The lattices $X^*(T)$ and $X^*(\hat{T})$ are identified with the cocharacter groups $X_*(\hat{T})$ and $X_*(T)$, respectively. If $\alpha \in \Delta$ then $\alpha^\vee$ will denote the corresponding coroot. We will denote by $\{\alpha_1, \cdots, \alpha_r\}$ the simple roots. As usual, Δ is partitioned into positive and negative roots, $\Delta = \Delta_+ \cup \Delta_-$ and similarly $\Delta^\vee$.

If $\alpha^\vee$ is a coroot, let $x_{\alpha^\vee} : \mathbb{G}_\mathrm{a} \longrightarrow G$ be the corresponding one-parameter subgroup. We normalize it so that its differential $dx_{\alpha^\vee} \in \mathfrak{g}_\mathbb{Z}$ is a Chevalley basis vector. By the theory of Chevalley groups there exist a homomorphism $i_{\alpha^\vee} : \mathrm{SL}(2) \longrightarrow G$ such that

$$x_{\alpha^\vee}(t) = i_{\alpha^\vee} \left(\begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right).$$

Let N be the unipotent subgroup generated by the images of the $x_{\alpha^\vee}$ where $\alpha^\vee$ runs through the positive coroots, and let N_- be the group generated by the $x_{-\alpha^\vee}$. Then $B = TN$ is the standard Borel subgroup of G, and $B_- = TN_-$ is the opposite Borel.

Let F be a nonarchimedean local field and $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and let $q = |\mathfrak{o}/\mathfrak{p}|$. The residue field will be denoted by $\mathbb{F}_q$. The group $G(F)$ has $K = G(\mathfrak{o})$ as a maximal compact subgroup. This is the group generated by the $x_{\alpha^\vee}(\mathfrak{o})$ with $\alpha^\vee \in \Delta^\vee$. The Weyl group W is $N(T)/T$. If w is an element of the Weyl group W, we will choose a fixed representative of w in $G(\mathfrak{o})$, and by abuse of notation we will denote this element also as w. Nothing will depend on this choice in any essential way. The long Weyl group element will be denoted w_0.

The group $X_*(T)$ of rational cocharacters is isomorphic to $T(F)/T(\mathfrak{o})$, where the one-parameter subgroup $\varphi \in X_*(T)$ corresponds to the coset $\varphi(\varpi)T(\mathfrak{o})$ with ϖ a prime element. A character τ of $T(F)$ is called *unramified* if it is trivial on $T(\mathfrak{o})$. We will let W act on unramified characters τ, so that

$$(w\tau)(t) = \tau(\omega^{-1}t\omega), \qquad w \in W, \qquad t \in T(F), \tag{5}$$

where ω is any representative of w in the normalizer of T. Since τ is unramified, this does not depend on the choice of representative ω.

As we have mentioned the group $X_*(T)$ of rational cocharacters of T is identified with the weight lattice Λ, which is the group $X^*(\hat{T})$ of rational characters of $\hat{T}$. Thus we have a surjection

$$T(F) \longrightarrow T(F)/T(\mathfrak{o}) \cong X_*(T) \cong X^*(\hat{T}) = \Lambda. \tag{6}$$

If $\lambda \in \Lambda$ we will choose a representative $a_\lambda \in T(F)$ of the corresponding coset in $T(F)/T(\mathfrak{o})$. Also if $\boldsymbol{z} \in \hat{T}(\mathbb{C})$ and $\lambda \in \Lambda$ we will denote by $\boldsymbol{z}^\lambda$ the application of the character λ to $\boldsymbol{z}$. Also if $t \in T(F)$ we may apply the homomorphism (6) to t and apply the resulting rational character of $\hat{T}$ to $\boldsymbol{z}$; we will denote the result by $\tau_{\boldsymbol{z}}(t)$. Thus $\tau_{\boldsymbol{z}}$ is an unramified character of T and $\boldsymbol{z} \longmapsto \tau_{\boldsymbol{z}}$ is an isomorphism of $\hat{T}(\mathbb{C})$ with the group of unramified characters of $T(F)$. The action (5) is compatible with natural action of W on $\hat{T}(\mathbb{C})$.

Let $\tau = \tau_{\boldsymbol{z}}$ be such an unramified character. Let $I(\tau) = I(\boldsymbol{z})$ be the space of the representation of $G(F)$ induced from $\tau_{\boldsymbol{z}}$. This is the space of locally constant functions $f : G(F) \longrightarrow \mathbb{C}$ that satisfy

$$f(bg) = (\delta^{1/2}\tau)(b)f(g), \qquad b \in B(F),$$

where $\delta : B(F) \longrightarrow \mathbb{R}^\times$ is the modular character. The action of $G(F)$ is by right translation. Thus the corresponding representation $\pi = \pi_{\boldsymbol{z}} : G(F) \longrightarrow \mathrm{GL}\big(I(\boldsymbol{z})\big)$ satisfies $\pi(g)f(x) = f(xg)$.

We pick a character ψ of $N_-(F)$. We assume that $\psi \circ i_{-\alpha^\vee} : F \longrightarrow \mathbb{C}$ is trivial on $\mathfrak{o}$ but no larger fractional ideal. We will consider the Whittaker functional $\Omega_{\boldsymbol{z}}$ on the module $M(\boldsymbol{z})$ with respect to the fractional ideal ψ. This is defined by the integral

$$\Omega_{\boldsymbol{z}}(f) := \int_{N_-(F)} f(n)\, \psi(n)^{-1}\, dn.$$

The integral is convergent if $|\boldsymbol{z}^\alpha| < 1$ for every $\alpha \in \Delta_+$. The functional can be extended to all $\boldsymbol{z}$ by analytic continuation.

Let J be the Iwahori subgroup of $G(F)$ which is the preimage of $B(\mathbb{F}_q)$ and $B_-(\mathbb{F}_q)$ under the mod p reduction map $K \longrightarrow G(\mathbb{F}_q)$. We will denote by $M(\tau) = M(\boldsymbol{z})$ the space of Iwahori fixed vectors in $I(\boldsymbol{z})$.

Lemma 1. $N_-(F) \cap BJ = N(\mathfrak{o})$.

Proof. We have the Iwahori factorization $J = N(\mathfrak{p})T(\mathfrak{o})N_-(\mathfrak{o})$. Since $N(\mathfrak{p})T(\mathfrak{o}) \subset B$ an element of $N_-(F) \cap BJ$ may be written as $n = bn'$ with $b \in B$ and $n' \in N_-(\mathfrak{o})$. Now $b \in B \cap N_-(F)$ so $b = 1$ and thus $n = n' \in N_-(\mathfrak{o})$. □

Lemma 2. *Let* $\lambda \in \Lambda$*, and let* $\Phi \in M(\boldsymbol{z})$*. Then* $\Omega_{\boldsymbol{z}}(\pi(a_{-\lambda})\Phi) = 0$ *unless* λ *is dominant.*

Compare Casselman and Shalika [8] Lemma 5.1.

Proof. Let $W(g) = \Omega_{\boldsymbol{z}}(\pi(g)\Phi)$. Then if $n \in N_-(F)$ and $k \in J$ we have $W(ngk) = \psi(n)W(g)$. If λ is not dominant, it is easy to see we may find $n \in N_-(\mathfrak{o})$ such that $\psi(a_{-\lambda} n a_\lambda) \neq 1$. Then $W(a_{-\lambda}) = W(a_{-\lambda} n) = W(a_{-\lambda} n a_\lambda \cdot a_{-\lambda}) = \psi(a_{-\lambda} n a_\lambda) W(a_{-\lambda})$ showing that $W(a_{-\lambda}) = 0$. □

If $w \in W$ let us define $\Phi_w^{\boldsymbol{z}} \in M(\boldsymbol{z})$ as follows. Every element of $G(F)$ may be written as $bw'k$ with $b \in B$, Weyl group element w' having its representative (also denoted w' by abuse of notation) in $N(T) \cap K$, and $k \in J$. Define

$$\Phi_w^{\boldsymbol{z}}(bw'k) := \begin{cases} \delta^{1/2}\tau_{\boldsymbol{z}}(b) & \text{if } w' = w, \\ 0 & \text{otherwise.} \end{cases}$$

These are a basis of $M(\boldsymbol{z})$.

Proposition 1. *Given an unramified character τ of $T(F)$ we have*

$$\Omega_{\boldsymbol{z}}(\pi(a_{-\lambda})\Phi_1^{\boldsymbol{z}}) = \begin{cases} \delta^{1/2}(a_\lambda)\boldsymbol{z}^{-\lambda} & \text{if } \lambda \text{ is dominant,} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Proof. By Lemma 2 we may assume that λ is dominant. Then

$$\Omega_{\boldsymbol{z}}(\pi(a_{-\lambda})\Phi_1^{\boldsymbol{z}}) = \int_{N_-(F)} \Phi_1^{\boldsymbol{z}}(na_{-\lambda})\psi(n)\, dn.$$

Make the variable change $n \to a_{-\lambda}na_\lambda$ which introduces a factor of $\delta(a_\lambda)$. So

$$\Omega_{\boldsymbol{z}}(\pi(a_{-\lambda})\Phi_1^{\boldsymbol{z}}) = \delta(a_\lambda) \int_{N_-(F)} \Phi_1^{\boldsymbol{z}}(a_{-\lambda}n)\psi(a_{-\lambda}na_\lambda)\, dn$$

which equals

$$\delta(a_\lambda)^{1/2}\boldsymbol{z}^{-\lambda} \int_{N_-(F)} \Phi_1^{\boldsymbol{z}}(n)\psi(a_{-\lambda}na_\lambda)\, dn.$$

By Lemma 1 we have $\Phi_1^{\boldsymbol{z}}(n) = 0$ unless $n \in N_-(\mathfrak{o})$, in which case $a_{-\lambda}na_\lambda \in N_-(\mathfrak{o})$ also, so $\psi(a_{-\lambda}na_\lambda) = 1$. The statement follows. □

The standard intertwining integral $\mathcal{A}_w : I(\boldsymbol{z}) \longrightarrow I(w\boldsymbol{z})$ is

$$\mathcal{A}_w^{\boldsymbol{z}}\Phi(g) = \int_{N(F)\cap wN_-(F)w^{-1}} \Phi(w^{-1}ng)\, dn.$$

The integral is convergent for $\tau = \tau_{\boldsymbol{z}}$ with $|\boldsymbol{z}^\alpha| < 1$ when $\alpha \in \Delta^+$. It makes sense for other $\boldsymbol{z}$ by meromorphic continuation in a suitable sense. Let $\tau = \tau_{\boldsymbol{z}}$ and define

$$C_\alpha(\tau) = \frac{1 - q^{-1}\boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^\alpha}. \tag{8}$$

Proposition 2. *For any $w \in W$,*

$$\Omega_{w\boldsymbol{z}} \circ \mathcal{A}_w^{\boldsymbol{z}} = \prod_{\substack{\alpha \in \Delta_+ \\ w\alpha \in \Delta_-}} \frac{1 - q^{-1}\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^\alpha}\Omega_{\boldsymbol{z}}. \tag{9}$$

Proof. We will derive this from Casselman and Shalika [8], Proposition 4.3. Let $\Omega'_{\boldsymbol{z}}$ be the Whittaker functional with respect to the upper triangular unipotent, which is the functional appearing in their formulas. Casselman and Shalika prove

$$\Omega'_{w\boldsymbol{z}} \circ \mathcal{A}^{\boldsymbol{z}}_w = \prod_{\substack{\alpha \in \Delta^+ \\ w\alpha \in \Delta^-}} \frac{1 - q^{-1}\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^{\alpha}} \Omega'_{\boldsymbol{z}}.$$

We take the character of $N(F)$ to be the conjugate of the character ψ of $N_-(F)$ by $\pi(w_0)$; thus

$$\Omega'(f) = \int_{N(F)} f(w_0 n)\, \psi(w_0 n w_0^{-1})\, dn = \int_{N_-(F)} f(n w_0) \psi(n)\, dn.$$

So $\Omega'_{\boldsymbol{z}} = \Omega_{\boldsymbol{z}} \circ \pi(w_0)$ and since $\mathcal{A}^{\boldsymbol{z}}_w$ is an intertwining operator, the statement follows. □

Proposition 3. *Let $\alpha = \alpha_i$ be a simple root and $s = s_i$ the corresponding simple reflection. Then*

$$\mathcal{A}^{s\boldsymbol{z}}_s \Phi^{s\boldsymbol{z}}_w + C_\alpha(\boldsymbol{z}) \Phi^{\boldsymbol{z}}_w = \begin{cases} \Phi^{\boldsymbol{z}}_w + \Phi^{\boldsymbol{z}}_{sw} & \text{if } sw > w, \\ q^{-1}(\Phi^{\boldsymbol{z}}_w + \Phi^{\boldsymbol{z}}_{sw}) & \text{if } sw < w. \end{cases}$$

The order is the Bruhat order. With s a simple reflection, $sw > w$ is equivalent to the condition $l(sw) = l(w) + 1$. See also Rogawski [29] who interprets the operator as a Kazhdan-Lusztig element in the affine Hecke algebra.

Proof. We will deduce this from Casselman [7], Theorem 3.4. We observe that Casselman's ϕ_w is $\pi(w_0)\Phi_{ww_0}$. The map $w \mapsto ww_0$ is inclusion reversing with respect to the Bruhat order. Thus the case $sw < w$ corresponds to Casselman's first formula, which is

$$\mathcal{A}^{\boldsymbol{z}}_s \Phi^{\boldsymbol{z}}_w = (C_\alpha(\boldsymbol{z}) - 1)\Phi^{s\boldsymbol{z}}_w + q^{-1}\Phi^{s\boldsymbol{z}}_{sw}.$$

Now we make use of the identity $C_\alpha(\boldsymbol{z}) + C_{-\alpha}(\boldsymbol{z}) = 1 + q^{-1}$ and write this

$$\mathcal{A}^{\boldsymbol{z}}_s \Phi^{\boldsymbol{z}}_w = (q^{-1} - C_{-\alpha}(\boldsymbol{z}))\Phi^{s\boldsymbol{z}}_w + q^{-1}\Phi^{s\boldsymbol{z}}_{sw}.$$

Then we replace $\boldsymbol{z}$ by $s\boldsymbol{z}$ and since $C_{-\alpha}(s\boldsymbol{z}) = C_\alpha(\boldsymbol{z})$ we obtain

$$\mathcal{A}^{s\boldsymbol{z}}_s \Phi^{s\boldsymbol{z}}_w = (q^{-1} - C_\alpha(\boldsymbol{z}))\Phi^{\boldsymbol{z}}_w + q^{-1})\Phi^{\boldsymbol{z}}_{sw}.$$

This gives the second case.

For the first case, assume that $sw > w$. We use Casselman's second identity but replace w by sw. So his second identity gives us

$$\mathcal{A}_s^{\boldsymbol{z}} \Phi_w^{\boldsymbol{z}} = \Phi_{sw}^{s\boldsymbol{z}} + (C_\alpha(\boldsymbol{z}) - q^{-1}) \Phi_w^{s\boldsymbol{z}} = \Phi_{sw}^{s\boldsymbol{z}} + (1 - C_{-\alpha}(\boldsymbol{z})) \Phi_w^{s\boldsymbol{z}}.$$

Again changing $\boldsymbol{z}$ to $s\boldsymbol{z}$ gives the second case of our identity. □

3 Whittaker functions and Hecke algebras

Let λ be a dominant weight. We do not want the $\boldsymbol{z}^{-\lambda}$ in (7); rather, we want $\boldsymbol{z}^\lambda$. So we will apply the functional Ω to the *contragredient* of $I(\boldsymbol{z})$. This is the representation $I(\boldsymbol{z}^{-1})$. Thus our principal object of study is

$$\mathcal{W}_{\lambda,w}(\boldsymbol{z}) = \delta^{-1/2}(a_\lambda) \, \Omega_{\boldsymbol{z}^{-1}}(\pi(a_{-\lambda}) \Phi_w^{\boldsymbol{z}^{-1}}). \tag{10}$$

We have divided by the constant $\delta^{1/2}(a_\lambda)$ just to keep it out of the formulas.

If s_i is a simple reflection, let $\mathfrak{D}_i$ and $\mathfrak{T}_i$ be the operators that were defined in (3) of the introduction. The factor $\delta^{-1/2}(a_\lambda)$ in (10) is independent of $\boldsymbol{z}$ and therefore commutes with the operators ∂_i, $\mathfrak{D}_i$ and $\mathfrak{T}_i$.

Proposition 4. *Let $s = s_i$ be a simple reflection. Then*

$$\Omega_{\boldsymbol{z}^{-1}}(\mathcal{A}_s^{s\boldsymbol{z}^{-1}} \Phi_w^{s\boldsymbol{z}^{-1}} + C_{-\alpha_i}(\boldsymbol{z}) \Phi_w^{\boldsymbol{z}^{-1}}) = \mathfrak{D}_i \Omega_{\boldsymbol{z}^{-1}} \Phi_w^{\boldsymbol{z}^{-1}}. \tag{11}$$

Proof. We use Proposition 2 with $\boldsymbol{z}$ replaced by $s\boldsymbol{z}^{-1}$. The left-hand side of (11) equals

$$\left(\frac{1 - q^{-1} \boldsymbol{z}^{-\alpha_i}}{1 - \boldsymbol{z}^{\alpha_i}} \right) \Omega_{s\boldsymbol{z}^{-1}} \Phi_w^{s\boldsymbol{z}^{-1}} + \left(\frac{1 - q^{-1} \boldsymbol{z}^{-\alpha_i}}{1 - \boldsymbol{z}^{-\alpha_i}} \right) \Omega_{\boldsymbol{z}^{-1}} \Phi_w^{\boldsymbol{z}^{-1}}$$

$$= (1 - q^{-1} \boldsymbol{z}^{-\alpha_i})(1 - \boldsymbol{z}^{-\alpha_i})^{-1} (\Omega_{\boldsymbol{z}^{-1}} \Phi_w^{\boldsymbol{z}^{-1}} - \boldsymbol{z}^{-\alpha_i} \Omega_{s\boldsymbol{z}^{-1}} \Phi_w^{s\boldsymbol{z}^{-1}}) = \mathfrak{D}_i \Omega_{\boldsymbol{z}^{-1}} \Phi_w^{\boldsymbol{z}^{-1}}.$$

□

Theorem 1. *(i) For any dominant weight λ,*

$$\mathcal{W}_{\lambda,1}(\boldsymbol{z}) = \boldsymbol{z}^\lambda.$$

(ii) Suppose that $s_i w > w$. Then

$$\mathcal{W}_{\lambda,s_i w}(\boldsymbol{z}) = \mathfrak{T}_i \mathcal{W}_{\lambda,w}(\boldsymbol{z}).$$

Proof. Part (i) follows from Proposition 1. Since $\mathfrak{T}_i = \mathfrak{D}_i - 1$, part (ii) follows from combining Proposition 4 with the result of Proposition 3 (with $\boldsymbol{z}^{-1}$ instead of $\boldsymbol{z}$). $\square$

In the remainder of the section, we show that the operators $\mathfrak{T}_i$ generate an algebra isomorphic to the (finite) Hecke algebra. Recall that to a parameter v, the Hecke agebra $\mathcal{H}_v$ is a $\mathbb{C}(v)$ algebra generated by T_i ($1 \leqslant i \leqslant r$) subject to the *quadratic relations*

$$T_i^2 = (v-1)T_i + v$$

and the *braid relations*

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots$$

where the number of terms on both sides is the order of $s_i s_j$. For the connection to Whittaker functions, set $v = q^{-1}$.

We first make use of the Whittaker function to give a short proof that the $\mathfrak{T}_i$ satisfy the braid relation. Let $\mathcal{D}$ be the ring of operators on $\mathcal{O}(\hat{T})$ of the form $\sum_{w \in W} f_w \cdot w$ where $f_w \in \mathcal{O}(\hat{T})$, and the multiplication is defined by $(f_1 \cdot w_1)(f_2 \cdot w_2) = f_1^{w_1} f_2 \cdot w_1 w_2$. The $\mathfrak{D}_i$ are naturally elements of this ring.

Lemma 3. *Suppose that $D \in \mathcal{D}$ and that D annihilates $\boldsymbol{z}^\lambda$ for every dominant weight λ. Then $D = 0$.*

Proof. Define the *support* of $f \in \mathcal{O}(\hat{T})$ to be the finite set of weights with nonzero coefficients in f. Let $D = \sum_{w \in W} f_w \cdot w$. We may choose the dominant weight λ so that the functions $f_w \boldsymbol{z}^{w\lambda}$ have disjoint support. Then $D\boldsymbol{z}^\lambda = 0$ implies that each $f_w = 0$ and so $D = 0$. $\square$

Proposition 5. *Let $s = s_i$ and s_j be simple reflections. Then the operators $\mathfrak{T}_i$ and $\mathfrak{T}_j$ satisfy the same braid relations as s_i and s_j. That is, if k is the order of $s_i s_j$ then*

$$\mathfrak{T}_i \mathfrak{T}_j \mathfrak{T}_i \cdots = \mathfrak{T}_j \mathfrak{T}_i \mathfrak{T}_j \cdots , \tag{12}$$

where k is the number of factors on both sides of this equation.

Proof. By Lemma 3 it is enough to show that these both have the same effect on $\boldsymbol{z}^\lambda$ where λ is a dominant weight. By Proposition 1, $\mathcal{W}_{\lambda,1}(\boldsymbol{z}) = \boldsymbol{z}^\lambda$. Applying either side of (12) to $\mathcal{W}_{\lambda,1}$ gives $\mathcal{W}_{\lambda,w}$, so the statement is clear. $\square$

The quadratic relations for the $\mathfrak{T}_i$ is simpler, and may be checked directly.

Proposition 6. *For any simple reflection s_i,*

$$\mathfrak{D}_i^2 = (1 + q^{-1})\mathfrak{D}_i, \quad \textit{or equivalently,} \quad \mathfrak{T}_i^2 = (q^{-1} - 1)\mathfrak{T}_i + q^{-1}.$$

Proof. We prove the relation on the $\mathfrak{D}_i$. Note that the Demazure operator ∂_i satisfies

$$\partial_i^2 = \partial_i, \qquad \partial_i \boldsymbol{z}^{-\alpha_i}\partial_i = -\partial_i.$$

On the other hand

$$\mathfrak{D}_i^2 = (1 - q^{-1}\boldsymbol{z}^{-\alpha_i})\partial_i(1 - q^{-1}\boldsymbol{z}^{-\alpha_i})\partial_i = (1 - q^{-1}\boldsymbol{z}^{-\alpha_i})(\partial_i^2 - \partial_i q^{-1}\boldsymbol{z}^{-\alpha_i}\partial_i)$$

and so $\mathfrak{D}_i^2 = (1 + q^{-1})\mathfrak{D}_i$. □

4 Extended affine Hecke algebra modules

The Hecke algebra $\mathcal{H}_v$ defined in the previous section may be expanded to the *extended affine Hecke algebra* $\tilde{\mathcal{H}}_v$. The algebra $\tilde{\mathcal{H}}_v$ is generated by $\mathcal{H}_v$ and a commutative subalgebra Θ, which is isomorphic to the group algebra of the weight latice Λ. As a vector space, $\tilde{\mathcal{H}}_v \cong \mathcal{H}_v \otimes \Theta$. Let ζ^λ $(\lambda \in \Lambda)$ be the basis vectors of Θ corresponding to its realization as the group algebra of Λ, so that $\zeta^{\lambda+\mu} = \zeta^\lambda\zeta^\mu$. Thus to describe the multiplication in $\tilde{\mathcal{H}}_v$ it is sufficient to explain how the generators T_i move past the ζ^λ. This is the *Bernstein relation*:

$$\zeta^\lambda T_i - T_i\zeta^{s_i\lambda} = \left(\frac{v-1}{1-\zeta^{-\alpha_i}}\right)(\zeta^\lambda - \zeta^{s_i\lambda}). \tag{13}$$

Note that the right-hand side is in Θ; the numerator is divisible by the denominator in this ring. Let $\mathcal{O}_v(\hat{T}) = \mathbb{C}(v) \otimes \mathcal{O}(\hat{T})$ be the ring of rational functions on $\hat{T}$ with the ground field extended to $\mathbb{C}(v)$.

We use the operators ∂_i, $\mathfrak{D}_i$ and $\mathfrak{T}_i$ to give an action of $\tilde{\mathcal{H}}_v$ on $\mathcal{O}_v(\hat{T})$ as follows. For ∂_i we still use the formula (2). For $\mathfrak{D}_i$ and $\mathfrak{T}_i$ we replace q^{-1} by v in (3) so that:

$$\mathfrak{D}_i = (1 - v\boldsymbol{z}^{-\alpha_i})\partial_i, \qquad \mathfrak{T}_i = \mathfrak{D}_i - 1.$$

It is easy to see that

$$\mathfrak{T}_i f = \frac{f - s_i f}{\boldsymbol{z}^{\alpha_i} - 1} - v\frac{f - \boldsymbol{z}^{-\alpha_i}s_i f}{\boldsymbol{z}^{\alpha_i} - 1}. \tag{14}$$

The operator $\mathfrak{T}_i$ is similar to but slightly different from *Demazure-Lusztig operators* defined by Lusztig [24]. For comparison, these are

$$f \longmapsto \frac{f - s_i f}{\boldsymbol{z}^{\alpha_i} - 1} - v \frac{f - \boldsymbol{z}^{\alpha_i} s_i f}{\boldsymbol{z}_i^{\alpha_i} - 1}.$$

Lusztig [24] used these operators to give a representation of the Hecke algebra on the equivariant K-theory of the flag variety, which was applied by Kazhdan and Lusztig [22, 23] to prove the Deligne-Langlands conjecture.

We note that $\mathcal{H}_v$ admits an algebra homomorphism $\mathrm{sgn} : \mathcal{H}_v \longrightarrow \mathbb{C}(v)$ in which $\mathfrak{T}_i \mapsto -1$. Indeed, this specialization is consistent with both the braid and quadratic relations. We may induce this representation to $\tilde{\mathcal{H}}_v$, obtaining what is called the *antispherical* representation, which we now describe. If V is any $\mathcal{H}_v$-module, let us call a vector $v \in V$ *antispherical* if

$$\phi \cdot v = \mathrm{sgn}(\phi)\, v, \qquad \phi \in \mathcal{H}_v.$$

The $\tilde{\mathcal{H}}_v$-module $\mathcal{M}_{\mathrm{anti}}$ of the antispherical representation is generated by an antispherical vector m_0, such that if V is any module containing an antispherical vector v, then there is a unique $\tilde{\mathcal{H}}_v$-module homomorphism $\mathcal{M}_{\mathrm{anti}} \longrightarrow V$ such that $m_0 \mapsto v$. Since it is characterized by a universal property, $\mathcal{M}_{\mathrm{anti}}$ is unique up to isomorphism. To prove existence, such a module may be constructed by quotienting $\tilde{\mathcal{H}}_v$ by the left ideal generated by the $\mathfrak{T}_i + 1$.

Theorem 2. *Let v be an indeterminate. Then there is a representation of $\tilde{\mathcal{H}}_v$ on $\mathcal{O}_v(\hat{T})$ in which $T_i f(\boldsymbol{z}) = \mathfrak{T}_i f(\boldsymbol{z})$ and $\zeta^\lambda f(\boldsymbol{z}) = \boldsymbol{z}^{-\lambda} f(\boldsymbol{z})$ for $\lambda \in \Lambda$. The resulting module is isomorphic to the antispherical module with antispherical vector $\boldsymbol{z}^{-\rho}$. Furthermore*

$$\mathcal{W}_{\lambda,w}(\boldsymbol{z}) = \begin{cases} T_w \zeta^{-\lambda} \cdot 1 & \text{if } \lambda \text{ is dominant,} \\ 0 & \text{otherwise.} \end{cases}$$

Closely related facts may be found in Arkhipov and Bezrukavnikov [2] and Reeder [27]. One would like to say that the antispherical vector corresponds to $\mathcal{W}_{-\rho,1}$. Now if λ is dominant, by (10) we know that that $\mathcal{W}_{\lambda,w}$ is, up to a normalization, the value of a Whittaker function at $a_{-\lambda}$. But this interpretation fails when $\lambda = -\rho$, because $-\rho$ is not dominant, and indeed the Whittaker function vanishes at a_ρ by Lemma 2. So the antispherical vector in the model corresponds to a "phantom" value of the Whittaker function at a point where the actual value is zero!

Proof. First, let us observe that the results of the previous section imply the braid and quadratic actions. For the quadratic relations, the proof of Proposition 6 works with q^{-1} replaced by an indeterminate v. For the braid relations, note that if v is specialized to q^{-1} where q is a prime power, then the braid relations are are proved in Proposition 5; since there are an infinite number of such q this implies that they are true as algebraic identities.

This proves that we have a representation of $\mathcal{H}_v$ on $\mathcal{O}_v(\hat{T})$. To show that we have a representation of $\tilde{\mathcal{H}}_v$, we must also check the Bernstein relation (13). Apply the left-hand side of (13) to $f(\boldsymbol{z})$. Writing f and s_if instead of $f(\boldsymbol{z})$ and $f(s_i\boldsymbol{z})$, this gives

$$\boldsymbol{z}^{-\lambda}\frac{f-s_if}{\boldsymbol{z}^{\alpha_i}-1} - v\boldsymbol{z}^{-\lambda}\frac{f-\boldsymbol{z}^{-\alpha_i}s_if}{\boldsymbol{z}^{\alpha_i}-1} - \frac{\boldsymbol{z}^{-s_i\lambda}f-\boldsymbol{z}^{-\lambda}s_if}{\boldsymbol{z}^{\alpha_i}-1} - v\frac{z^{-s_i\lambda}f-\boldsymbol{z}^{-\alpha_i-\lambda}s_if}{\boldsymbol{z}^{\alpha_i}-1}.$$

The terms involving s_if all cancel. We rewrite the remaining terms by multiplying each numerator and denominator by -1 to obtain

$$\left[-\boldsymbol{z}^{-\lambda}\frac{1}{1-\boldsymbol{z}^{\alpha_i}} + v\boldsymbol{z}^{-\lambda}\frac{1}{1-\boldsymbol{z}^{\alpha_i}} + \frac{\boldsymbol{z}^{-s_i\lambda}}{1-\boldsymbol{z}^{\alpha_i}} + v\frac{z^{-s_i\lambda}}{1-\boldsymbol{z}^{\alpha_i}}\right] f \quad = \quad \left(\frac{v-1}{1-\zeta^{-\alpha_i}}\right)(\zeta^\lambda - \zeta^{s_i\lambda})f.$$

It remains for us to show that $\mathcal{O}_v(\hat{T}) \cong \mathcal{M}_{\text{anti}}$. First we observe that there is an antispherical vector $\boldsymbol{z}^{-\rho}$, where ρ is the Weyl vector (half the sum of the positive roots). Indeed, $\boldsymbol{z}^{-\rho}$ is annihilated by ∂_i since $s_i\boldsymbol{z}^{-\rho} = \boldsymbol{z}^{\alpha_i}\boldsymbol{z}^{\rho}$, and so $\mathfrak{D}_i\boldsymbol{z}^{-\rho} = 0$, which implies that the vector $\boldsymbol{z}^{-\rho}$ is antispherical.

By the universal property of the antispherical module, we have a homomorphism $\mathcal{M}_{\text{anti}} \longrightarrow \mathcal{O}_v(\hat{T})$ such that $m_0 \longmapsto \boldsymbol{z}^{-\lambda}$. We will argue that this map is injective. First, we have

$$\tilde{\mathcal{H}}_v = \bigoplus_{\lambda\in\Lambda} \zeta^\lambda \mathcal{H}_v,$$

so $\mathcal{M}_{\text{anti}}$ is spanned by the vectors $\zeta^\lambda m_0$, which are mapped to $\boldsymbol{z}^{-\rho-\lambda}$. These are a basis of $\mathcal{O}_v(\hat{T})$. From this, it is clear that the map is both injective and surjective. □

Proposition 7. *There is a unique action of* $\widetilde{\mathcal{H}}_{q^{-1}}$ *on* $M(\tau)$ *in which*

$$T_s \cdot \Phi_w = \begin{cases} \Phi_{sw} & \text{if } sw > w, \\ q^{-1}\Phi_{sw} + (q^{-1}-1)\Phi_w & \textit{if } sw < w, \end{cases} \tag{15}$$

and for $\lambda \in \Lambda$,

$$\zeta^\lambda \cdot \Phi_1 = \boldsymbol{z}^\lambda \Phi_1.$$

For any dominant weight λ*, the map from* $M(\boldsymbol{z}^{-1})$ *to* $\mathcal{O}(\hat{T})$

$$\mathbb{W}_\lambda : \phi \mapsto \mathcal{W}_{\lambda,\phi}(\boldsymbol{z}) := \delta^{-1/2}(a_\lambda)\, \Omega_{\boldsymbol{z}^{-1}}(\pi(a_{-\lambda})\phi^{\boldsymbol{z}^{-1}}) \tag{16}$$

is an intertwiner of $\tilde{\mathcal{H}}_{q^{-1}}$ *modules.*

Proof. The fact that (15) defines an action of the finite Hecke algebra $\mathcal{H}_{q^{-1}}$ on $M(\tau)$ follows from the fact that both $\mathcal{H}_{q^{-1}}$ and $M(\tau)$ are vector spaces with bases indexed by W; so $T_w \mapsto \Phi_w$ defines a vector space isomorphism. Therefore we may transport the action of $\mathcal{H}_{q^{-1}}$ to an action on $M(\tau)$ by this isomorphism, and this regular representation is the action (15).

To check that (16) is an $\mathcal{H}_{q^{-1}}$-module homomorphism, it suffices to verify that for any simple reflection s_i and any $w \in W$,

$$\mathbb{W}_\lambda(T_i \cdot \Phi_w) = \mathfrak{T}_i \mathbb{W}_\lambda(\Phi_w) \tag{17}$$

The case where $s_i w > w$ is just Theorem 1, Part (ii). The case $s_i w < w$ follows by the same proof using Propositions 3 and 4.

We claim that the image of $\mathbb{W}_\lambda$ in $\mathcal{O}(\hat{T})$ is closed under the action of $\tilde{\mathcal{H}}_{q^{-1}}$. Indeed, it is clear that the image of Φ_1 lies in a one-dimensional subspace invariant under Θ, and since $\tilde{\mathcal{H}}_{q^{-1}} = \mathcal{H}_{q^{-1}}\Theta$, the statement follows. Since (16) is obviously injective on $M(\tau)$ we may therefore pull back the action of $\tilde{\mathcal{H}}_{q^{-1}}$ on $\mathcal{O}(\hat{T})$ and obtain an action on $M(\tau)$. □

5 Bruhat order

In addition to the functions $\mathcal{W}_{\lambda,w}(\boldsymbol{z})$ defined by (10) it is also natural to sum over the Bruhat order and write

$$\tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}) = \sum_{y \leqslant w} \mathcal{W}_{\lambda,w}(\boldsymbol{z}).$$

In this section, we present formulas for $\tilde{\mathcal{W}}_{\lambda,w}$ using the operator $\mathfrak{D}_i$ and combinatorics of the Bruhat order.

Of particular note is the spherical Whittaker function $\tilde{\mathcal{W}}_{\lambda,w_0}$ computed by Casselman and Shalika [8]. In our modified setup the Casselman-Shalika formula states that

$$\tilde{\mathcal{W}}_{\lambda,w_0}(\boldsymbol{z}) = \prod_{\alpha\in\Delta_+}(1-q^{-1}\boldsymbol{z}^{-\alpha})\chi_\lambda(\boldsymbol{z}),$$

where χ_λ is the irreducible character of $\hat{G}(\mathbb{C})$ with highest weight λ. An alternate proof of this fact will be given in the next section.

Proposition 8. *Let s be a simple reflection and $w_1, w_2 \in W$.*

(i) Assume that $sw_1 < w_1$ and $sw_2 < w_2$. Then $w_1 \leqslant w_2$ if and only if $sw_1 \leqslant w_2$ if and only if $sw_1 \leqslant sw_2$.

(ii) Assume that $sw_1 > w_1$ and $sw_2 > w_2$. Then $w_1 \geqslant w_2$ if and only if $sw_1 \geqslant w_2$ if and only if $sw_1 \geqslant sw_2$.

Proof. Part (i) is a well-known property of Coxeter groups, called property $Z(s, w_1, w_2)$ by Deodhar [13]. Note that $w \longmapsto ww_0$ is an order reversing bijection of W. Applying this gives (ii). □

Suppose that $s = s_i$ is a left ascent of $w \in W$: $sw > w$. Then we will define

$$H(w,s) = \{u \in W | u, su \leqslant w\}.$$

Proposition 9. *The set $H(w,s)$ is cofinal in W in the sense that if $u \in H(w,s)$ and $t \leqslant u$ then $t \in H(w,s)$.*

Proof. We have $t \leqslant u$ with both $u, su \leqslant w$. We wish to show that $t \in H(w,s)$. We may assume without loss of generality that $su < u$. For if not, then $u < su$ so $t \leqslant su$. Thus interchanging u and su if necessary, we may assume that $su < u$. Also without loss of generality, $t < st$ since otherwise both t, st are $\leqslant u \leqslant w$ as required. Now taking $w_1 = su$ and $w_2 = t$ in Proposition 8 (ii), we see that both $t, st \leqslant u$ and so $t \in H(w,s)$. □

Define an integer-valued function $c_{w,s}$ on W by

$$c_{w,s}(u) = \sum_{\substack{t \in H(w,s) \\ t \geqslant u}} (-1)^{l(t)-l(u)}.$$

Theorem 3. *Let $\alpha = \alpha_i$ be a simple root, and let $s = s_i$ denote the corresponding reflection. Assume that $sw > w$. Then*

$$\tilde{\mathcal{W}}_{\lambda,sw}(\boldsymbol{z}) = (1-q^{-1}\boldsymbol{z}^{-\alpha})\,\partial_i\tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}) - q^{-1}\sum_{u\in H(w,s)}\mathcal{W}_{\lambda,u}(\boldsymbol{z}). \tag{18}$$

Equivalently,

$$\tilde{\mathcal{W}}_{\lambda,sw}(\boldsymbol{z}) = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\,\partial_i\tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}) - q^{-1}\sum_{u\in H(w,s)} c_{w,s}(u)\,\tilde{\mathcal{W}}_{\lambda,u}(\boldsymbol{z}). \tag{19}$$

Proof. By Proposition 4

$$\delta^{1/2}(a_\lambda)(1 - q^{-1}\boldsymbol{z}^{-\alpha})\partial_i\tilde{\mathcal{W}}_{\lambda,w} = \sum_{x\leqslant w}\Omega_{\boldsymbol{z}^{-1}}(\mathcal{A}_s^{s\boldsymbol{z}^{-1}}\Phi_x^{s\boldsymbol{z}^{-1}} + C_{-\alpha}(\boldsymbol{z})\Phi_x^{\boldsymbol{z}^{-1}}).$$

We split the sum into two parts according as $sx > x$ or $sx < x$ and use Proposition 3. We have

$$\sum_{\substack{x\geqslant w\\ sx<x}}\Omega_{\boldsymbol{z}^{-1}}(\mathcal{A}_s^{s\boldsymbol{z}^{-1}}\Phi_x^{s\boldsymbol{z}^{-1}} + C_{-\alpha}(\boldsymbol{z})\Phi_x^{\boldsymbol{z}^{-1}}) = \delta^{1/2}(a_\lambda)\sum_{\substack{x\geqslant w\\ sx<x}}(\mathcal{W}_{\lambda,x}(\boldsymbol{z}) + \mathcal{W}_{\lambda,sx}(\boldsymbol{z})).$$

By Proposition 8 (ii) with $w_1 = w$ and $w_2 = x$, we see that

$$\bigcup_{\substack{x\geqslant w\\ sx>x}}\{x, sx\} = \{u\in W | u \geqslant sw\}.$$

Therefore this contribution equals $\delta^{1/2}(a_\lambda)\,\tilde{\mathcal{W}}_{\lambda,sw}(\boldsymbol{z})$.

On the other hand, let us consider the contributions from $sx < x$. By Propositions 3 and 4 these contribute

$$q^{-1}\,\delta^{1/2}(a_\lambda)\sum_{\substack{x\geqslant w\\ sx<x}}(\mathcal{W}_{\lambda,x}(\boldsymbol{z}) + \mathcal{W}_{\lambda,sx}(\boldsymbol{z})) = q^{-1}\,\delta^{1/2}(a_\lambda)\sum_{u\in H(w,s)}\mathcal{W}_{\lambda,u}(\boldsymbol{z}).$$

This proves (18). By Möbius inversion (Verma [31], Deodhar [13]) we may write

$$\Phi_u^{\boldsymbol{z}} = \sum_{t\leqslant u}(-1)^{l(t)-l(u)}\tilde{\Phi}_t^{\boldsymbol{z}},$$

and substituting this gives (19). □

The function $c_{w,s}$ has a tendency to take on only a few nonzero values. It vanishes off $H(w,s)$. But even if $H(w,s)$ contains many elements, $c_{w,s}(u)$ will typically vanish for most of these. This sparseness means there are usually only a few terms on the right-hand side in (19). For example, in the group SL_4, with Cartan type A_3, if we consider the pairs w, s where s is a left ascent of w, we find sixteen such pairs where $c_{w,s}$ is always zero. Thus for these pairs the identity (19) takes the form

$$\tilde{\mathcal{W}}_{\lambda,sw}(\boldsymbol{z}) = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\,\partial_i \tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}).$$

There are seventeen pairs (w, s) such that $c_{w,s}(u) \neq 0$ for only one particular u. Then

$$\tilde{\mathcal{W}}_{\lambda,sw}(\boldsymbol{z}) = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\partial_i \tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}) - q^{-1}\tilde{\mathcal{W}}_{\lambda,u}(\boldsymbol{z}).$$

Finally, there are three cases where

$$\tilde{\mathcal{W}}_{\lambda,sw}(\boldsymbol{z}) = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\partial_i \tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}) - q^{-1}\tilde{\mathcal{W}}_{\lambda,u}(\boldsymbol{z}) - q^{-1}\tilde{\mathcal{W}}_{\lambda,v}(\boldsymbol{z}) + q^{-1}\tilde{\mathcal{W}}_{\lambda,t}(\boldsymbol{z})$$

These cases are given by the following by the following table.

w	s	u	v	t
$s_1s_2s_3s_2s_1$	s_2	$s_1s_2s_3s_1$	$s_2s_3s_2s_1$	$s_2s_3s_1$
$s_2s_1s_3s_2$	s_1	$s_1s_2s_1$	$s_1s_3s_2$	s_1s_2
$s_2s_1s_3s_2$	s_3	$s_2s_3s_2$	$s_1s_3s_2$	s_3s_2

6 Specializations

There are two interesting specializations of the parameter v, besides $v = q^{-1}$ which produces the Whittaker function. We may also take $v = 0$ and $v = 1$. Using information we get from these specializations, we will prove two well-known interesting results: the Casselman-Shalika formula for the spherical Whittaker function, and Demazure's formula (22) below expressing the character of an irreducible representation as a Demazure character.

We've proved using Whittaker functions that the operator $\mathfrak{T}_i$ satisfies the braid relation:

$$\mathfrak{T}_i\mathfrak{T}_j\mathfrak{T}_i \cdots = \mathfrak{T}_j\mathfrak{T}_i\mathfrak{T}_j \cdots$$

where the number of terms on both sides equals the order of s_is_j. Recall that

$$\mathfrak{T}_i f = \frac{f - s_i f}{\boldsymbol{z}^{\alpha_i} - 1} - q^{-1}\frac{f - \boldsymbol{z}^{-\alpha_i} s_i f}{\boldsymbol{z}^{\alpha_i} - 1}.$$

The two components of this are the divided difference operator

$$D_i f = \frac{f - s_i f}{\boldsymbol{z}^{\alpha_i} - 1}$$

and $-q^{-1}\boldsymbol{z}^{-\alpha_i}\partial_i$, with ∂_i the Demazure operator. It is easy to check that $D_i = \partial_i - 1$. We will denote by ρ the Weyl vector (half the sum of the positive roots).

Proposition 10. *Both D_i and ∂_i satisfy the braid relation.*

Proof. For D_i we may just specialize $q^{-1} \to 0$ in the braid relation for $\mathfrak{T}_i$. To deduce the braid relation for ∂_i, note that $-D_i$ and ∂_i are conjugate in the ring of endomorphisms of $\mathcal{O}(\hat{T})$. Indeed, if $f \in \mathcal{O}(T)$ define $\theta f(\boldsymbol{z}) = \boldsymbol{z}^{-\rho} f(-\boldsymbol{z})$. We have $\theta^2 = 1$ and $\theta(-D_i)\theta = \partial_i$, so ∂_i also satisfies the braid relation. □

Now we may define $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ where $w = s_{i_1} \cdots s_{i_k}$ a reduced decomposition of $w \in W$. We may emphasize the dependence on q by writing, instead of $W_{\lambda,w}(\boldsymbol{z})$ and $\tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z})$ the notations $\mathcal{W}_{\lambda,w}(\boldsymbol{z}, q^{-1})$ and $\tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}, q^{-1})$.

Proposition 11. *For any dominant weight λ and $w \in W$, then $\mathcal{W}_{\lambda,w}(\boldsymbol{z}, 1) = (-1)^{l(w)}\boldsymbol{z}^{w(\lambda+\rho)-\rho}$.*

Proof. If $w = 1$, we know this. We argue by induction on $l(w)$. Specializing to $q = 1$ the operator $\mathfrak{T}_i$ becomes

$$\mathfrak{T}_i f = \frac{f - s_i f}{\boldsymbol{z}^{\alpha_i} - 1} - \frac{f - \boldsymbol{z}^{-\alpha_i} s_i f}{\boldsymbol{z}^{\alpha_i} - 1} = \frac{\boldsymbol{z}^{-\alpha_i} - 1}{\boldsymbol{z}^{\alpha_i} - 1} s_i f = -\boldsymbol{z}^{-\alpha_i} s_i f.$$

So it is sufficient to show that the function $f_w(\boldsymbol{z}) = (-1)^{l(w)}\boldsymbol{z}^{w(\lambda+\rho)-\rho}$ satisfies the recursion

$$-\boldsymbol{z}^{-\alpha_i} s_i f_w(\boldsymbol{z}) = f_{s_i w}(\boldsymbol{z})$$

when $s_i w > w$ and this follows from $s_i\rho = \rho - \alpha_i$. □

Proposition 12. *We have*

$$\tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}, 0) = \partial_w \boldsymbol{z}^\lambda.$$

Proof. This is true if $w = 1$, so we argue by induction on $l(w)$. We have proven the recursion

$$\tilde{\mathcal{W}}_{\lambda,s_i w}(\boldsymbol{z}, q^{-1}) = (1 - q^{-1}\boldsymbol{z}^{-\alpha_i})\, \partial_i \tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}, q^{-1}) - q^{-1} \sum_{u \in H(s,w)} \mathcal{W}_{\lambda,u}(\boldsymbol{z}, q^{-1}).$$

Specializing $q^{-1} \to 0$ we have

$$\tilde{\mathcal{W}}_{\lambda,s_i w}(\boldsymbol{z}, 0) = \partial_i \tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}, 0),$$

and the statement follows. □

Let $\mathcal{E}$ be the ring of finite linear combinations of $\boldsymbol{z}^\mu$ with $\mu \in \Lambda$ in which the coefficients are polynomials in q^{-1}. The ring $\mathcal{E}$ is a principal ideal domain.

Lemma 4. *Let $f(\boldsymbol{z}, q^{-1}) \in \mathcal{E}$. Suppose that for each simple reflection s_i we have $f(\boldsymbol{z}) = (1-q^{-1}\boldsymbol{z}^{-\alpha})f_i(\boldsymbol{z}, q^{-1})$ where $f_i \in \mathcal{E}$ and f_i is s_i-invariant. Then*

$$f(\boldsymbol{z}, q^{-1}) = \prod_{\alpha \in \Delta_+} (1 - q^{-1}\boldsymbol{z}^{-\alpha}) f_0(\boldsymbol{z}, q^{-1}) \tag{20}$$

where $f_0 \in \mathcal{E}$ and f_0 is W-invariant.

Proof. The factors $(1 - q^{-1}\boldsymbol{z}^{-\alpha})$ with $\alpha \in \Delta_+$ are coprime in $\mathcal{E}$. First let us show that f is divisible by each of them. If not, let α be such that $(1 - q^{-1}\boldsymbol{z}^{-\alpha})$ does not divide f. Write $\alpha = \sum n_i\alpha_i$ where $n_i \in \mathbb{Z}$ and call $\sum n_i$ the *height* of α. We assume the counterexample α is minimal with respect to the height. By hypothesis, α is not simple, so $\mathrm{ht}(\alpha) > 1$. Thus we may find s_i such that $\alpha = s_i(\beta)$ where β is a positive root of lower height. By induction $f(\boldsymbol{z})$ is divisibe by $(1-q^{-1}\boldsymbol{z}^{\beta})$. However $f(\boldsymbol{z}) = (1-q^{-1}\boldsymbol{z}^{-\alpha_i})f_i(\boldsymbol{z})$ where $f_i(\boldsymbol{z})$ is s_i-invariant. Since β and $s_i(\beta)$ are both positive, $\beta \neq \alpha_i$ and so f_i is divisible by $(1 - q^{-1}\boldsymbol{z}^{-\beta})$. Since it is not divisible by $(1 - q^{-1}\boldsymbol{z}^{-\alpha})$ this is a contradiction.

Thus we have (20) for f_0 in $\mathcal{E}$. To see that f_0 is invariant under W, write

$$f_0(\boldsymbol{z}) = \prod_{\substack{\alpha \in \Delta_+ \\ \alpha \neq \alpha_i}} (1 - q^{-1}\boldsymbol{z}^{-\alpha})^{-1} f_i(\boldsymbol{z}),$$

where f_i is s_i-invariant, and the factors in the product are permuted by s_i, so f_0 is s_i-invariant for every simple reflection s_i. □

Let $\mathcal{W}_\lambda^\circ(\boldsymbol{z}) := \tilde{\mathcal{W}}_{\lambda,w_0} = \sum_w \mathcal{W}_{\lambda,w}(\boldsymbol{z})$ be the spherical Whittaker function.

Theorem 4. (Casselman-Shalika) *Let λ be a dominant weight. Then*

$$\mathcal{W}_\lambda^\circ(\boldsymbol{z}) = \prod_{\alpha \in \Delta_+} (1 - q^{-1}\boldsymbol{z}^{-\alpha}) \partial_{w_0} \boldsymbol{z}^\lambda. \tag{21}$$

This can be compared with the Casselman-Shalika formula as it is usually stated when we show in Theorem 5 that $\partial_{w_0}\boldsymbol{z}^\lambda$ is the character of the irreducible representation of $\hat{G}(\mathbb{C})$ with highest weight λ.

Proof. Let

$$g_i = \sum_{\substack{w \in W \\ s_i w > w}} \mathcal{W}_{\lambda,w}(\boldsymbol{z}), \qquad f_i = \partial_i g_i.$$

Since f_i is in the image of ∂_i it is s_i-invariant. Moreover $\mathcal{W}_\lambda^\circ = \mathfrak{D}_i g_i = (1 - q^{-1}\boldsymbol{z}^{-\alpha_i}) f_i$. By the Lemma it follows that

$$\prod_{\alpha \in \Delta_+} (1 - q^{-1}\boldsymbol{z}^{-\alpha}) f_0(\boldsymbol{z})$$

where f_0 is W-invariant. We have

$$\mathcal{W}_\lambda^\circ(\boldsymbol{z}) = \sum_w \mathfrak{T}_w \boldsymbol{z}^\lambda$$

and each term is a polynomial in q^{-1} of degree $l(w)$. So $\mathcal{W}_\lambda^\circ(\boldsymbol{z})$ is a polynomial in q^{-1} of degree $l(w_0) = |\Delta_+|$. It follows that f_0 has degree zero, that is, is independent of q.

Since f_0 is independent of q, we may compute it by setting $q^{-1} \to 0$. The spherical Whittaker function $\mathcal{W}_\lambda^\circ$ equals $\tilde{\mathcal{W}}_{\lambda,w_0}$. So by Proposition 12 we have $f_0 = \partial_{w_0} \boldsymbol{z}^\lambda$. □

Let λ be a dominant weight, and as before let $\chi_\lambda(\boldsymbol{z})$ be the irreducible character of $\hat{G}(\mathbb{C})$ with highest weight λ.

Theorem 5. (Demazure) *Let λ be a dominant weight. Then*

$$\partial_{w_0} \boldsymbol{z}^\lambda = \chi_\lambda(\boldsymbol{z}). \tag{22}$$

Proof. By Proposition 11

$$\tilde{\mathcal{W}}_{\lambda,w}(\boldsymbol{z}, 1) = \sum_{u \leqslant w} (-1)^{l(u)} \boldsymbol{z}^{u(\lambda+\rho)-\rho}.$$

So we may take $w = w_0$ in (21) and specialize to $q = 1$ to obtain

$$\prod_{\alpha \in \Delta_+} (1 - \boldsymbol{z}^{-\alpha}) \partial_{w_0} \boldsymbol{z}^\lambda = \sum_{u \in W} (-1)^{l(u)} \boldsymbol{z}^{u(\lambda+\rho)-\rho}.$$

Moving the product to the other side and applying the Weyl character formula we obtain the result. □

7 Fibers of Bott-Samelson resolutions

In this section, we will give some relationships between Whittaker functions and the geometry of Schubert and Bott-Samelson varieties. Let $\mathfrak{w} = (s_{h_1}, \cdots, s_{h_k})$ be a reduced decomposition of $w = s_{h_1} \cdots s_{h_k}$ into a product of simple reflections. Bott and Samelson gave a smooth projective variety $Z_{\mathfrak{w}}$ together with a morphism $Z_{\mathfrak{w}} \longrightarrow X_w$ that is a birational equaivalence. If X_w is singular, this gives a resolution of its singularities. On the other hand, we have seen that the Whittaker function $\tilde{\mathcal{W}}_{w,\lambda}(\boldsymbol{z})$ is "roughly" equal to $\mathfrak{D}_{h_1} \cdots \mathfrak{D}_{h_k} \boldsymbol{z}^{\lambda}$; there are correction terms, and these follow the same combinatorics as the fibers of the Bott-Samelson map.

Let G be a complex reductive group, and let B be a Borel subgroup. In the application to Whittaker functions, G will be the group formerly denoted $\hat{G}(\mathbb{C})$, but we suppress the hat in this section.

Let $X = G/B$ be the flag variety. If w is an element of the Weyl group W, let Y_w be the image of BwB in X. The closure

$$X_w = \bigcup_{u \leqslant w} Y_u$$

is the closed Schubert cell.

To define the Bott-Samelson variety $Z_{\mathfrak{w}}$, let P_i be the parabolic subgroup generated by B and s_i. The group B^k acts on $P_{h_1} \times \cdots \times P_{h_k}$ on the right by

$$(p_1, \cdots, p_k)(b_1, \cdots, b_k) = (p_1 b_1, b_1^{-1} p_2 b_2, \cdots, b_{k-1} p_k b_k). \tag{23}$$

Then $Z_{\mathfrak{w}}$ is the quotient variety. The multiplication map $P_{h_1} \times \cdots \times P_{h_k} \longrightarrow G$ induces a rational map $Z_{\mathfrak{w}} \longrightarrow X_w$ that is a birational equivalence.

The fibers of the map $Z_{\mathfrak{w}} \longrightarrow X_w$ are constant over each open Bruhat cell Y_u with $u \leqslant w$. The fiber over Y_w (containing the generic point) consists of a single point. For other Y_u with $u \leqslant w$ the cohomology of the fiber was described combinatorially by Deodhar [14]. To explain this description, let us choose a subword of $\mathfrak{w}$ representing u. This means that we have a sequence

$$1 \leqslant j_1 < j_2 < \cdots < j_l \leqslant k \tag{24}$$

with $u = s_{h_{j_1}} \cdots s_{h_{j_l}}$. This is not assumed to be a reduced decomposition. There may be more than one such subword, and we will sum over these.

We may alternatively specify a sequence $\sigma_1, \sigma_2, \cdots, \sigma_k, \sigma_{k+1}$ of Weyl group elements such that $\sigma_{k+1} = 1$, and each σ_i equals either σ_{i+1} or $s_{h_i} \sigma_{i+1}$, and

$\sigma_1 = u$. Such data are equivalent to giving a sequence (24). Indeed, given such as sequence define σ by downward induction from $\sigma_{k+1} = 1$ with $\sigma_i = s_{h_i}\sigma_{i+1}$ if i is in the sequence (24), or $\sigma_i = \sigma_{i+1}$ if it is not. Following Deodhar, we define the *defect* $d(\sigma)$ to be the number of i such that $s_{h_i}\sigma_{i+1} < \sigma_{i+1}$. We also write $\pi(\sigma) = u$, i.e. $\pi(\sigma) = \sigma_1$.

Now let F_u be the fiber of the map $Z_{\mathfrak{w}} \longrightarrow X_w$ over a point in Y_u. The *Poincaré* polynomial $P_{\mathfrak{w},u}(q)$ is the polynomial of degree equal to the complex dimension of F_u whose n-th coefficient is the dimension if $H^{2n}(F_u)$.

Proposition 13. *To any $u, w \in W$ with $u \leqslant w$,*

$$P_{\mathfrak{w},u}(q) = \sum_{\pi(\sigma)=u} q^{d(\sigma)}.$$

Proof. This was stated without proof in Deodhar [14], where it was an observation of the referee. A proof of an equivalent formula may be found in Gaussent [15]. Note that Deodhar parses the word representing u from left to right (so his $\sigma_0 = 1$ and $\sigma_i = \sigma_{i-1}$ or $\sigma_{i-1}s_{h_i}$), while it is more convenient for us to parse it from right to left. This does not affect the Poincaré polynomial. $\square$

We will now relate this to Whittaker functions. Let us adopt a notation that is suggestive of an analogy between these and Schubert varieties. If $w \in W$, define the operators $\boldsymbol{Y}_w := \mathfrak{T}_w$ and $\boldsymbol{X}_w = \sum_{u \leqslant w} \boldsymbol{Y}_{u,\lambda}$ on $\mathcal{O}(\hat{T})$ so that

$$\mathcal{W}_{w,\lambda}(\boldsymbol{z}) = \boldsymbol{Y}_w(\boldsymbol{z}^\lambda) \quad \text{and} \quad \tilde{\mathcal{W}}_{w,\lambda}(\boldsymbol{z}) = \boldsymbol{X}_w(\boldsymbol{z}^\lambda).$$

Finally, as noted in the introduction, for a reduced word $\mathfrak{w} = (s_{h_1}, \cdots, s_{h_k})$, let

$$\boldsymbol{Z}_{\mathfrak{w}} = \mathfrak{D}_{h_1} \cdots \mathfrak{D}_{h_k}, \quad \text{with} \quad \mathfrak{D}_i = (1 - q^{-1}\boldsymbol{z}^{-\alpha_i})\partial_i.$$

Our theme is that the relationship between the functions $\boldsymbol{Z}_{\mathfrak{w}}$ and $\boldsymbol{X}_w$ is the same as the relationship between the Bott-Samelson and Schubert varieties $Z_{\mathfrak{w}}$ and X_w. The idea is that $Z_{\mathfrak{w}}$ is built up from a point by successive $\mathbb{P}^1$ fiberings, and $\mathfrak{D}_i$ corresponds to such an operation. If we accept this analogy, $\boldsymbol{Z}_{\mathfrak{w}}$ is analogous to the Bott-Samelson variety $Z_{\mathfrak{w}}$.

Moreover $\boldsymbol{Y}_w$ and $\boldsymbol{X}_w$ are analogous to the open and closed Schubert varieties Y_w and X_w. The next result shows that we can express $\boldsymbol{Z}_{\mathfrak{w}}$ in terms of the $\boldsymbol{Y}_u$ with $u \leqslant w$ by multiplying in a factor corresponding to the fiber of $Z_{\mathfrak{w}} \longrightarrow X_w$ over the open Schubert cell Y_u. This factor is just the Poincaré polynomial of the fiber.

Theorem 6. *To any w in W with reduced decomposition $\mathfrak{w}$,*

$$\boldsymbol{Z}_{\mathfrak{w}} = \sum_{u \leqslant w} P_{\mathfrak{w},u}(q^{-1})\boldsymbol{Y}_u.$$

Proof. Combining Proposition 3 with Proposition 4, we see that applying $\mathfrak{D}_i$ to $\boldsymbol{Y}_y$ (with $y \in W$) gives $\boldsymbol{Y}_y + \boldsymbol{Y}_{s_i y}$ if $s_i y > y$; or the same thing multiplied by q^{-1} if $s_i y < y$. Thus applying $\mathfrak{D}_{h_1} \cdots \mathfrak{D}_{h_k}$ to $\boldsymbol{Y}_1(\boldsymbol{z}^\lambda) = \boldsymbol{z}^\lambda$ (Theorem 1 (i)) and collecting the coefficients of $\boldsymbol{Y}_u$, it is clear that there will be a contribution for each sequence $\sigma_1, \cdots, \sigma_k, \sigma_{k+1}$ such that $\sigma_{k+1} = 1$ and σ_i is either $s_{h_i}\sigma_{i+1}$ or σ_{i+1}. Moreover, there will be a multiplication by a power of q^{-1} each time we encounter a descent. The total power of q^{-1} is just Deodhar's defect $d(\sigma)$. □

There is a variant of this result that we will prove from scratch. If s is a left ascent of w then we have a *partial Bott-Samelson variety* $Z_{s,w}$ which is the quotient $(P \times X_w)/B$ where if $s = s_i$ then $P = P_i$ and B acts on the right by $(p, x) \cdot b = (pb^{-1}, bx)$.

The map $\mu(p, x) = p \cdot x$ is obviously compatible with the action of B on $P \times X_w$, hence induces a morphism $\mu : Z_{s,w} \longrightarrow X_{sw}$. It is a birational equivalence. The fiber over an open Schubert cell Y_u (with $u \leqslant sw$) is either a single point or a $\mathbb{P}^1$, and we will find a combinatorial criterion to distinguish these cases. Let T be a maximal torus of G contained in B. Since the fibers of μ are constant on Schubert cells $Y_t \subset X_{sw}$ with $t \in W$, it suffices to study the fiber $\mu^{-1}(y_t)$, where $y_t \in Y_t^T$ is the unique T-fixed point in the Schubert cell Y_t. Since the fiber $\mu^{-1}(y_t)$ is either a single point or has dimension 1, it is determined by its Euler characteristic $\chi(\mu^{-1}(y_t))$, and this is what we will compute.

Lemma 5. *Let V be a projective complex algebraic variety with a T action whose fixed point set V^T consists of isolated points. Then $\chi(V) = \#V^T$.*

Proof. Choosing a regular element λ of $\mathrm{Hom}(\mathbb{C}^\times, T)$, it follows that V has a $\mathbb{C}^\times$ action with the same fixed point set, that is, $V^{\mathbb{C}^\times} = V^T$. This $\mathbb{C}^\times$ action defines a Bialynicki-Birula cellular decomposition of V, with cells $\{U_x\}_{x \in V^{\mathbb{C}^\times}}$ defined by

$$U_x = \{z \in V \mid \lim_{\varepsilon \to 0} \varepsilon \cdot z = x\},$$

one cell for each x. See Bialynicki-Birula, Carrell and McGovern [3]. Since the cells are all of even real dimension, the Euler characteristic of V is simply the number of cells - that is, the number of fixed points. □

Proposition 14. *The fiber of μ over Y_u is $\mathbb{P}^1$ if and only if both $u, su \leqslant w$, and is a point otherwise.*

Proof. In view of Lemma 5, in order to compute $\chi(\mu^{-1}(y_t))$, we need to compute the number of fixed points in the set $\mu^{-1}(y_t)^T$. This is straightforward since the fixed point set $Z_{s,w}^T$ equals $\{(u,t) \mid u \in \langle s \rangle, t \leqslant w\}$ and the map $\mu^T : Z_{s,w}^T \longrightarrow X_{sw}^T$ is multiplication $(u,t) \mapsto ut$. Discussions of these facts may be found in many places, usually for "standard" Bott-Samelson varieties rather than these partial ones. See, for example Brion [5].

Now, from these two facts, we compute $(\mu^T)^{-1}(y_t)$ for $t \leqslant w$. In general, we have

$$(\mu^T)^{-1}(y_t) = \{(u,x) \mid u \in \langle s \rangle,\ x \leqslant w \text{ and } ux = t\}.$$

Since $\langle s \rangle$ has order two, there are at most two points in $(\mu^T)^{-1}(y_t)$. One of them is the point $y_{(1,t)}$, which is the image of the affine point $(1,t)$ in $\mathbb{P}^1$. The other possibility is the $y_{(s,st)}$; but this point is only a point of $Z_{s,w}^T$ if $st \leqslant w$. Thus we conclude that $(\mu^T)^{-1}(y_t)$ is in bijection with the elements z such that t and st are less than or equal to w.

The map μ is an isomorphism over the big cell Y_{sw}. Thus it remains to study the fibers over the cells Y_u with $u \leqslant w$.

It now suffices to to show that the fiber over u is a $\mathbb{P}^1$ if and only if both $u, su \leqslant w$, and is a point otherwise. Thus we must show that the Euler characteristic of $\mu^{-1}(y_u)$ is equal to 2 if and only if both $u, su \leqslant w$, and is equal to one otherwise. By Lemma 5, we must show that the cardinality of $(\mu^T)^{-1}(y_u)$ is equal to 2 if and only if both $u, su \leqslant w$, and is equal to 1 otherwise. But, as explained above, $(\mu^T)^{-1}(y_u)$ is the one element set $y_{(1,u)}$ unless both $u, su \leqslant w$, in which case $(\mu^T)^{-1}(y_u)$ is the 2 element set $\{y_{(1,u)}, y_{(s,su)}\}$. □

Now let $H(w,s) = \{u \in W | u, su \leqslant w\}$ and $c_{w,s}$ be as in Theorem 3. We may write (19) in our suggestive notation as (for $sw > w$)

$$\mathfrak{D}_i \boldsymbol{X}_w = \boldsymbol{X}_{sw} + q^{-1} \sum_{u \in H(w,s)} c_{w,s}(u) \boldsymbol{X}_u. \tag{25}$$

As in Proposition 9 the set $H(w,s)$ has the property that if $u \in H(w,s)$ and $t \leqslant u$ then $t \in H(w,s)$.

Proposition 15. *Let $s = s_i$ be a left ascent of w. Then*

$$\{y \in X_{sw} \mid \mu^{-1}(y) \text{ is nontrivial}\} = \sum_{u \in H(w,s)} c_{w,s}(u) X_u. \tag{26}$$

The notation must be explained. We are thinking of the varieties in (26) as multisets, and the identity is in the sense of inclusion-exclusion. In other words, if $y \in X$ and we sum the coefficients $c_{w,s}(u)$ over u such that $y \in X_u$, we will get 1 if $\mu^{-1}(y)$ is nontrivial, and 0 otherwise.

Since the fiber of the map $Z_{s,w} \longrightarrow X_{sw}$ over the points in Y_u with $u \in H(w,s)$ is $\mathbb{P}^1$, these points will contribute $1+q^{-1}$ to the Poincaré polynomial of the fiber, while the other points in X_{sw} will contribute 1. If now $P_u(q^{-1})$ is the Poincaré polynomial of the fiber over $y \in X_{sw}$, we may rewrite this identity (26) as

$$\sum_{u \leqslant sw} P_u(q^{-1}) Y_u = X_{sw} + q^{-1} \sum_{u \in H(w,s)} c_{w,s}(u) X_u,$$

and now we recognize this as analogous to (25).

Proof. Let $y \in X$. Let $t \in W$ such that $y \in Y_t$. Then

$$\sum_{\substack{u \in H(w,s) \\ y \in X_u}} c_{w,s}(u) = \sum_{\substack{u \in H(w,s) \\ t \leqslant u}} c_{w,s}(u).$$

It follows from Möbius inversion (Verma [31] or Deodhar [13]) that given $y \in X$ that this is 1 if $t \in H(w,s)$ and 0 otherwise. Thus the statement follows from Proposition 14. □

8 Nonsymmetric Macdonald polynomials

In this section, we explain how the Iwahori Whittaker functions presented earlier are limits of non-symmetric Macdonald polynomials. To do so, we use the results and notation of Ion [19], with one exception – for ease of comparison, we write $\boldsymbol{z}^\lambda$ for an element of the lattice $\mathcal{O}(\hat{T})$ rather than e^λ. Recall that in an earlier section, we showed that for any dominant weight λ, the Iwahori Whittaker function is given by

$$\mathcal{W}_{\lambda,w}(\boldsymbol{z}) = \mathfrak{T}_w(\boldsymbol{z}^\lambda)$$

where $\mathfrak{T}_{s_i} = \mathfrak{T}_i$ is given by (14) with $v = q^{-1}$. Conjugating by $\boldsymbol{z}^\rho$, viewed as translation by ρ in $\mathcal{O}(\hat{T})$, gives

$$\boldsymbol{z}^\rho \, \mathcal{T}_i \, \boldsymbol{z}^{-\rho} = -s_i + (1 - q^{-1}) \frac{s_i - 1}{1 - \boldsymbol{z}^{\alpha_i}}.$$

In Ion's notation the "basic representation" of Cherednik, a faithful representation of the double affine Hecke algebra on $\mathcal{O}(\hat{T})$, has T_i acting according to Section 2.2 of [19] (see p. 3491) by

$$T_i \cdot \boldsymbol{z}^\lambda = t_i^{1/2} \boldsymbol{z}^{s_i(\lambda)} + (t_i^{1/2} - t_i^{-1/2}) \frac{\boldsymbol{z}^\lambda - \boldsymbol{z}^{s_i(\lambda)}}{1 - \boldsymbol{z}^{-\alpha_i}}$$

or in our operator notation, letting $\bar{T}_i := t_i^{1/2} T_i$:

$$\bar{T}_i = t_i s_i + (t_i - 1) \frac{1 - s_i}{1 - \boldsymbol{z}^{-\alpha_i}}.$$

Recalling that

$$\overline{T_i}^{-1} = \overline{T_i} - (t_i - 1),$$

then the action of $\bar{T}_i^{-1}$ in the module is given by

$$\bar{T}_i^{-1} = s_i + (1 - t_i) \frac{1 - s_i}{1 - \boldsymbol{z}^{\alpha_i}},$$

so that by setting $t_i = q^{-1}$ for all $i > 0$ and for any $w \in W$,

$$\bar{T}_i^{-1} = -\boldsymbol{z}^\rho \, \mathfrak{T}_i \, \boldsymbol{z}^{-\rho}, \quad \text{and} \quad \bar{T}_{w^{-1}}^{-1} = (-1)^{\ell(w)} \boldsymbol{z}^\rho \, \mathfrak{T}_w \, \boldsymbol{z}^{-\rho}. \tag{27}$$

with $\mathfrak{T}_i$ as in (14).

Ion's main result in [19] (Theorem 4.8) states that the "dual standard basis" $\{\bar{T}_{w_\lambda^{-1}}^{-1} \omega_{\tilde{\lambda}} \cdot 1\}_{\lambda \in P}$ is the family $\{\tilde{E}_\lambda(0, t)\}_{\lambda \in P}$, a normalized version of the non-symmetric Macdonald polynomial. Here w_λ is an *affine* Weyl group element corresponding to λ and $\omega_{\tilde{\lambda}}$ is an operator corresponding to a minuscule weight associated to λ. Thus, given the relation (27), it is natural to suspect that Iwahori Whittaker functions $\mathcal{W}_{\lambda,w}(\boldsymbol{z})$ are related to the polynomials $\tilde{E}_\lambda(0, t)$. However, the action of the Hecke algebra from the Whittaker functional is naturally given in terms of the Bernstein presentation for $\widetilde{\mathcal{H}} = \mathcal{H} \ltimes P$, so we are not able to use Ion's result directly.

Instead, we make use of the relation (see the proof of Theorem 4.8 in [19]):

$$\tilde{E}_\lambda(q,t) = \chi(w_0)^{-1} T_{w_0} w_0(\tilde{E}_\lambda(q^{-1},t^{-1})). \tag{28}$$

Here χ is the map that sends each generator T_i to $t_i^{1/2}$. This identity can be used to relate special values of $E_\lambda(0,t)$ and $E_\lambda(\infty,t^{-1})$, and then take advantage of the properties of $E_\lambda(\infty,t)$.

Then $\tilde{E}_\lambda$ is related to E_λ by Definition 3.10 of [19], which uses the character:

$$\xi(w) = t^{\ell(w)/2} \quad \text{to define} \quad \tilde{E}_\lambda(q,t) := \xi(w_\lambda^\circ)^{-1} E_\lambda(q,t),$$

where w_λ° is the minimal length element for which λ is taken to an antidominant weight. (Note that having identified all $t_i = t$, then $\xi = \chi$.) Thus (28) makes sense as stated for the unnormalized $E_\lambda(q,t)$ as well.

Theorem 7. *The Whittaker function $\mathcal{W}_{\lambda,w}(\boldsymbol{z})$ with $q^{-1} = t$ is equal to*

$$(-t)^{\ell(w)} \boldsymbol{z}^{-\rho} w_0 E_{w_0 w(\lambda+\rho)}(0,t^{-1}).$$

Proof. According to (27), $\mathcal{W}_{\lambda,w}(\boldsymbol{z})$ is equal to $(-1)^{\ell(w)} \boldsymbol{z}^{-\rho}\, \bar{T}_{w^{-1}}^{-1} \boldsymbol{z}^{\lambda+\rho}$, so it suffices to show

$$t^{-\ell(w)} \bar{T}_{w^{-1}}^{-1} \boldsymbol{z}^{\lambda+\rho} = w_0 E_{w_0 w(\lambda+\rho)}(0,t^{-1}). \tag{29}$$

To connect with non-symmetric Macdonald polynomials $E_\lambda(\infty,t)$, note that the right-hand side of (29) may be rewritten

$$w_0 E_{w_0 w(\lambda+\rho)}(0,t^{-1}) = \chi(w_0) w_0 T_{w_0}^{(t^{-1})} w_0 E_{w_0 w(\lambda+\rho)}(\infty,t), \tag{30}$$

using (28) with $t \mapsto t^{-1}$ where $T_{w_0}^{(t^{-1})}$ denotes the operator with t replaced by t^{-1}. An easy verification on simple reflections shows that $w_0 T_{w_0}^{(t^{-1})} w_0 = T_{w_0}^{-1}$. So we may write the above right-hand side of (30) as

$$\chi(w_0) T_{w_0}^{-1} E_{w_0 w(\lambda+\rho)}(\infty,t).$$

Since $\lambda+\rho$ is strictly dominant, Theorem 3.1 of [19], equation (20) applies and the above further simplifies to

$$\chi(w)^{-1} T_{w_0}^{-1} T_{w_0 w} E_{\lambda+\rho}(\infty,t).$$

Further by Proposition 3.6 of [19], to any μ dominant, $E_\mu(\infty,t) = \boldsymbol{z}^\mu$, so the above equals

$$\chi(w)^{-1} T_{w_0}^{-1} T_{w_0 w} \boldsymbol{z}^{\lambda+\rho}.$$

By the length increasing relation on the T_i, we have $T_{w_0w}T_{w^{-1}} = T_{w_0}$. Substituting in $T_{w_0}T_{w^{-1}}^{-1}$ for T_{w_0w} in the expression above, we get

$$\chi(w)^{-1}T_{w_0}^{-1}T_{w_0}T_{w^{-1}}^{-1}\boldsymbol{z}^{\lambda+\rho} = \chi(w)^{-1}T_{w^{-1}}^{-1}\boldsymbol{z}^{\lambda+\rho} = t^{-\ell(w)}\overline{T}_{w^{-1}}^{-1}\boldsymbol{z}^{\lambda+\rho},$$

and (29) follows. □

References

[1] H. H. Andersen. Schubert varieties and Demazure's character formula. *Invent. Math.*, 79(3):611–618, 1985.

[2] S. Arkhipov and R. Bezrukavnikov. Perverse sheaves on affine flags and Langlands dual group, with an appendix by Bezrukavnikov and I. Mirković. *Israel J. Math.* (170), 135–183, 2009.

[3] A. Białynicki-Birula, J. B. Carrell, and W. M. McGovern. *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, of *Encyclopaedia of Mathematical Sciences* **131**. Springer-Verlag, 2002.

[4] R. Bott and H. Samelson. Applications of the theory of Morse to symmetric spaces. *Amer. J. Math.*, 80:964–1029, 1958.

[5] M. Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., 33–85. Birkhäuser, 2005.

[6] B. Brubaker, D. Bump, and S. Friedberg. Unique functionals and representations of Hecke algebras. *Pacific J. Math.*, 260(2): 381–394, 2012.

[7] W. Casselman. The unramified principal series of $\mathfrak{p}$-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[8] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.

[9] I. Cherednik. Double affine Hecke algebras and Macdonald's conjectures. *Ann. of Math. (2)*, 141(1):191–216, 1995.

[10] I. Cherednik and X. Ma. A new take on spherical, Whittaker, and Bessel functions. *Selecta Math.* 19(3):737–817,819–864, 2013.

[11] I. Cherednik and D. Orr. One-dimensional nil-daha and Whittaker functions I. *Transform. Groups* (17), 4:953–987, 2012.

[12] M. Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup.* (4), 7:53–88, 1974.

[13] V. V. Deodhar. Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function. *Invent. Math.*, 39(2):187–198, 1977.

[14] V. V. Deodhar. A combinatorial setting for questions in Kazhdan-Lusztig theory. *Geom. Dedicata* 36(1):95–119, 1990.

[15] S. Gaussent. The fibre of the Bott-Samelson resolution. *Indag. Math. (N.S.)* 12(4):453468, 2001.

[16] T. Haines, R. Kottwitz, and A. Prasad. Iwahori-Hecke algebras. *J. Ramanujan Math. Soc.* (25), 2:113–145, 2010.

[17] B. Ion. Nonsymmetric Macdonald polynomials and Demazure characters. *Duke Math. J.*, 116(2): 299 – 318, 2003.

[18] B. Ion. Nonsymmetric Macdonald polynomials and matrix coefficients for unramified principal series. *Adv. Math.* 201(1):36–62, 2006.

[19] B. Ion. Standard bases for affine parabolic modules and nonsymmetric Macdonald polynomials. *J. Algebra* 319(8):3480–3517, 2008.

[20] N. Iwahori and H. Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups. *Inst. Hautes Études Sci. Publ. Math.*, 25:5–48, 1965.

[21] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.

[22] D. Kazhdan and G. Lusztig. Equivariant K-theory and representations of Hecke algebras. *Invent. Math.*, 80(2):209-231, 1985.

[23] D. Kazhdan and G. Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Invent. Math.*, 87(2):153-215, 1987.

[24] G. Lusztig. Equivariant K-theory and representations of Hecke algebras. *Proc. Amer. Math. Soc.*, 94(2):337–342, 1985.

[25] I. G. Macdonald. Affine Hecke algebras and orthogonal polynomials. volume 157 of *Cambridge Tracts in Mathematics*, Cambridge University Press, 2003.

[26] M. Reeder. On certain Iwahori invariants in the unramified principal series. *Pacific J. Math.*, 153(2):313–342, 1992.

[27] M. Reeder. Isogenies of Hecke algebras and a Langlands correspondence for ramified principal series representations, *Representation Theory*, 6:101–126, 2002.

[28] F. Rodier, Modèle de Whittaker des représentations admissibles des groupes réductifs p-adiques déployés. *C. R. Acad. Sci. Paris Sér. A-B* 275:1045–1048, 1972.

[29] J. D. Rogawski. On modules over the Hecke algebra of a p-adic group. *Invent. Math.*, 79(3):443–465, 1985.

[30] W. Soergel, Wolfgang Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen, *J. Inst. Math. Jussieu* 6(3):501–525, 2007.

[31] D.-N. Verma. Möbius inversion for the Bruhat ordering on a Weyl group. *Ann. Sci. École Norm. Sup. (4)*, 4:393–398, 1971.

[32] B. Webster and G. Williamson, A geometric model for Hochschild homology of Soergel bimodules. *Geom. Topol.* 12(2):12431263, 2008.