More applications of multiple Dirichlet series

Gautam Chinta

Bretton Woods, NH
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Nonvanishing n^{th} order twists of a GL_2 form

Theorem (Fearnley-Kisilevsky, David-F-K) Fix n prime. Let E be an elliptic curve defined over $\mathbb{Q}$. Suppose $L^{alg}\left(\frac{1}{2}, E\right) \not\equiv 0 \pmod{n}$. Then $L^{alg}\left(\frac{1}{2}, E \otimes \chi\right)$ is nonzero for infinitely many Dirichlet characters of order n.

Proof Arithmetic of modular symbols and Cebotarev density theorem.

Arithmetic interpretation Let F be a cyclic extension of $\mathbb{Q}$ of degree n. Then a Galois character χ of $Gal(F/\mathbb{Q})$ corresponds to a Dirichlet character of order n. The Birch/Swinnerton-Dyer conjecture predicts that

$$L\left(\frac{1}{2}, E \otimes \chi\right) \text{ is nonzero}$$

iff

$$\text{rank of } E(F)^{\chi} \text{ is zero.}$$

Nonvanishing n^{th} order twists, (cont.)

Theorem(Brubaker, Bucur, C, Frechette, Hoffstein) Let K be a number field containing the n^{th} roots of unity. Let π be a cuspidal, self-dual automorphic representation of $GL_2(\mathbb{A}_K)$. Suppose $L\left(\frac{1}{2}, \pi\right) \neq 0$. Then there exist infinitely many idele class characters χ of K of order n s.t. $L\left(\frac{1}{2}, \pi \otimes \chi\right) \neq 0$.

So, BBCFH + Modularity + BSD implies:

Let E/K be a rank zero elliptic curve. Then there exist infinitely many degree n cyclic extensions F of K s.t. $\#E(F) < \infty$.

Nonvanishing n^{th} order twists, (cont.)

Remarks

(i) $n = 2$ (Waldspurger, Kohnen-Zagier) relates quadratic twists of L-series of π to Fourier coefficients of a metaplectic cuspidal automorphic representation. Don't know how to do this for $n > 2$!

(ii) Note necessity of assumption $L\left(\frac{1}{2}, \pi\right) \neq 0$ in both [FK] and [BBCFH]. Theorem should be true without this assumption. In fact, "almost all" twists should be nonzero when $n > 2$. (See e.g. [DFK] where a random matrix model is given for predicting frequency of vanishing twists.)

(iii) (Diaconu-Tian) Let p be a prime number, F a totally real field of odd degree s.t. $[F(\mu_p) : F] = 2$. Let W_δ be the twisted Fermat curve

$$W_\delta : x^p + y^p = \delta.$$

Then there exist infinitely many $\delta \in F^\times / F^{\times p}$ for which W_δ has no F-rational solutions.

The proof of this result is based on Zhang's extension of the Gross-Zagier formula to totally real fields and on Kolyvagin's technique of Euler systems. Then, a double Dirichlet series is used to show that a certain family of twisted L-series has nonvanishing central value infinitely often.

Idea of proof of BBCFH

Consider the double Dirichlet series

$$\begin{aligned} Z(s,w;\pi) &= \sum \frac{L\left(s,\pi\otimes\chi_d^{(n)}\right)}{|d|^w}\overline{G(1,\chi_d)} \\ &= \sum \frac{a_m D(w,m)}{|m|^s} \end{aligned}$$

where $D(w,m)$ is the Gauss sum Dirichlet series of Kubota, which is essentially the m^{th} Fourier coefficient of $\tilde{E}^{(n)}(z,W)$.

Some properties of $Z(s,w;\pi)$:

- Two functional equations:

$$\begin{aligned} (s,w) &\mapsto (1-s, w+2s-1) \\ (s,w) &\mapsto (w+s-\tfrac{1}{2}, 1-w). \end{aligned}$$

- Meromorphic continuation to $\mathbb{C}^2$.

- simple pole at $w=\frac{1}{2}+\frac{1}{n}$.

Idea of proof of BBCFH, (cont.)

Residue of $Z(s,w;\pi)$ at $w=\frac{1}{2}+\frac{1}{n}$: is a constant multiple of

$$L\left(s+\frac{1}{2n},\pi\otimes\theta^{(n)}\right).$$

Therefore

$$\sum_{|d|<x} L\left(\tfrac{1}{2},\pi\otimes\chi_d^{(n)}\right)\overline{G(1,\chi_d)}$$
$$\sim *L\left(\tfrac{1}{2}+\frac{1}{2n},\pi\otimes\theta^{(n)}\right)x^{\frac{1}{2}+\frac{1}{n}}+O_\epsilon(x^{\frac{1}{2}+\frac{1}{n}-\epsilon}).$$

How to show $L\left(\frac{1}{2}+\frac{1}{2n},\pi\otimes\theta^{(n)}\right)$ nonzero??

But nonvanishing assumption $L\left(\frac{1}{2},\pi\right) \neq 0$ provides another means of showing infinitely many nonvanishing central twists. Fix $s = \frac{1}{2}$. Then we have functional equation

$$Z\left(\frac{1}{2},w;\pi\right) \mapsto Z\left(\frac{1}{2},1-w;\pi\right).$$

But a nonzero Dirichlet series with such a functional equation must have infinitely many nonzero coefficients!

Recently an unconditional nonvanishing theorem for $n = 3$ has been established by [Brubaker-Friedberg-Hoffstein]. It remains an open problem to prove such a statement for $n \geq 4$.

Determination of modular forms by twists of critical values

Theorem (Luo-Ramakrishnan) Let f, g be two Hecke newforms for a congruence subgroup of $SL_2(\mathbb{Z})$. Suppose there exists a nonzero constant c s.t.

$$L\left(\frac{1}{2}, f \otimes \chi_d\right) = cL\left(\frac{1}{2}, g \otimes \chi_d\right)$$

for all quadratic characters χ_d. Then $f = cg$.

Theorem is more general than stated: applies to any critical values, also to other families of twists, ...

Theorem of Luo and Ramakrishnan

- An application to a question of Kohnen: let g_1, g_2 be two newforms in the Kohnen subspace $S^+_{k+\frac{1}{2}}$ with Fourier coefficients $b_1(n), b_2(n)$ respectively. Suppose
$$b_1^2(|D|) = b_2^2(|D|)$$
for almost all fundamental discriminants with $(-1)^k D > 0$. Then $g_1 = \pm g_2$, i.e. you can't just switch some of the signs of the coefficients and get another eigenform. Proof uses Waldspurger's formula relating the square of $b_j(|D|)$ to a suitable multiple of a twisted central value.

- Similar theorem holds for central derivatives in the case of negative root number [L-R]. By the theorem of Gross-Zagier, this allows one to determine an elliptic curve by heights of Heegner points.

Extensions of Luo-Ramakrishnan

- Ji Li, in his recent thesis, extends [LR] to π_1, π_2 cuspidal automorphic representations of $GL_2(\mathbb{A}_K)$, for K an arbitrary number field.

- (C, Diaconu) extends [LR] to symmetric squares of cusp forms on $GL_2(\mathbb{A}_{\mathbb{Q}})$.

All of these theorems are proved by considering twisted averages of twists of central L-values. The result of J. Li should also extend to cover the case of determining π by twisted central derivatives. Over a number field, the averaging method employed by [LR] (originating in the work of Iwaniec and Murty-Murty) will not work.

Proof of Chinta-Diaconu

By contrast with J. Li's result, the result of [CD] is valid only over $\mathbb{Q}$. It relies on GL_3 averaging techniques of Bump-Friedberg-Hoffstein and Diaconu-Goldfeld-Hoffstein. It would be interesting to see if the methods of Soundararajan will work here.

Idea: for r prime, consider the twisted averages

$$\sum_{0<d<x} L\left(\tfrac{1}{2}, \pi_1 \otimes \chi_d\right) \chi_r(d) P_d\left(\tfrac{1}{2}, \pi_1\right) \sim a_r(\pi_1) x \log x$$

$$\sum_{0<d<x} L\left(\tfrac{1}{2}, \pi_2 \otimes \chi_d\right) \chi_r(d) P_d\left(\tfrac{1}{2}, \pi_2\right) \sim a_r(\pi_2) x \log x$$

It follows that $a_r(\pi_1) = a_r(\pi_2)$ **provided** we can remove the weights $P_d\left(\tfrac{1}{2}, \pi_i\right)$ from the summation.

Proof of Chinta-Diaconu (cont.)

In sieving to to remove the weights, we require

- Lindelöf on average: $\sum_{|d|<x} |L\left(\frac{1}{2}, \pi \otimes \chi_d\right)| <<_\epsilon x^{5/4+\epsilon}$
- Work of Kim and Kim-Shahidi establishing the automorphy of the symmetric fourth power of a GL_2 form

Of course,

$$\sum_{|d|<x} |L\left(\frac{1}{2}, \pi \otimes \chi_d\right)| <<_\epsilon x^{1+\epsilon}$$

is expected. Our proof of the bound above is valid only over $\mathbb{Q}$, as we appeal to a character sum estimate of Heath-Brown. It would be of great interest to see what types of bounds can be proved over an arbitrary number field.

Mean values of quadratic zeta functions

Recall result of Jutilla, Takhtadzjan- Vinogradov:

$$\sum_{0<\pm d<X} L(1/2, \chi_d) \sim c_1 X \log X + c_2 x + O(x^{1-\epsilon}).$$

Multiplying the LHS by $\zeta(s)$, the sum will range over Dedekind zeta functions of quadratic extensions of $\mathbb{Q}$, ordered by conductor.

The quadratic multiple Dirichlet series associated to A_5 will give an analogous mean value result for zeta functions of biquadratic extensions.

Dynkin diagrams and multiple Dirichlet series

Given a simply-laced Dynkin diagram, vertices $v_1, \cdots, v_r$. Construct a multiple Dirichlet series which is roughly of the form:

$$\sum_{n_1,n_2,\ldots,n_r=1}^{\infty} \frac{\left[\prod_{j\,>\,i,\ v_j \text{ adjacent to } v_i} \left(\frac{n_i}{n_j}\right)\right]}{n_1^{s_1} \cdots n_r^{s_r}}.$$

Functional equations:

$$\sigma_i : s_i \mapsto 1 - s_i, s_j \mapsto s_j + s_i - 1/2,$$

if v_j is adjacent to v_i. The other s_k are left unchanged.

We have the Coxeter relations

$$\sigma_i^2 = 1, \qquad (\sigma_i \sigma_j)^{\epsilon(i,j)} = 1,$$

where $\epsilon(i,j) = 3$ if v_i and v_j are adjacent nodes in the Dynkin diagram, and $\epsilon(i,j) = 2$ if they are not.

The region of absolute convergence of the heuristic multiple Dirichlet series contains a Weyl chamber. Translating this region by the group of functional equations will yield the analytic continuation of Z to $\mathbb{C}^r$.

The multiple Dirichlet series constructed in this manner are conjectured to coincide with the Whittaker coefficients of Eisenstein series on the metaplectic double cover of the split simply connected semisimple group associated with this Dynkin diagram. (Bump-Hoffstein verified for A_2.)

See Brubaker-Bump-Chinta-Friedberg-Hoffstein

Biquadratic zeta functions

This heuristic can be precisely realized for A_5. The associated multiple Dirichlet series is roughly of the form $Z(s_1, s_2, s_3, s_4, s_5) =$

$$\sum_{d_2, d_4} \frac{L(s_1, \chi_{d_2}) L(s_3, \chi_{d_2 d_4}) L(s_5, \chi_{d_4})}{d_2^{s_2} d_4^{s_4}}.$$

Specialize to $s_1, s_3, s_5 = 1/2$, analytically continue, and contour integration:

Theorem(C)

$$\sum_{\substack{d_1, d_2 > 0 \\ d_1 d_2 < x \\ odd}} a(d_1, d_2) L_2(\tfrac{1}{2}, \chi_{d_1}) L_2(\tfrac{1}{2}, \chi_{d_2}) L_2(\tfrac{1}{2}, \chi_{d_1 d_2})$$

$$\sim \frac{\zeta_2(\frac{3}{2}) \zeta_2(2)^3}{48} X \log^4 X,$$

as $X \to \infty$.