Function field example of a quadratic double Dirichlet series

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The rational function field $\mathbb{F}_q(t)$

Notation:

- q is an odd prime power, congruent to 1 mod 4 (for simplicity)

- $\mathbb{F}_q[t]$ = polynomial ring in t with coefficients in the finite field $\mathbb{F}_q$. This is a PID. The nonzero prime ideals of $\mathbb{F}_q[t]$ are generated by irreducible polynomials.

- $\mathbb{F}_q(t)$ quotient field

- Define $\mathbf{N}(f) = |f| = q^{\deg f}$ for $f \in \mathbb{F}_q[t]$

The zeta function of $\mathbb{F}_q[t]$

- $\zeta(s)$ defined by Euler product or Dirichlet series

$$\prod_{\substack{P \in \mathbb{F}_q[t] \\ irred,monic}} \left(1 - \frac{1}{|P|^s}\right)^{-1} = \sum_{\substack{f \in \mathbb{F}_q[t] \\ monic,nonzero}} \frac{1}{|f|^s}$$

- Geometric series: $\zeta(s) =$

$$\sum_{n=0}^{\infty} \frac{\#\text{ of monic polys of deg } n}{q^{ns}} = \frac{1}{1 - q^{1-s}}$$

- Functional equation

$$\zeta^*(s) := \frac{1}{1 - q^{-s}} \zeta(s) = q^{2s-1} \zeta^*(1-s)$$

Quadratic residue symbol

For f an irreducible, monic polynomial in $\mathbb{F}_q[t]$, define

$$\chi_f(g) = \left(\frac{f}{g}\right) = g^{(|f|-1)/2}(\bmod f).$$

Thus $\chi_f(g) = \pm 1$ for f, g relatively prime.

If f_1, f_2 are two monic polynomials s.t. $f_1 f_2$ is squarefree, we define $\chi_{f_1 f_2} = \chi_{f_1}\chi_{f_2}$. Thus χ_f now makes sense whenever f is monic and squarefree.

Quadratic Reciprocity Let $f, g \in \mathbb{F}_q[t]$ be monic, squarefree and relatively prime. Then

$$\left(\frac{f}{g}\right) = \left(\frac{g}{f}\right)$$

Quadratic Dirichlet L-series

We define the L-series associated to the quadratic residue symbol χ_f by

$$\begin{aligned} L(s,\chi_f) &= \prod_P \left(1 - \frac{\chi_f(P)}{|P|^s}\right)^{-1} \\ &= \sum_{g\neq 0} \frac{\chi_f(g)}{|g|^s} \end{aligned}$$

Functional equation: Define

$$L^*(s,\chi_f) = \begin{cases} \frac{1}{1-q^{-s}} L(s,\chi_f) & \text{if deg } f \text{ even} \\ L(s,\chi_f) & \text{if deg } f \text{ odd} \end{cases}$$

Then, $L^*(s,\chi_f)$

$$= \begin{cases} q^{2s-1}|f|^{1/2-s} L^*(1-s,\chi_f) & \text{if deg } f \text{ even} \\ q^{2s-1}|qf|^{1/2-s} L^*(1-s,\chi_f) & \text{if deg } f \text{ odd} \end{cases}$$

The A_2 quadratic double Dirichlet Series

We wish to construct a double Dirichlet series of the form

$$Z(s,w) = \sum_{\substack{f \in \mathbb{F}_q[t] \\ monic, nonzero}} \frac{L(s,\chi_f)}{|f|^w} = \sum\sum \frac{\left(\frac{f}{g}\right)}{|f|^w |g|^s}$$

We want to define the quadratic residue symbols in such a way that

- the definition agrees with our old definition when fg is squarefree
- summing over g (resp. f) produces an L-series in s (resp. w) with an Euler product and satisfying the "right" functional equation

It turns out that there is a unique way to do this.

The A_2 quadratic double Dirichlet Series (cont.)

Let

$$Z(s,w) = \sum_f \sum_g \frac{\chi_{f_0}(\widehat{g})a(g,f)}{|f|^w |g|^s}$$

where

- f_0 is the squarefree part of f,
- $\widehat{g}$ is the part of g relatively prime to f, and
- the coefficients $a(g,f)$ should be multiplicative and chosen to ensure the proper functional equations.

The weighting coefficients $a(g, f)$

What does this last condition mean? Multiplicativity means

$$a(g, f) = \prod_{\substack{P^\alpha || g \\ P^\beta || f}} a(P^\alpha, P^\beta).$$

Thus

$$L(s, \widehat{\chi}_f) := \sum_g \frac{\chi_{f_0}(g) a(g, f)}{|g|^s}$$

has the Euler product

$$\prod_P \left(\sum_{k=0}^{\infty} \frac{\chi_{f_0}(\widehat{P}^k) a(P^k, f)}{|P|^{ks}} \right) = L(s, \chi_{f_0}) \mathcal{Q}_f(s),$$

say, where $\mathcal{Q}_f(s)$ is a finite Euler product supported in the primes dividing f to order greater than 1.

Weighting polynomials and functional equations

Functional Equation: We want $L(s, \widehat{\chi}_f)$

$$= \begin{cases} q^{2s-1}\frac{1-q^{-s}}{1-q^{s-1}}|f|^{1/2-s}L(1-s,\widehat{\chi}_f) & \text{if deg } f \text{ even} \\ q^{2s-1}|qf|^{1/2-s}L(1-s,\widehat{\chi}_f) & \text{if deg } f \text{ odd} \end{cases}$$

It follows that the weighting polynomials must satisfy the functional equation

$$\mathcal{Q}_f(s) = \left|\frac{f}{f_0}\right|^{\frac{1}{2}-s} \mathcal{Q}_f(1-s).$$

Examples Let P be an irreducible polynomial of norm p

(i) $\mathcal{Q}_1(s) = \mathcal{Q}_P(s) = 1$

(ii) $\mathcal{Q}_{P^2}(s) = 1 - \frac{1}{p^s} + \frac{p}{p^{2s}}$

(iii) $\mathcal{Q}_{P^3}(s) = 1 + \frac{p}{p^{2s}}$

(iv) $\mathcal{Q}_{P^4}(s) = 1 - \frac{1}{p^s} + \frac{p}{p^{2s}} - \frac{p}{p^{3s}} + \frac{p^2}{p^{4s}}$

A generating function

Reformulate the functional equations of the $\mathcal{Q}$ in terms of the coefficients $a(P^k, P^l)$.

Fix an irreducible polynomial P of norm p and let $x = p^{-s}, y = p^{-w}$. Construct the generating series

$$H(x,y) = \sum_{k,l=0}^{\infty} a(P^k, P^l) x^k y^l.$$

Summing over one index (say k) while leaving the other fixed , we get the P-part of $L(s, \widehat{\chi}_{P^l})$:

$$\sum_k a(P^k, P^l) x^k = \begin{cases} \mathcal{Q}_{P^l}(x) & \text{if } l \text{ odd} \\ \frac{1}{1-x}\mathcal{Q}_{P^l}(x) & \text{if } l \text{ even} \end{cases}$$

Recall that the weighting polynomials satisfy

$$\mathcal{Q}_{P^{2l+i}}(x) = (x\sqrt{p})^{2l} \mathcal{Q}_{P^{2l+i}}\left(\frac{1}{px}\right)$$

for $i = 0, 1$.

An axiomatic description of the generating function $H(x,y)$

By virtue of the functional equations satisfied by the $\mathcal{Q}$ the generating series $H(x,y)$ will satisfy a certain functional equation. We describe this now, together with the limiting behavior and x,y symmetry of H.

(A1) $H(x,y) = H(y,x)$

(A2) $H(x,0) = 1/(1-x)$

(A3) The auxiliary functions

$$\begin{aligned} H_0(x,y) &:= (1-x)\left[H(x,y) + H(x,-y)\right], \\ H_1(x,y) &:= \frac{1}{y}\left[H(x,y) - H(x,-y)\right] \end{aligned}$$

are invariant under the transformation

$$(x,y) \mapsto \left(\tfrac{1}{px}, xy\sqrt{p}\right).$$

The generating function $H(x, y)$ and functional equations of $Z(s, w)$

There is a unique power series in x, y satisfying **A1, A2** and **A3**:

$$H(x, y) = \frac{1 - xy}{(1 - x)(1 - y)(1 - px^2y^2)}.$$

With the $a(P^k, P^l)$ defined implicitly by the above generating series, the double Dirichlet series $Z(s, w)$ will satisfy functional equations

$$(s, w) \mapsto (1 - s, w + s - \tfrac{1}{2})$$
$$(s, w) \mapsto (s + w - \tfrac{1}{2}, 1 - w)$$

These two functional equations generate a group G, isomorphic to the dihedral group of order 6.

The region of absolute convergence of the double Dirichlet series contains (essentially) a fundamental domain (or Weyl chamber) for the action of G on $\mathbb{C} \times \mathbb{C}$. Translating this region by the group of functional equations yields the analytic continuation of $Z(s, w)$ to all of $\mathbb{C}^2$.

Application:mean values of L-functions

Analytic properties of a Dirichlet series can often be translated (via contour integration or Tauberian theorems) into information about partial sums of the coefficients of the series.

For example, let $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a holomorphic function of s for $\mathrm{Re}(s) > \sigma \in \mathbb{R}$. Suppose that $F(s)$ has a pole of order $r+1$ at $s = \sigma$ with leading term c and is otherwise holomorphic for $\mathrm{Re}(s) > \sigma - \epsilon$. Then, under some mild growth restrictions on F,

$$\sum_{n<X} a_n \sim \frac{c}{r!} X (\log X)^r.$$

One application of the theory of multiple Dirichlet series is to deduce mean value properties for special values of L-functions from the analytic properties of a multiple Dirichlet series.

To describe this in this simple example, we first need to compute the poles and residues of $Z(s, w)$.

Poles of $Z(s,w)$

The double Dirichlet series

$$Z(s,w) = \sum_f \frac{L(s,\widehat{\chi}_f)}{|f|^w}$$

has an obvious pole at $s = 1$ coming from the pole of the ζ-function when f is a perfect square. Translating by the group G of functional equations gives the complete set of polar divisors of $Z(s,w)$:

$$s = 1, w = 1, s + w = 3/2.$$

(The other translates of $s = 1$ by the group G do not produce further poles as they get cancelled out by the poles of the gamma function.)

The residue at $w = 1$

We will use the expression

$$Z(s,w) = \sum_g \frac{L(w,\chi_{g_0})\mathcal{Q}_g(w)}{|g|^s}$$

and knowledge of the weighting polynomials to compute the residue of $Z(s,w)$ at $w = 1$.

The numerator $L(w,\chi_{g_0})\mathcal{Q}_g(w)$ of the summand has a simple pole at $w = 1$ iff g is a perfect square. In this case, the residue of $L(w,\chi_{g_0})\mathcal{Q}_g(w)$ is simply $c \cdot \mathcal{Q}_g(1)$, where c is the residue of the zeta function. Now, $\mathcal{Q}_g(w) = \prod_{P^{2\alpha}||g} \mathcal{Q}_{P^{2\alpha}}(w)$.

¿From the explicit computation of $H(x,y)$ we find that

$$\sum_{k=0}^{\infty} \frac{\mathcal{Q}_{P^{2k}(1)}}{p^{2ks}} = \frac{1}{1-p^{-2s}},$$

and hence $\mathcal{Q}_{P^{2k}}(1) = 1$ for all k, P, which implies

$$\operatorname*{Res}_{w=1} Z(s,w) = R_1(s) = c\zeta(2s).$$

The pole of $Z\left(\frac{1}{2}, w\right)$ at $w = 1$

To compute mean values of $L\left(\frac{1}{2}, \widehat{\chi}_f\right)$ we need to understand the polar structure of $Z\left(\frac{1}{2}, w\right)$ as a function of w. The location of the first pole ($w = 1$) is immediate from what we have already done. The computation of the order is a little more involved.

In a neighborhood of $\left(\frac{1}{2}, 1\right)$ the double Dirichlet series looks like

$$Z(s, w) = \frac{R_1(s)}{w - 1} + \frac{R_2(s)}{w + s - \frac{3}{2}} + Y(s, w),$$

where $Y(s, w)$ is holomorphic in a neighborhood of $\left(\frac{1}{2}, 1\right)$.

The pole of $Z\left(\frac{1}{2}, w\right)$ at $w = 1$

Using the facts that $R_1(s)$ has a simple pole at $s = \frac{1}{2}$ and that $Z\left(\frac{1}{2}, w\right)$ is holomorphic for $w > 1$ we deduce that $R_2(s)$ must also have a simple pole $s = \frac{1}{2}$ which cancels the pole from R_1. Therefore, we have

$$Z(s,w) = \frac{A_1}{(w-1)(s-\frac{1}{2})} + \frac{A_2}{w-1} - \frac{A_1}{(w+s-\frac{3}{2})(s-\frac{1}{2})} + \frac{B_2}{w+s-\frac{3}{2}} + Y(s,w)$$

for some constants A_1, A_2, B_2. Setting $s = \frac{1}{2}$ we conclude that

$$Z\left(\frac{1}{2}, w\right) = \frac{A_1}{(w-1)^2} + \frac{A_1'}{w-1} + O(1)$$

in a neighborhood of $w = 1$, where $A_1' = A_2 + B_2$.

Mean values of $L\left(\frac{1}{2}, \widehat{\chi}_f\right)$

By contour integration, it follows that

$$\sum_{|f|<x} L\left(\frac{1}{2}, \widehat{\chi}_f\right) = A_1 x \log x + A_1' x + o(x)$$

as $x \to \infty$.

Since A_1 is nonzero, it follows that $L\left(\frac{1}{2}, \chi_f\right)$ is nonzero infinitely often.

Special values (and in particular, nonvanishing special values) of L-functions are often of arithmetic interest.

Computing $Z(s, w)$

As noted earlier, for a function field, the group of functional equations satisfied by the double Dirichlet series $Z(s, w)$ will force it to be a rational function. So what is it? I will describe a method of determining $Z(s, w)$ which works also for the FHL-series (constructed from n^{th} order Dirichlet L-functions), but I can't seem to get it to work in any other case. I'll continue to work with the quadratic case below.

Goal With $a(g, f)$ defined as above, express

$$Z(s, w) = \sum_f \sum_g \frac{\chi_{f_0}(\widehat{g})a(g, f)}{|f|^w |g|^s}$$

as a rational function of $x = q^{-s}, y = q^{-w}$.

Computing $Z(s, w)$, (cont.)

Recall: if $f = f_0 f_1^2$ with f_0 squarefree,

$$L(s, \widehat{\chi}_f) = \sum_{\substack{g \in \mathbb{F}_q[t] \\ \text{monic}}} \frac{a(g, f)\chi_{f_0}(\widehat{g})}{|g|^s} = L(s, \chi_{f_0})\mathcal{Q}_f(s).$$

Because of the functional equation $L(s, \widehat{\chi}_f)$ satisfies, it is either

- if f is not a perfect square, a polynomial of degree $n - 1$ in q^{-s}, or
- if f is a perfect square, then

  $$L(s, \widehat{\chi}_f) = \mathcal{Q}_f(s)\zeta(s)$$

  where $\mathcal{Q}_f(s)$ is a polynomial in q^{-s} of degree n with $\mathcal{Q}_f(1) = 1$.

Convolutions of rational fucntions

Let $R_1(x,y)$ and $R_2(x,y)$ be two rational functions, regular at the origin

$$R_1(x,y) = \sum_{j,k\geq 0} b_1(j,k)x^j y^k$$
$$R_2(x,y) = \sum_{j,k\geq 0} b_2(j,k)x^j y^k.$$

Then we let let $R_1 \star R_2$ denote the power series defined by

$$(R_1 \star R_2)(x,y) = \sum_{j,k\geq 0} b_1(j,k)b_2(j,k)x^j y^k.$$

Then $(R_1 \star R_2)(x,y)$ is again a rational function of x and y. Indeed, write $(R_1 \star R_2)(x,y) =$

$$\int\int R_1(z_1,z_2)R_2\left(\frac{x}{z_1},\frac{y}{z_2}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

and evaluate the integral by partial fractions.

Computing $Z(s,w)$, (cont.)

By the remarks about the degree of $L(s,\widehat{\chi}_f)$, we know

$$\sum_{deg\ \ g=m} a(g,f) = 0$$

if deg $f \leq m$, unless f is a perfect square. We write

$$Z(s,w) = Z_0(s,w) + Z_0(w,s) - Z_1(s,w)$$

where

$$Z_0(s,w) = \sum_{m \geq n \geq 0} \frac{1}{q^{ns}q^{mw}} \sum_{\substack{deg\ \ f=n \\ deg\ \ g=m}} a(g,f)$$

and

$$Z_1(s,w) = \sum_{n \geq 0} \frac{1}{q^{ns}q^{nw}} \sum_{\substack{deg\ \ f=n \\ deg\ \ g=n}} a(g,f)$$

Computing $Z(s, w)$, (cont.)

The nice thing now is that in evaluating Z_0 we only have to worry about when f is a perfect square. In this case, the character is $\chi_f(g)$ is not present, and we have a stronger multiplicativity statement which translates into an Euler product for a closely related series.

More precisely, let

$$Y_0(s, w) = \sum_{\substack{f,g \text{ monic} \\ f \text{ a perfect square}}} \frac{a(g, f)}{|f|^w |g|^s}.$$

Then Y_0 has an Euler product, and using our knowledge of $a(P^k, P^l)$ we may compute

$$Y_0(s, w) = \frac{1 - q^{1-s-2w}}{(1 - q^{1-2w})(1 - q^{1-s})(1 - q^{2-2s-2w})}.$$

Computing $Z(s, w)$, (concl.)

The rest is easy: Since $Z_0 = Y_0 \star K$, for

$$K(x, y) = \sum_{m \geq n \geq 0} x^n y^m = \frac{1}{(1-x)(1-xy)},$$

we may compute

$$Z_0(s, w) = \frac{1}{(1-q^{1-w})(1-q^{3-2s-2w})}.$$

By a similar argument, we find

$$Z_1(s, w) = \frac{1}{(1-q^{3-2s-2w})}.$$

Putting everything together, we arrive at

$$Z(s, w) = \frac{1-q^{2-s-w}}{(1-q^{1-s})(1-q^{1-w})(1-q^{3-2s-2w})}$$

or after setting $x = q^{-s}, y = q^{-w}$,

$$\boxed{Z(s, w) = \frac{1-q^2xy}{(1-qx)(1-qy)(1-q^3x^2y^2)}}$$