Construction of Cuspidal Representations

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Some references are to *Hecke Algebras* and *Philosophy of Cusp Forms*, two chapters of a forthcoming book which have been posted as a single file. I suggest reading *Philosophy of Cusp Forms* and using *Hecke Algebras* as a reference for one minor point.

1 The Number of Cuspidal Representations

There is a duality between the characters and conjugacy classes of the symmetric group S_n – both are parametrized by partitions of n. This duality extends to $\mathrm{GL}(n, \mathbb{F}_q)$. In this duality, we observe the following:

central conjugacy classes	1-dimensional representations
regular semisimple conjugacy classes	induced from distinct cuspidals
regular semisimple conjugacy classes, eigenvalues in $\mathbb{F}_{q^n}$	cuspidal representations
unipotent conjugacy classes	unipotent representations
conjugacy classes whose characteristic polynomial is a power of an irreducible	monatomic representations

We want to divide the conjugacy classes of $\mathrm{GL}(n, \mathbb{F}_q)$ into "types." Roughly, two conjugacy classes have the same type if their rational canonical forms have the same shape. For example, $\mathrm{GL}(2)$ has four distinct shapes of conjugacy classes. They are

$$\left\{\begin{pmatrix} a & \\ & b \end{pmatrix} \Big| a \neq b\right\}, \qquad \left\{\begin{pmatrix} a & \\ & a \end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix} a & 1 \\ & a \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} & 1 \\ -\nu^{1+q} & \nu + \nu^q \end{pmatrix}\right\},$$

where the last consists of the conjugacy classes of matrices whose eigenvalues are ν and ν^q, where $\nu \in \mathbb{F}_{q^2} - \mathbb{F}_q$.

These four types of conjugacy classes correspond to the four types of irreducible representations: the $q+1$-dimensional principal series, induced from a pair of distinct characters of $\mathrm{GL}(1)$; the one-dimensional representations $\chi\circ\det$, where χ is a character of $\mathbb{F}_q^\times$; the q-dimensional representations obtained by tensoring the Steinberg representation with a one-dimensional character; and the $q-1$-dimensional cuspidal representations.

By formalizing this duality we will be able to compute the number of cuspidal representations.

Let $f(X) = X^d + a_{d-1}X^{d-1} + \ldots + a_0$ be a monic irreducible polynomial over $\mathbb{F}_q$ of degree d. Let

$$U(f) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-1} \end{pmatrix}$$

be the rational canonical form. Let

$$U_r(f) = \begin{pmatrix} U(f) & I_d & 0 & \cdots & 0 \\ 0 & U(f) & I_d & & \\ 0 & 0 & U(f) & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & U(f) \end{pmatrix},$$

an array of $r\times r$ blocks, each of size $d\times d$. If $\lambda = (\lambda_1,\cdots,\lambda_t)$ is a partition of r, so that $\lambda_1 \geqslant \cdots \geqslant \lambda_t$ are nonnegative integers with $|\lambda| = \sum_i \lambda_i = r$, let

$$U_\lambda(f) = \begin{pmatrix} U_{\lambda_1}(f) & & \\ & \ddots & \\ & & U_{\lambda_t}(f) \end{pmatrix}.$$

Then every conjugacy class of $\mathrm{GL}(k,\mathbb{F}_q)$ has a representative of the form

$$\begin{pmatrix} U_{\lambda^1}(f_1) & & \\ & \ddots & \\ & & U_{\lambda^m(f_m)} \end{pmatrix},$$

where the f_i are distinct monic irreducible polynomials, and each $\lambda^i = (\lambda^i_1, \lambda^i_2, \cdots)$ is a partition. The conjugacy class is unchanged if the f_i and λ^i are permuted, but otherwise, they are uniquely determined.

Thus the conjugacy class is determined by the following data: a pair of sequences $r_1,\cdots,r_m$ and $d_1,\cdots,d_m$ of integers, and for each $1\leqslant i\leqslant m$ a partition λ^i of r_i and a monic irreducible polynomial $f_i\in\mathbb{F}_q[X]$ of degree d_i, such that no $f_i = f_j$ if $i\neq j$. The data $(\{r_i\},\{d_i\},\{\lambda^i\},\{f_i\})$ and $(\{r_i'\},\{d_i'\},\{(\lambda')^i\},\{f_i'\})$

parametrize the same conjugacy class if and only if they both have the same length m, and there exists a permutation $\sigma \in S_m$ such that $r_i' = r_{\sigma(i)}$, $d_i' = d_{\sigma(i)}$, $(\lambda')^i = \lambda^{\sigma(i)}$ and $f_i' = f_{\sigma(i)}$.

We say two conjugacy classes are of the same *type* if the parametrizing data have the same length there exists a permutation $\sigma \in S_m$ such that $r_i' = r_{\sigma(i)}$, $d_i' = d_{\sigma(i)}$, $(\lambda')^i = \lambda^{\sigma(i)}$. (The f_i and f_i' are allowed to differ.) The types of conjugacy classes depends of course on k but is independent of q.

Lemma 1 *Let $\{n_1, n_2, \cdots\}$ be a sequence of numbers, and for each N_k let X_k be a set of cardinality N_k. Let $X = \bigcup X_k$ (disjoint). Let Σ_k be the following set. An element of Σ_k consists of a 4-tuple $(\{r_i\}, \{d_i\}, \{\lambda^i\}, \{x_i\})$, where $\{r_i\} = \{r_1, \cdots, r_m\}$ and $\{d_i\} = \{d_1, \cdots, d_m\}$ are sequences of positive integers, such that $\sum r_i d_i = k$, together with a sequence $\{\lambda^i\}$ of partitions of r_i and an element $x_i \in X_{d_i}$, such that no x_i are equal. Define an equivalence relation $\sim$ on Σ_k in which two elements are considered equivalent if they can be obtained by permuting the data, that is, if $\sigma \in S_m$ then*

$$(\{r_i\}, \{d_i\}, \{\lambda^i\}, \{x_i\}) \sim (\{r_{\sigma(i)}\}, \{d_{\sigma(i)}\}, \{\lambda^{\sigma(i)}\}, \{x_{\sigma(i)}\}).$$

Let M_k be the number of equivalence classes. Then the sequence of numbers N_k is determined by the sequence of numbers M_k.

Proof By induction on k, we may assume that the cardinalities $N_1, \cdots, N_{k-1}$ are determined by the M_k. Let M_k' be the cardinality of the set of equivalence classes of $(\{r_i\}, \{d_i\}, \{\lambda^i\}, \{x_i\}) \in \Sigma_k$ in which no $x_i \in X_k$. Clearly M_k' depends only on the cardinalities $N_1, \cdots, N_{k-1}$ of the sets $X_1, \cdots, X_{k-1}$ from which the x_i are to be drawn, so (by induction) it is determined by the M_i. Now we claim that $N_k = M_k - M_k'$. Indeed, if given $(\{r_i\}, \{d_i\}, \{\lambda^i\}, \{x_i\}) \in \Sigma_k$ of length m, if any $x_i \in X_k$, then since $\sum_{i=1}^m r_i d_i = k$, we must have $m = 1$, $r_1 = 1$, $d_1 = k$, and the number of such elements is exactly N_k. □

Theorem 1 *The number of cuspidal representations of $\mathrm{GL}(k, \mathbb{F}_q)$ equals the number of irreducible monic polynomials of degree k over $\mathbb{F}_q$.*

Proof We can apply the Lemma with X_k either the set of cuspidal representations of S_k or with the set of monic irreducible polynomials of degree k over $\mathbb{F}_q$. We will show that in the first case, M_k is the number of irreducible representations of $\mathrm{GL}(k, \mathbb{F}_q)$, while in the second, M_k is the number of conjugacy classes. Since these are equal, the result follows.

If X_k is the set of cuspidal representations of $\mathrm{GL}(k, \mathbb{F}_q)$, from each element $(\{r_i\}, \{d_i\}, \{\lambda^i\}, \{x_i\}) \in \Sigma_k$ we can build an irreducible representation of $\mathrm{GL}(k, \mathbb{F}_q)$ as follows. First, since x_i is a cupidal representation of $\mathrm{GL}(d_i, \mathbb{F}_q)$ we can build the x_i-monatomic representations of $\mathrm{GL}(d_i r_i, \mathbb{F}_q)$ by decomposing $x_i^{\circ r_i}$. By Corollary 49.1, the irreducible constituents of $x_i^{\circ r_i}$ are parametrized by partitions of r_i, so x_i and λ^i parametrize an x_i-monotomic representation π_i of $\mathrm{GL}(r_i d_i, \mathbb{F}_q)$. Let $\pi = \pi_1 \circ \ldots \circ \pi_m$. By Theorem 49.4, every irreducible representation of $\mathrm{GL}(k,\mathbb{F}_q)$ is constructed uniquely (up to permutation of the π_i) in this way.

On the other, take X_k to be the set of monic irreducible polynomials of degree k over $\mathbb{F}_q$. We have explained above how the conjugacy classes of $\mathrm{GL}(k,\mathbb{F}_q)$ are parametrized by such data. $\square$

The degrees of the cuspidal representations are computed in Exercise 49.6 of *The Philosophy of Cusp Forms*, or Section 7 of *Notes on Representations of* $\mathrm{GL}(r)$ *over a finite field*. It is shown that the degree of a cuspidal representation of $\mathrm{GL}(k,\mathbb{F}_q)$ is

$$(q^{k-1}-1)(q^{k-2}-1)\cdots(q-1).$$

2 Frobenius Reciprocity for parabolic induction

If G is a finite group and H a subgroup then Frobenius reciprocity has two equivalent forms. Let (π,V) be a representation of G and (σ,W) is a representation of H, and if $\mathrm{Ind}(W)$ is the induced module of G, then

$$\mathrm{Hom}_G(V,\mathrm{Ind}(W))\cong\mathrm{Hom}_H(G,W).$$

Since the inner product of the characters of two representation is the dimension of the space of intertwining operators for the two representations, this can also be written

$$\left\langle\chi_\pi,\chi_\sigma^G\right\rangle_G=\langle\chi_\pi,\chi_\sigma\rangle_H\,,$$

where χ_π and χ_σ are the characters of π and σ.

Both versions of Frobenius reciprocity have adapations to parabolic induction. Let $P=MU$ be a parabolic subgroup of $G=\mathrm{GL}(k,\mathbb{F}_q)$, where M is the Levi factor and U is the unipotent radical.

Proposition 1 *Let (π,V) be a representation of G and let (σ,W) be a representation of M. Let V^U be the space of U-invariants in V. Since M normalizes U, V^U is an M-module. On the other, hand, we may parabolically induce W to a representation* $\mathrm{Ind}(\sigma)$ *of G. We have*

$$\mathrm{Hom}_G(V,\mathrm{Ind}(W))\cong\mathrm{Hom}_M(V^U,W).$$

Proof By ordinary Frobenius reciprocity

$$\mathrm{Hom}_G(V,\mathrm{Ind}(W))\cong\mathrm{Hom}_P(V,W).$$

We need to show that

$$\mathrm{Hom}_P(V,W)\cong\mathrm{Hom}_M(V^U,W).$$

Let V_0 be the span of elements of the form $w-\pi(u)w$ with $u\in U$. It is not hard to show that $V=V^U\oplus V_0$, as M-modules. Since by the nature of parabolic induction P acts trivially on W, any U-equivariant map $V \longrightarrow W$ annihilates V_0. Thus

$$\mathrm{Hom}_P(V, W) \cong \mathrm{Hom}_M(V/V_0, W) \cong \mathrm{Hom}_M(V^U, W).$$

$\square$

Let χ be a character of G, and let σ be a character of M. Let $\mathrm{Ind}(\sigma)$ be the character of the representation of G parabolically induced from σ, and let χ_U be the function on M defined by

$$\chi_U(m) = \frac{1}{|U|} \sum_{u \in U} \chi(mu).$$

Proposition 2 *(i) We have*

$$\langle \chi, \mathrm{Ind}(\sigma) \rangle_G = \langle \chi_U, \sigma \rangle_M . \tag{1}$$

(ii) χ_U is a class function on M.
(iii) χ_U is the character of the representation of M on V^U.

Proof For (i), by ordinary Frobenius reciprocity,

$$\langle \chi, \mathrm{Ind}(\sigma) \rangle_G = \langle \chi, \sigma \rangle_P = \frac{1}{|MU|} \sum_{m \in M} \sum_{u \in U} \chi(mu)\sigma(mu).$$

Noting that in parabolic induction σ is extended to P in such a way that $\sigma(mu) = \sigma(m)$, the result follows.

For (ii), we first check that χ_U is a class function. If $m, n \in M$ we have $\chi_U(nmn^{-1}) = |U|^{-1} \sum \chi_U(nmn^{-1}u)$. Since M normalizes U, we may replace u by nun^{-1} in the sum and use the fact that χ_U is a class function. Now the inner product of χ_U with every irreducible character of M is an integer by (i), so χ_U is a character.

For (iii), both χ_U as we've defined it and the character of the representation of M on V^U satisfy the same identity (1); for the character of V^U this statement uses Proposition 1. $\square$

Exercise 1 Let $1 \leqslant k, r < n$. Let σ_1, σ_2 be monatomic characters of $\mathrm{GL}(r, \mathbb{F}_q)$ and $\mathrm{GL}(n-r, \mathbb{F}_q)$ with respect to a pair of distinct cuspidal representations. Let σ denote the character of the representation $\sigma_1 \circ \sigma_2$ of $\mathrm{GL}(n, \mathbb{F}_q)$, which is irreducible by Theorem **??**. Let $m_1 \in \mathrm{GL}(k, \mathbb{F}_q)$ and $m_2 \in \mathrm{GL}(n-k, \mathbb{F}_q)$. Let U be the unipotent radical of the standard parabolic subgroup P of $\mathrm{GL}(n, \mathbb{F}_q)$ with Levi factor $M = \mathrm{GL}(k, \mathbb{F}_q) \times \mathrm{GL}(n-k, \mathbb{F}_q)$. if $k = r, k \neq n-r$,

$$\frac{1}{|U|} \sum_{u \in U} \sigma \left(u \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \right) = \begin{cases} \sigma_1(m_1)\sigma_2(m_2) & \text{if } k = r, k \neq n-r, \\ \sigma_1(m_2)\sigma_2(m_1) & \text{if } k = n-r, k \neq r, \\ \sigma_1(m_1)\sigma_2(m_2) + \sigma_1(m_2)\sigma_2(m_1) & \text{if } k = r = n-r. \end{cases}$$

Hint: Both sides are class functions, so it is sufficient to compare the inner products with $\rho_1 \otimes \rho_2$ where ρ_1 and ρ_2 are irreducible representations of $\mathrm{GL}(k, \mathbb{F}_q)$ and $\mathrm{GL}(n-k, \mathbb{F}_q)$ respectively. Using Proposition 2 this amounts to comparing $\sigma_1 \circ \sigma_2$ and $\rho_1 \circ \rho_2$.

3 Decomposition of Green's generalized character

Let $\theta : \overline{\mathbb{F}}_q^\times \longrightarrow \mathbb{C}^\times$ be a character. As Green proved, there exists a generalized character χ_n of $\mathrm{GL}(n, \mathbb{F}_q)$ defined by

$$\chi_n(g) = \sum_{i=1}^{n} \theta(\alpha_i),$$

where α_i are the eigenvalues of g.

In this section we will assume that θ is injective, and determine the decomposition of χ_n. We will find a unique cuspidal representation in its decomposition.

Proposition 3 *Suppose that χ is an irreducible representation of* $\mathrm{GL}(n, \mathbb{F}_q)$*. Let g be a regular semisimple element whose eigenvalues are in $\mathbb{F}_{q^n}$. If $\chi(g) \neq 0$, then χ is monatomic.*

Proof If χ is not monatomic, then by Theorem 49.6 of *The Philosophy of Cusp Forms*, $\chi = \chi_1 \circ \ldots \circ \chi_r$, where the χ_i are monatomic characters of smaller $\mathrm{GL}(r_i, \mathbb{F}_q)$. If P is the parabolic subgroup with Levi factor $\prod \mathrm{GL}(r_i, \mathbb{F}_q)$, no conjugate of g is in P, so the induced character χ vanishes at g. $\square$

If $n = r + t$, let $P_{r,t}$ be the standard parabolic subgroup with Levi factor $M_{r,t} = \mathrm{GL}(r, \mathbb{F}_q) \times \mathrm{GL}(t, \mathbb{F}_q)$, and let $U_{r,t}$ be its unipotent radical. If π and π' are irreducible representations of $\mathrm{GL}(r, \mathbb{F}_q)$ and $\mathrm{GL}(t, \mathbb{F}_q)$, let $\pi \circ \pi'$ be the representation of $\mathrm{GL}(n, \mathbb{F}_q)$ parabolically induced from the representation $\pi \otimes \pi'$ of $M_{r,t}$. We will also use the same notation $\circ$ for parabolic induction of characters. We will denote by 1_r the trivial character of $\mathrm{GL}(r, \mathbb{F}_q)$.

We let $T_{(n)}$ be the anisotropic maximal torus, which is the image of the embedding $\mathbb{F}_{q^n}^\times \longrightarrow \mathrm{GL}(n, \mathbb{C})$ induced by an embedding of $\mathbb{F}_{q^n}$ into $\mathrm{Mat}_n(\mathbb{C})$ as an $\mathbb{F}_q$-algebra. More generally, if λ is a partition of n, let

$$T_\lambda = \prod T_{\lambda_i}.$$

Let T_λ^{reg} denote the subset of elements of T_λ which are regular, that is, which have distinct eigenvalues.

Proposition 4 *The inner product $\langle \chi_k, \chi_k \rangle = k$.*

Proof We will first prove that this is true for q sufficiently large, then show that it is true for all q. The idea of the proof is to show that as a function of q, the inner product is $k + O(q^{-1})$. Since it is an integer, it must equal k when q is sufficiently large.

The number of elements of $G = \mathrm{GL}(k, \mathbb{F}_q)$ is $q^{k^2} + O(q^{k^2-1})$. This is clear since G is the complement of the determinant locus in $\mathrm{Mat}_k(\mathbb{F}_q) \cong \mathbb{F}_q^{k^2}$. The set G_{reg} of regular semisimple elements also has order $q^{k^2}+O(q^{k^2-1})$ since it is the complement of the discriminant locus. Since $|\chi_k(g)| \leqslant k$ for all g,

$$\langle \chi_k, \chi_k \rangle = \frac{1}{|G|} \sum_{g \in G_{\mathrm{reg}}} |\chi_k(g)|^2 + O(q^{-1}).$$

This in turn equals

$$\frac{1}{|G|} \sum_{\lambda a \text{ partition of } k} \sum_{g \in T_\lambda^{\mathrm{reg}}} |\chi_k(g)|^2 + O(q^{-1}) =$$

$$\frac{1}{|G|} \sum_{\lambda} [G : N_G(T_\lambda)] \sum_{g \in T_\lambda} |\chi_k(g)|^2 + O(q^{-1}),$$

since the complement of the T_λ^{reg} in T_λ is of codimension one. We note that the restriction of χ_k to T_λ is the sum of k distinct characters, so

$$\sum_{g \in T_\lambda} |\chi_k(g)|^2 = k|T_\lambda|.$$

Thus the inner product is

$$k \times \frac{1}{|G|} \sum_{\lambda} [G : N_G(T_\lambda)]|T_\lambda| + O(q^{-1}).$$

We have

$$\frac{1}{|G|} \sum_{\lambda} [G : N_G(T_\lambda)]|T_\lambda| = \frac{1}{|G|} \sum_{\lambda} [G : N_G(T_\lambda)]|T_\lambda^{\mathrm{reg}}| + O(q^{-1})$$
$$= \frac{1}{|G|}|G_{\mathrm{reg}}| + O(q^{-1}) = 1 + O(q^{-1}).$$

The result is now proved for q sufficiently large.

To prove the result for all q, we will show that the inner product $\langle \chi_k, \chi_k \rangle$ is a polynomial in q. This will follow if we can show that if S is the subset of G consisting of the union of conjugacy classes of a single type, then $[G : C_G(g)]$ is constant for $g \in S$ and

$$\sum_{g \in S} |\chi_k(g)|^2 \tag{2}$$

is a polynomial in q. We note that for each type, the index of the centralizer (**??**) is the same for all such matrices, and that this index is polynomial in q. Thus it is sufficient to show that the sum over the representatives (**??**) is a polynomial in q. Moreover, the value of χ_k is unchanged if every instance of a $U_r(f)$ is replaced with r blocks of $U(f)$, so we may restrict ourselves to semisimple conjugacy classes in confirming this. Thus if $k = \sum d_i r_i$, we consider the sum (2), where the sum is over all matrices

$$\begin{pmatrix} U_{(r_1)}(f_1) & & \\ & \ddots & \\ & & U_{(r_m)}(f_m) \end{pmatrix},$$

where f_i are *distinct* irreducible polynomials, each of size d_i, and $U_{(r)}(f)$ is the sum of r blocks of $U(f)$. It is useful to conjugate these matrices so that they are all elements of the same torus T_λ for some λ. The set S is then a subset of T_λ characterized by exclusion from certain (non-maximal) subtori.

Let us look at an example. Suppose that $\lambda = (2,2,2)$ and $k = 6$. Then S consists of elements of T_λ, which may be regarded as $(\mathbb{F}_{q^2})^\times$ of the form (α, β, γ), where α, β and γ are distinct elements of $\mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times$. Now if we sum (2) over *all* of T_λ we get a polynomial in q, namely $6(q^2-1)^3$. On the other hand, we must subtract from this three contributions when one of α, β and γ is in $\mathbb{F}_q^\times$. These are subtori of the form $T_{(2,2,1)}$. We must also subtract three contributions from subgroups of the form $T_{(2,2)}$ in which two of α, β and γ are equal. Then we must add back contributions with have been subtracted twice, etc.

In general, the set S will consist of the set T_λ minus subtori $T_1, \cdots, T_N$. If I is a subset of $\{1, \cdots, N\}$ let $T_I = \bigcap_{i \in I} T_i$. We now use the inclusion-exclusion principle in the form

$$\sum_{g \in S} |\chi_k(g)|^2 = \sum_{g \subset T_\lambda} |\chi_k(g)|^2 + \sum_{\varnothing \neq I \subseteq \{1, \cdots, N\}} (-1)^{|I|} \sum_{g \in T_I} |\chi_k(g)|^2.$$

Each of the sums on the right is easily seen to be a polynomial in q, and so is (2). $\square$

Theorem 2 *For each k there exists a cuspidal $\sigma_k = \sigma_{k,\theta}$ of $\mathrm{GL}(k, \mathbb{F}_q)$ such that if g is a regular semisimple element of $\mathrm{GL}(k, \mathbb{F}_q)$ whose eigenvalues are the Galois conjugates of $\nu \in \mathbb{F}_{q^k}^\times$, then*

$$\sigma_{k,\theta}(g) = (-1)^{k-1} \sum_{i=0}^{k-1} \theta(\nu^{q^i}).$$

If 1_k denotes the trivial character of $\mathrm{GL}(k, \mathbb{F}_q)$, then

$$\chi_n = \sum_{k=1}^{n} (-1)^{k-1} \sigma_k \circ 1_{n-k}.$$

Note that $\sigma_k \circ 1_{n-k}$ is an irreducible character of $\mathrm{GL}(n, \mathbb{F}_q)$ by Theorem 49.6 of *The Philosophy of Cusp Forms*. So this gives the expression of χ_n in terms of irreducibles.

Proof Proof is by induction on k. We assume that the existence of σ_k and the decomposition of σ_k are known for $k < n$.

We will show first that

$$\langle \chi_n, \sigma_k \circ 1_{n-k} \rangle = (-1)^{k-1}. \tag{3}$$

Let $P = MU$ be the standard parabolic subgroup with Levi factor $M = \mathrm{GL}(k, \mathbb{F}_q) \times \mathrm{GL}(n-k, \mathbb{F}_q)$ and unipotent radical U. If $m \in M$ and $u \in U$, then as matrices in $\mathrm{GL}(n,\mathbb{F}_q)$, m and mu have the same characteristic polynomials, so $\chi_n(mu)=\chi_n(m)$. Thus in the notation of the previous section, with $\chi=\chi_n$, we have $\chi_U=\chi$ restricted to M. Therefore

$$\langle\chi_n,\sigma_k\circ 1_{n-k}\rangle_G=\langle\chi_n,\sigma_k\otimes 1_{n-k}\rangle_M\,.$$

Let

$$m=\begin{pmatrix} m_1 & \\ & m_2\end{pmatrix}\in M,\qquad m_1\in\mathrm{GL}(k,\mathbb{F}_q),m_2\in\mathrm{GL}(n-k,\mathbb{F}_q).$$

Clearly $\chi_n(m)=\chi_k(m_1)+\chi_{n-k}(m_2)$. Now using induction hypothesis, χ_{n-k} does not the trivial character of $\mathrm{GL}(n-k,\mathbb{F}_q)$ so orthogonal to 1_{n-k} on $\mathrm{GL}(n-k,\mathbb{F}_q)$; so we can ignore $\chi_{n-k}(m_2)$. Thus

$$\langle\chi_n,\sigma_k\circ 1_{n-k}\rangle_G=\langle\chi_k,\sigma_k\rangle_{\mathrm{GL}(k,\mathbb{F}_q)}\,.$$

By the induction hypothesis, χ_k contains σ_k with multiplicity $(-1)^{k-1}$, and so (3) is proved.

Now $\sigma_k\circ 1_{n-1}$ is an irreducible representation of $\mathrm{GL}(n,\mathbb{F}_q)$ by Theorem 49.6 of *The Philosophy of Cusp Forms*. So we have exhibited $n-1$ irreducible characters, each of which occurs in χ_n with multiplicity ±1. Since $\langle\chi_n,\chi_n\rangle=n$, there must be one remaining irreducible character σ_n such that

$$\chi_n=\sum_{k=1}^{n-1}(-1)^{k-1}\sigma_k\circ 1_{n-k}\pm\sigma_n. \tag{4}$$

We note that σ_n must be cuspidal since the other irreducible constituents are not, and χ_n does not vanish on the regular semisimple conjugacy class, so by Proposition 3 it is monatomic and irreducible, hence cuspidal.

To evaluate the sign in (4), we compare the values at the identity to get the relation

$$n=\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}_{(q)}\prod_{j=1}^{k-1}(q^j-1)\pm\prod_{j=1}^{n-1}(q^j-1),$$

where

$$\binom{n}{k}_{(q)}=\frac{\prod_{j=1}^{n}(q^j-1)}{\left(\prod_{j=1}^{k}(q^j-1)\right)\left(\prod_{j=1}^{n-k}(q^j-1)\right)}$$

is the Gaussian binomial coefficient. Plugging $q=0$ to this identity shows that the missing sign must be $(-1)^{n-1}$. $\square$

4 An Example

Here is the character table of $\mathrm{GL}(2,\mathbb{F}_3)$, a group of order 48 which is a double cover of $\mathrm{PGL}(2,\mathbb{F}_q)\cong S_4$.

		$1a$	$2a$	$6a$	$3a$	$4a$	$8a$	$8b$	$2b$
	χ_1	1	1	1	1	1	1	1	1
	χ_2	1	1	1	1	1	-1	-1	-1
cuspidal	χ_3	2	2	-1	-1	2	0	0	0
cuspidal	χ_4	2	-2	1	-1	0	$-\sqrt{2}i$	$\sqrt{2}i$	0
cuspidal	χ_5	2	-2	1	-1	0	$\sqrt{2}i$	$-\sqrt{2}i$	0
	χ_6	3	3	0	0	-1	1	1	-1
Steinberg	χ_7	3	3	0	0	-1	-1	-1	1
	χ_8	4	-4	-1	1	0	0	0	0
Green($\lvert\theta\rvert = 8$)	$\chi_8 - \chi_4$	2	-2	-2	2	0	$\sqrt{2}i$	$-\sqrt{2}i$	0
Green($\lvert\theta\rvert = 4$)	$\chi_1 + \chi_7 - \chi_3$	2	2	2	2	-2	0	0	2

We constructed the two cuspidal representations χ_4 and χ_5 in the last section. Indeed, taking θ to be a character of order 8 of $\mathbb{F}_9^\times$ (or of $\bar{\mathbb{F}}_3^\times$) produces the Green generalized character $\chi_8 - \chi_4$ or $\chi_8 - \chi_5$.

The cuspidal representation χ_3 also occurs as a constituent of Green's irreducible character, taking θ to be a character of order 4 of $\mathbb{F}_9^\times$ (or of $\bar{\mathbb{F}}_3^\times$). However this θ restricted to $\mathbb{F}_3^\times$ is trivial, so the resulting Green character has three irreducible constituents. We see that the method of proof in the preceding section definitely breaks down.

Also note that the cuspidal representation χ_3 is only distinguished from $\chi_1 + \chi_2$ by its values on the non-semisimple conjugacy classes $6a$ and $3a$.

5 Cuspidal characters

Without attempting a completely general result, we will now give a variation of Theorem 2 which is sufficient to construct all cuspidal representations of $\mathrm{GL}(k, \mathbb{F}_q)$ when k is prime.

If $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a partition of k, consider the conjugacy class in S_k which contains as a representative disjoint cycles of lengths λ_1, $\lambda_2, \cdots$. The centralizer of a representative σ_λ has order z_λ, where if λ contains r_1 parts equal to 1, r_2 parts equal to 2, etc, then

$$z_\lambda = \prod_j (j^{r_j}\, r_j!).$$

To see this, note that if $c_1, \cdots, c_r$ are the $r = r_j$ disjoint j-cycles in the decomposition of σ_λ into disjoint cycles, then $C_{S_k}(\sigma_\lambda)$ contains a subgroup of order j^r generated by $c_1, \cdots, c_r$, and it also contains a subgroup of order $r!$ isomorphic to S_r which in its action by conjugation permutes the c_i.

Proposition 5 *We have*

$$z_\lambda = [N_{\mathrm{GL}(k,\mathbb{F}_q)}(T_\lambda) : T_\lambda].$$

Proof The normalizer is generated by the Frobenius elements in each T_λ, generating a group of order $\prod j^{r_j}$, together with the Weyl group of the centralizer in $\mathrm{GL}(k, \mathbb{F}_q)$ of the center of $\mathrm{GL}(\lambda_1, \mathbb{F}_q) \times \mathrm{GL}(\lambda_2, \mathbb{F}_q) \times \cdots$, a group isomorphic to $S_{r_1} \times S_{r_2} \times \cdots$. □

Proposition 6 *Let $\theta : \bar{\mathbb{F}}_q^\times \longrightarrow \mathbb{C}^\times$ be a character. Assume that the restriction of θ to $\mathbb{F}_q^\times$ is trivial, but that for any $0 < d \leqslant k$, the restriction of θ to $\mathbb{F}_{q^d}^\times$ does not factor through the norm map $\mathbb{F}_{q^d}^\times \longrightarrow \mathbb{F}_{q^r}^\times$ for any proper divisor r of d. Then*

$$\langle \chi_k, \chi_k \rangle = k + 1.$$

Proof The proof is similar to Proposition 4. It is sufficient to show this for sufficiently large q. As in that Proposition, the sum is

$$\frac{1}{|G|} \sum_{\lambda a \text{ partition of } k} [G : N_G(T_\lambda)] \sum_{g \in T_\lambda} |\chi_k(g)|^2 + O(q^{-1}).$$

We note that $[N_G(T_\lambda) : T_\lambda] = z_\lambda$. With our assumptions if the partition λ contains r parts of size 1, the restriction of χ_k to T_λ consists of r copies of the trivial character, and $k - r$ copies of other characters, all distinct. Indeed, remembering that χ_k consists of the sum of $\theta(\alpha$

The inner product of χ_k with itself on T_λ is thus $k - r + r^2$. The sum is thus

$$\sum_\lambda \frac{1}{z_\lambda}(k + r^2 - r) + O(q^{-1}).$$

We can interpret this as a sum over the symmetric group. If $\sigma \in S_k$, let $r(\sigma)$ be the number of fixed points of σ. In the conjugacy class of shape λ, there are $k!/z_\lambda$ elements, and so

$$\sum_\lambda \frac{1}{z_\lambda}(k + r^2 - r) = \frac{1}{k!} \sum_{\sigma \in S_k} (k + r(\sigma)^2 - r(\sigma)).$$

Now $r(\sigma) = 1_k + \boldsymbol{s}$, where $\boldsymbol{s}$ is the character of an irreducible representation of degree We note that $r(\sigma)^2 - r(\sigma)$ is the value of the character $\boldsymbol{s}^2 + \boldsymbol{s}$, so the sum is

$$\left\langle k1_k + \boldsymbol{s}^2 + \boldsymbol{s}, 1_k \right\rangle = k \left\langle 1_k, 1_k \right\rangle + \left\langle \boldsymbol{s}^2, 1_k \right\rangle + \left\langle \boldsymbol{s}, 1_k \right\rangle$$

where the inner product is now over the symmetric group. Clearly $\langle 1_k, 1_k \rangle = 1$ and $\langle \boldsymbol{s}, 1_k \rangle = 0$. Since the character $\boldsymbol{s}$ is real and 1_k is the constant function equal to 1,

$$\left\langle \boldsymbol{s}^2, \boldsymbol{h}_k \right\rangle = \langle \boldsymbol{s}, \boldsymbol{s} \rangle = 1,$$

and the result follows. □

Theorem 3 *Suppose that n is a prime, and let $\theta : \mathbb{F}_{q^n}^\times \longrightarrow \mathbb{C}^\times$ be a character which does not factor through the norm map $\mathbb{F}_{q^n}^\times \longrightarrow \mathbb{F}_{q^r}^\times$ for any proper divisor r of n. Then there exists a cuspidal character $\sigma_{n,\theta}$ of $\mathrm{GL}(n, \mathbb{F}_q)$ such that if g is a regular semisimple elment with eigenvalues $\nu, \nu^q, \cdots \in \mathbb{F}_{q^n}$ then*

$$\sigma_{n,\theta}(g) = (-1)^{n-1} \sum_{i=0}^{n-1} \theta(\nu^{q^i}). \tag{5}$$

This gives a complete list of the cuspidal characters of $\mathbb{F}_{q^n}$.

The assumption that n is prime is unnecessary.

Proof We can extend θ to a character of $\bar{\mathbb{F}}_q$ without enlarging the kernel. Thus the kernel of θ is contained in $\mathbb{F}_{q^n}^\times$ and does not contain the kernel of any norm map $\mathbb{F}_{q^n}^\times \longrightarrow \mathbb{F}_{q^r}^\times$ for any proper divisor r of n. There are now two cases.

If χ is nontrivial on $\mathbb{F}_q^\times$, then we may proceed as in Theorem 2. We are not in the case of that Theorem, since we have not assumed that the kernel of θ is trivial, and we do not guarantee that the sequence of cuspidals σ_k that we construct can be extended to all k. However if $d \leqslant k$, our assumptions guarantee that the restriction of θ to $\mathbb{F}_{q^d}^\times$ does not factor through the norm map to $\mathbb{F}_{q^r}$ for any proper divisor r of d, since after all the kernel of θ is contained in $\mathbb{F}_{q^n}$, whose intersection with $\mathbb{F}_{q^d}$ is just $\mathbb{F}_q$ since n is prime and $d < n$. We get $\langle \chi_k, \chi_k \rangle = k$ for $k \leqslant n$, and proceeding as in Theorem we get a sequence of cuspidal representations σ_k of $\mathrm{GL}(k, \mathbb{F}_q)$ with $k \leqslant n$ such that

$$\chi_k = \sum_{r=1}^{k} (-1)^{r-1} \sigma_r \circ 1_{k-r}.$$

If θ is trivial on $\mathbb{F}_q^\times$, it is still true that the restriction of θ to $\mathbb{F}_{q^d}$ does not factor through the norm map to $\mathbb{F}_{q^r}$ for any proper divisor r of $d \leqslant k$ whenever $k \leqslant n$. So $\langle \chi_k, \chi_k \rangle = k+1$ by Theorem 6. Now we can proceed as before except that $\sigma_1 = 1_1$ so $\sigma_1 \circ 1_{k-1}$ is not irreducible – it is the sum of two irreducible representations $\boldsymbol{s}_{(k-1,1)}(q)$ and $\boldsymbol{s}_{(k)}(q)$ of $\mathrm{GL}(k, \mathbb{F}_q)$, in the notation of *Hecke operators*. Of course $\boldsymbol{s}_{(k)}(q)$ is the same as 1_k in the notation we've been using. The rest of the argument goes through as in Theorem 2. In particular the inner product formula $\langle \chi_k, \chi_k \rangle = k+1$ together with fact that $1_1 \circ 1_{k-1}$ accounts for two representations in the decomposition of χ_k guarantees that σ_k, defined to be $\chi_k - \sum_{r<k} (-1)^r \sigma_r \circ 1_{k-r}$ is irreducible

The cuspidal characters we've constructed are linearly independent by (5). They are equal in number to the total number of cuspidal representations, and so we have constructed all of them. $\square$